Appendix A

1.1 Derivatives of the group velocities with respect to anisotropic parameters

The sensitivity of the group velocity with respect to the elastic moduli has been discussed by some researchers (Zhou and Greenhalgh, 2005; Li et al., 2013). Here we derived the analytical expressions of the derivatives explicitly, which includes both the elastic moduli and Thomsen-type parameters.

The relation between the group and phase velocities in VTI medium is (Thomsen, 1986):

$$g_Q^2 = v_Q^2 + (\frac{\partial v_Q}{\partial \theta})^2, \quad \text{(A-1)}$$

where v_Q is the phase velocity and θ is the phase angle.

The derivative of the group velocities with respect to anisotropic parameters $\boldsymbol{m}$ is (Zhou and Greenhalgh, 2005):

$$\frac{\partial g_Q}{\partial \boldsymbol{m}} = \frac{v_Q}{g_Q}[\frac{\partial v_Q}{\partial \boldsymbol{m}} + \frac{1}{v_Q}(\frac{\partial v_Q}{\partial \theta})(\frac{\partial^2 v_Q}{\partial \boldsymbol{m} \partial \theta})]. \quad \text{(A-2)}$$

Then the derivatives we need to find are $\frac{\partial v_Q}{\partial \boldsymbol{m}}$, $\frac{\partial v_Q}{\partial \theta}$ and $\frac{\partial^2 v_Q}{\partial \boldsymbol{m} \partial \theta}$.

The group angle φ is given by (Tang and Li, 2008; Li et al., 2013):

$$\varphi = \theta + \Delta\theta,$$

$$\tan(\Delta\theta) = \frac{1}{v_Q}\frac{\partial v_Q}{\partial \theta}. \quad \text{(A-3)}$$

In the following derivation, the assumption $\frac{\partial \theta}{\partial c_{ij}} = 0$ is used. We assume that the perturbation of the phase angle is of high order to the traveltime perturbation (Zhang &Toksoz, 1998; Li et al., 2013).

A.1 c_{ij} parametrization

If $\boldsymbol{m}$ is represented by elastic moduli c_{ij}, the solution of phase velocities for a VTI medium is (Thomsen, 1986):

$$v_{qP,qSV} = \{\frac{1}{2}\,[c_{33} + c_{55} + (c_{11} - c_{33})sin^2\theta \pm D(\theta)]\}^{\frac{1}{2}},$$

$$v_{SH} = \{c_{55} + (c_{66} - c_{55})sin^2\theta\}^{\frac{1}{2}}, \qquad \text{(A-4)}$$

where $+$ in the right side corresponds to qP wave and $-$ corresponds to qSV wave. $D(\theta)$ is:

$$D(\theta) = \{\,(c_{33} - c_{55})^2 + D_1 sin^2\theta + D_2 sin^4\theta\}^{\frac{1}{2}}$$

$$D_1 = 2[2(c_{13} + c_{55})^2 - (c_{33} - c_{55})(c_{11} + c_{33} - 2c_{55})]$$

$$D_2 = (c_{11} + c_{33} - 2c_{55})^2 - 4(c_{13} + c_{55})^2. \qquad \text{(A-5)}$$

Then the term $\frac{\partial v_Q}{\partial c_{ij}}$ is given by:

$$\frac{\partial v_{qP,qSV}}{\partial c_{ij}} = \frac{1}{4v_{qP,qSV}}\left(\frac{\partial P}{\partial c_{ij}} \pm \frac{\partial D(\theta)}{\partial c_{ij}}\right),$$

$$\frac{\partial v_{SH}}{\partial c_{ij}} = \begin{cases} \frac{1}{2v_{SH}}cos^2\theta, & ij = 55 \\ \frac{1}{2v_{SH}}sin^2\theta, & ij = 66 \\ 0, & others \end{cases}, \qquad \text{(A-6)}$$

where $P = c_{33} + c_{55} + (c_{11} - c_{33})sin^2\theta$. The derivatives $\frac{\partial P}{\partial c_{ij}}$ and $\frac{\partial D(\theta)}{\partial c_{ij}}$ can be directly derived:

$$\frac{\partial P}{\partial c_{ij}} = \begin{cases} sin^2\theta, & ij = 11 \\ cos^2\theta, & ij = 33 \\ 1, & ij = 55 \\ 0, & others \end{cases}$$

$$\frac{\partial D(\theta)}{\partial c_{ij}} =$$

$$\frac{1}{2D(\theta)}\begin{cases} -2(c_{33}-c_{55})sin^2\theta + 2(c_{11}+c_{33}-2c_{55})sin^4\theta, & ij = 11 \\ 2(c_{33}-c_{55}) - 2(c_{11}+2c_{33}-3c_{55})sin^2\theta + 2(c_{11}+c_{33}-2c_{55})sin^4\theta, & ij = 33 \\ -2(c_{33}-c_{55}) + 2(c_{11}+3c_{33}+4c_{13})sin^2\theta - 4(c_{11}+c_{33}-2c_{13})sin^4\theta, & ij = 55. \\ 8(c_{13}+c_{55})[sin^2\theta - sin^4\theta], & ij = 13 \\ 0, & others \end{cases}$$

(A-7)

The terms $\frac{\partial v_Q}{\partial \theta}$ in Eq. A-2 is given by

$$\frac{\partial v_{qP,qSV}}{\partial \theta} = \frac{1}{4v_{qP,qSV}}\left(\frac{\partial P}{\partial \theta} \pm \frac{\partial D(\theta)}{\partial \theta}\right),$$

$$\frac{\partial v_{SH}}{\partial \theta} = \frac{(c_{66}-c_{55})}{2v_{SH}} sin2\theta \ ,$$

$$\frac{\partial P}{\partial \theta} = (c_{11}-c_{33})sin2\theta,$$

$$\frac{\partial D(\theta)}{\partial \theta} = \frac{1}{2D(\theta)}\{D_1 sin2\theta + 2D_2 sin^2\theta sin2\theta\}. \qquad \text{(A-8)}$$

The term $\frac{\partial^2 v_Q}{\partial c_{ij}\partial \theta}$ can be solved with Eqn. A-7 and A-8:

$$\frac{\partial^2 v_{qP,qSV}}{\partial c_{ij}\partial \theta} = -\frac{1}{v_{qP,qSV}}\frac{\partial v_{qP,qSV}}{\partial \theta}\frac{\partial v_{qP,qSV}}{\partial c_{ij}} + \frac{1}{4v_{qP,qSV}}\left(\frac{\partial^2 P}{\partial c_{ij}\partial \theta} \pm \frac{\partial^2 D(\theta)}{\partial c_{ij}\partial \theta}\right),$$

$$\frac{\partial^2 v_{SH}}{\partial c_{ij}\partial \theta} = \begin{cases} \frac{-1}{2v_{SH}}\left[sin^2\theta + 2\frac{\partial v_{SH}}{\partial \theta}\frac{\partial v_{SH}}{\partial c_{ij}}\right], & ij = 55 \\ \frac{1}{2v_{SH}}\left[sin^2\theta - 2\frac{\partial v_{SH}}{\partial \theta}\frac{\partial v_{SH}}{\partial c_{ij}}\right], & ij = 66, \\ 0, & others \end{cases}$$

$$\frac{\partial^2 P}{\partial c_{ij}\partial \theta} = \begin{cases} sin2\theta, & ij = 11 \\ -sin2\theta, & ij = 33, \\ 0, & others \end{cases}$$

$$\frac{\partial^2 D(\theta)}{\partial c_{ij}\partial \theta} = \frac{1}{2D(\theta)}\left\{\frac{\partial D_1}{\partial c_{ij}} sin2\theta + 2\frac{\partial D_2}{\partial c_{ij}} sin^2\theta sin2\theta - 2\frac{\partial D(\theta)}{\partial c_{ij}}\right\}, \qquad \text{(A-9)}$$

where $\frac{\partial D_1}{\partial c_{ij}}$ and $\frac{\partial D_2}{\partial c_{ij}}$ are given by:

$$\frac{\partial D_1}{\partial c_{ij}} = \begin{cases} -2(c_{33}-c_{55}), & ij = 11 \\ 2(c_{11}+2c_{33}-3c_{55}), & ij = 33 \\ 2(c_{11}+3c_{33}+4c_{13}), & ij = 55, \\ 8(c_{13}+c_{55}), & ij = 13 \\ 0, & others \end{cases}$$

$$\frac{\partial D_1}{\partial c_{ij}} = \begin{cases} 2(c_{11}+c_{33}-2c_{55}), & ij = 11 \\ 2(c_{11}+c_{33}-2c_{55}), & ij = 33 \\ -4(c_{11}+c_{33}-2c_{13}), & ij = 55, \\ -8(c_{13}+c_{55}), & ij = 13 \\ 0, & others \end{cases} \quad \text{(A-10)}$$

1.2 Thomsen-type parametrization

The relationship between Thomsen notation and stiffness coefficients of VTI media is given by (Thomsen, 1986):

$$V_{P0} = \sqrt{c_{33}},\ V_{S0} = \sqrt{c_{55}},$$

$$\varepsilon = \frac{c_{11}-c_{33}}{2c_{33}},\ \gamma = \frac{c_{66}-c_{44}}{2c_{44}},\ \delta = \frac{(c_{13}+c_{44})^2-(c_{33}-c_{44})^2}{2c_{33}(c_{33}-c_{44})}. \quad \text{(A-11)}$$

Then the phase velocities become:

$$v_{qP} = \{{V_{P0}}^2\ [1+\varepsilon sin^2\theta + Q(\theta)]\}^{\frac{1}{2}},$$

$$v_{qSV} = \{{V_{S0}}^2\ \left[1+(\frac{V_{P0}}{V_{S0}})^2\varepsilon sin^2\theta - (\frac{V_{P0}}{V_{S0}})^2 Q(\theta)\right]\}^{\frac{1}{2}},$$

$$v_{SH} = \{{V_{S0}}^2(1+2\gamma)sin^2\theta\}^{\frac{1}{2}}, \quad \text{(A-12)}$$

where

$$Q(\theta) = \frac{1}{2}\mathrm{F}\{[1+\frac{4\delta^*}{F^2}sin^2\theta cos^2\theta + \frac{4(F+\varepsilon)\varepsilon}{F^2}sin^4\theta]^{\frac{1}{2}} - 1\},$$

$$\mathrm{F} = 1-(\frac{V_{P0}}{V_{S0}})^2,\quad \delta^* = (2\delta-\varepsilon)F. \quad \text{(A-13)}$$

We can solve for $\frac{\partial v_Q}{\partial \boldsymbol{m}}$, $\frac{\partial v_Q}{\partial \theta}$ and $\frac{\partial^2 v_Q}{\partial \boldsymbol{m}\partial \theta}$ with Eqn. A-12. Alternatively, we can replace the terms in A.1 with the following relationships:

$$c_{33} = {V_{P0}}^2,\ c_{55} = {V_{S0}}^2$$

$$c_{11} = (2\varepsilon + 1){V_{P0}}^2,\ c_{66} = (2\gamma + 1){V_{S0}}^2,$$

$$c_{13} = \{({V_{P0}}^2 - {V_{S0}}^2)[{V_{P0}}^2(2\delta + 1) - {V_{S0}}^2]\}^{\frac{1}{2}} - {V_{S0}}^2. \qquad \text{(A-14)}$$