Appendix

This article studied the unsteady incompressible flow of a couple stress Casson fluid through a porous material in a channel. The direction of the flow is considered along the x-axis. The distance between the plates is d. The length of both the plates is infinite. The fluid and both plates are at rest at time $t_1 = 0$, with ambient temperature and concentration T_1 and C_1, respectively. After time $t_1 = 0^+$, the left plate at ($y_1 = 0$) begins to oscillate with characteristic velocity U and frequency ω, while the right plate at $y = d$ stays at rest. The temperature and concentration of the left plate are also raised to $T_1 + (T_d - T_1)At_1$ and $C_1 + (C_d - C_1)At_1$, respectively as displayed in Fig. 1.

In this appendix, the unsteady incompressible flow of couple stress Casson fluid is considered.

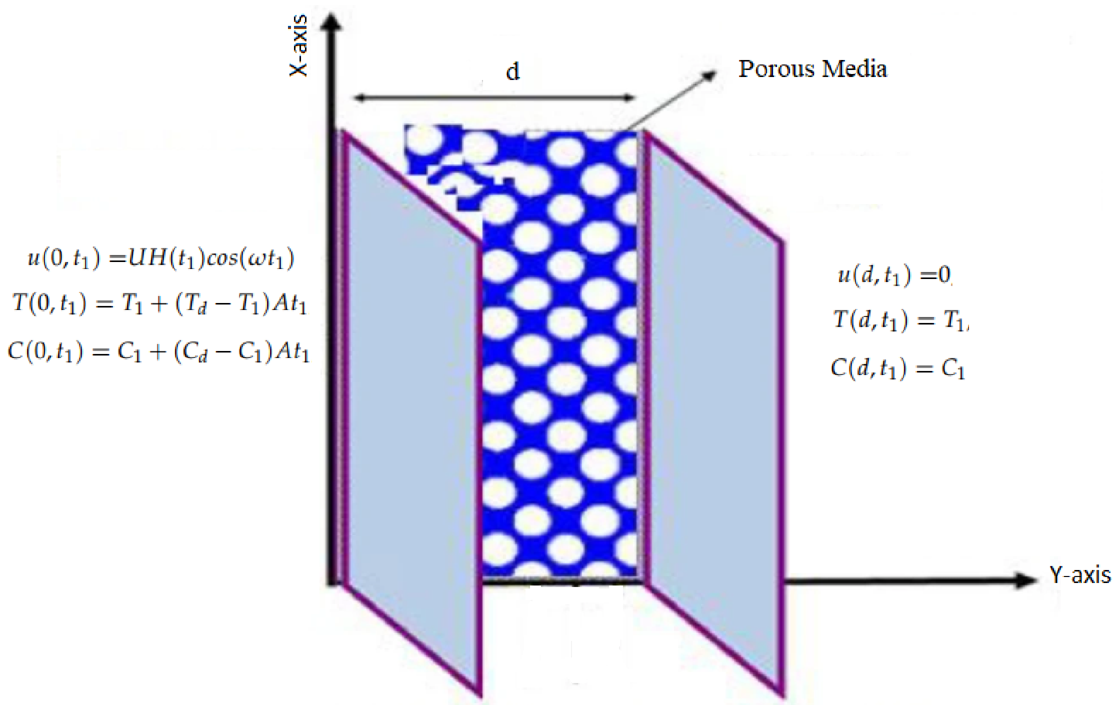

Figure 1: *Geometry of the problem.*

The governing equations for unsteady couple stress Casson fluid flow between infinite parallel plates are given by:

The equation of continuity is defined as:

$$\frac{\partial \rho}{\partial t} + \tilde{\nabla} \cdot (\rho \vec{V}) = 0, \tag{1}$$

Here "del" or "nabla" ∇ is a vector operator, t, ρ and $\vec{V}$ shows time variable, density, and the velocity vector field receptively.

The velocity vector field is defined as $\vec{V} = (u, v, w)$ with u, v and w are the velocity components along the axis i.e x, y and z axis.

For constant density, the flow becomes incompressible flow. Then Equation of continuity reduces to:

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2}$$

In the present work, one dimensional and unidirectional flow is considered. Thus, the velocity field $\vec{V}$ can be expressed as [?]:

$$\vec{V} = (u(y,t), 0, 0). \tag{3}$$

For our flow, the velocity, temperature, and concentration profiles are expressed as:

$$\vec{V} = (u(y,t), 0, 0), \qquad T = (T(y,t), 0, 0), \qquad C = (C(y,t), 0, 0). \tag{4}$$

for incompressible Casson is stated as:

$$\tau_{ij} = \begin{cases} 2\left(\mu_p + \frac{p_y}{\sqrt{2\pi}}\right)\mathbf{e_{ij}}, & \pi > \pi_b \\ 2\left(\mu_p + \frac{p_y}{\sqrt{2\pi}}\right)\mathbf{e_{ij}}, & \pi > \pi_b \end{cases} \tag{5}$$

where, τ_{ij} is the$(i,j)^{th}$ components of the stress tensor, p_y is yield stress of the fluid, μ_p is the dynamic viscosity for plastic non-Newtonian fluid, $\mathbf{e_{ij}}$ is the $(i,j)^{th}$ components of the deformation rate for non-Newtonian fluid, π_b is the critical value for deformation rate, π is the multiplication of deformation rate with itself, and $\pi = \mathbf{e_{ij}}\mathbf{e_{ij}}$.

The governing equation for incompressible flow of couple stress fluid is given by:

$$\rho\frac{D\vec{V}}{Dt} = \text{div } \tilde{\mathbf{T}} - \eta\vec{\nabla}^4\vec{V} + \rho\tilde{\mathbf{f}}. \tag{6}$$

Here $\rho\vec{f}$ are the body forces. The mathematical expression for body forces can be expressed as:

$$\rho\tilde{\mathbf{f}} = \vec{J} \times \vec{B} + \vec{R} + \rho\vec{g}, \tag{7}$$

As no magnetic field is considered, therefor:

$$\vec{J} \times \vec{B} = 0. \tag{8}$$

Substituting equation (8) in equation (7), we obtain:

$$\rho\tilde{\mathbf{f}} = \vec{R} + \rho\vec{g}, \tag{9}$$

The Cauchy stress tensor for incompressible non-Newtonian Casson fluid is:

$$\mathbf{T} = -P\mathbf{I} + \mu D. \tag{10}$$

where, I, P, μ and D is identity tensor, and rate of strain tensor respectively.
The rate of strain tensor is mathematically define as:

$$D = \frac{1}{2}(J + J^t), \tag{11}$$

where, the velocity gradient J and its transpose J^t is give below:

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}, \qquad J^t = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix}. \tag{12}$$

For velocity field defined in equation (3), the equation (11) can be written as:

$$J = \begin{pmatrix} 0 & \frac{\partial u}{\partial y} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad J^t = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

Substituting equation (13) in equation (11), we have:

$$D = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

Also the identity tensor in matrix notation is given by:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{then} \qquad p\mathbf{I} = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}. \tag{15}$$

Now, substituting equations (14) and (15) in equation (10), we have:

$$\mathbf{T} = \begin{pmatrix} -p & \frac{\mu}{2}\frac{\partial u}{\partial y} & 0 \\ \frac{\mu}{2}\frac{\partial u}{\partial y} & -p & 0 \\ 0 & 0 & -p \end{pmatrix}. \tag{16}$$

The divergence of Cauchy stress tensor give in equation (16) is stated as:

$$\vec{\nabla}.\mathbf{T} = \begin{pmatrix} -\frac{\partial p}{\partial x} + \frac{\mu}{2}\frac{\partial^2 u}{\partial y^2} \\ -\frac{\partial p}{\partial y} \\ -\frac{\partial p}{\partial z} \end{pmatrix}. \tag{17}$$

As the flow is in x direction, therefore $\frac{\partial p}{\partial y}$ and $\frac{\partial p}{\partial y}$ both are zero. The equation (17) reduce to:

$$\vec{\nabla}.\mathbf{T} = \left(-\frac{\partial p}{\partial x} + \frac{\mu}{2}\frac{\partial^2 u}{\partial y^2}\right) i. \tag{18}$$

The pressure is given by:

$$p = p_d + p_h. \tag{19}$$

Here, p_d and p_h indicates dynamic and hydrostatic pressures respectively. As in free convection, the fluid flow is due to only hydrostatic pressure. Thus, for constant density, we have:

$$\frac{\partial p}{\partial x} = \frac{\partial p_d}{\partial x} = -\rho_\infty g, \tag{20}$$

where ρ_∞ is the ambient density of the fluid.
Substituting eqauation (**??**) into eqauation (**??**), we have:

$$\vec{\nabla}.\mathbf{T} = \left(\rho_\infty g + \frac{\mu}{2}\frac{\partial^2 u}{\partial y^2}\right) i. \tag{21}$$

Incorporating equations (9) and ((21)) into equation (6) and using the velocity profile define in equation (3); we have:

$$\rho\frac{\partial u}{\partial t} = \frac{\mu}{2}\frac{\partial^2 u}{\partial y^2} + \rho_\infty g - \eta\frac{\partial^4 u}{\partial y^4} + \vec{R} + \rho\vec{g}. \tag{22}$$

Considering equation (5) for Casson fluid, the momentum equation for the flow ca be written as:

$$\rho\frac{\partial u}{\partial t} = \left(\mu_p + \frac{p_y}{\sqrt{2\pi}}\right)\frac{\partial^2 u}{\partial y^2} + \rho_\infty g - \eta\frac{\partial^4 u}{\partial y^4} + \vec{R} + \rho\vec{g}. \tag{23}$$

Equation (23) can be written as:

$$\rho\frac{\partial u}{\partial t} = \mu_p\left(1 + \frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2} + \rho_\infty g - \eta\frac{\partial^4 u}{\partial y^4} + \vec{R} + \rho\vec{g}, \tag{24}$$

here $\beta = \frac{\mu_p\sqrt{2\pi}}{p_y}$ is the Casson parameter for the fluid.
The mathematical form of Darcy's resistance for Casson fluid is $R = \mu_p\left(1 + \frac{1}{\beta}\right)\frac{\phi}{k_1}$. In addition, the gravitational acceleration is to the flow of fluid. Therefor equation (24) can be written as:

$$\rho\frac{\partial u}{\partial t} = \mu_p\left(1 + \frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2} - \eta\frac{\partial^4 u}{\partial y^4} + \mu_p\left(1 + \frac{1}{\beta}\right)\frac{\phi}{k_1}u + \vec{g}(\rho_\infty - \rho). \tag{25}$$

Equation (25) can be written in more simple form as:

$$\rho\frac{\partial u}{\partial t} = \mu\left(1 + \frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2} - \eta\frac{\partial^4 u}{\partial y^4} + \mu\left(1 + \frac{1}{\beta}\right)\frac{\phi}{k_1}u + \vec{g}(\rho_\infty - \rho). \tag{26}$$

Joseph Valentin Boussinesq $(1842 - 1929)$ developed the Boussinesq approximation. This approximation states that "Variation in density is only important in buoyancy term i.e ρg and in the rest of equation it can be ignored".

For pure substance (materials free of impurities), the density ρ can be expressed as:

$$\rho = \rho(p, T, C). \tag{27}$$

Where p, T, and C indicates the pressure, temperature and concentration respectively.

Taking total differential of equation (27); we have:

$$d\rho = \frac{\partial \rho}{\partial p}\, dp + \frac{\partial \rho}{\partial T}\, dT + \frac{\partial \rho}{\partial C}\, dC. \tag{28}$$

Which can be expressed as:

$$d\rho = K_T \rho dp - \beta_T \rho dT + K_C \rho dp - \beta_C \rho dC, \tag{29}$$

where, K_T and K_C indicates temperature and concentration compressibility with uniform temperature and concentration respectively.

The expression for thermal coefficient β_T at constant temperature are given by:

$$K_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right) \approx \frac{1}{\rho}\left(\frac{\Delta \rho}{\Delta p}\right). \tag{30}$$

The above equation can be written as:

$$\Delta \rho = K_T \rho \Delta p. \tag{31}$$

Similarly;

$$\beta_T = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right) \approx -\frac{1}{\rho}\left(\frac{\Delta \rho}{\Delta p}\right), \tag{32}$$

The above equation can be written as:

$$\Delta \rho = -\beta_T \rho \Delta p \tag{33}$$

The concentration expansion β_C at constant temperature is:

$$K_C = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right) \approx \frac{1}{\rho}\left(\frac{\Delta \rho}{\Delta p}\right). \tag{34}$$

The above equation can be written as:

$$\Delta \rho = K_C \rho \Delta p. \tag{35}$$

Similarly;

$$\beta_C = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right) \approx -\frac{1}{\rho}\left(\frac{\Delta \rho}{\Delta p}\right), \tag{36}$$

The above equation can be written as:

$$\Delta\rho = -\beta_C \rho \Delta p. \tag{37}$$

In free convection flow, thermal and mass compressibility have constant temperature and concentration. Thus, the effect of K_T and K_T can be ignored in equation (29); we can write:

$$d\rho = -\beta_T \rho dT - \beta_C \rho dC, \tag{38}$$

which can be expressed as:

$$\rho_\infty - \rho = -\beta_T \rho (T_\infty - T) - \beta_C \rho (C_\infty - C). \tag{39}$$

The above equation can be written as:

$$\rho_\infty - \rho = \beta_T \rho (T - T_\infty) + \beta_C \rho (C - C_\infty). \tag{40}$$

Incorporating equation (40) in equation (26); we have:

$$\rho \frac{\partial u}{\partial t} = \mu \left(1 + \frac{1}{\beta}\right) \frac{\partial^2 u}{\partial y^2} - \eta \frac{\partial^4 u}{\partial y^4} + \mu \left(1 + \frac{1}{\beta}\right) \frac{\phi}{k_1} u + \beta_T \rho \vec{g} (T - T_\infty) + \beta_C \rho \vec{g} (C - C_\infty) \tag{41}$$