Supplementary Material

APPENDIX A: HISTORICAL NOTES

Equations such as Eq. 11 in the main article have historically arisen in nineteenth century classical physics and twentieth century quantum physics. The former example is associated with the "equation of telegraphy" and the latter example with the "Klein-Gordon equation." In the following paragraphs, these two examples are briefly discussed.

After a successful submarine telegraph cable was laid across the English Channel in 1851, plans were made to lay one across the entire Atlantic from Ireland to Newfoundland. After failures in the late 1850's, two submarine cables came into operation in September 1866. William Thomson (1824–1907), later (after 1892) known as Lord Kelvin, was actively engaged in the scientific and engineering aspects of these projects, even participating in the laying of the cables aboard the vessel *Great Eastern*. One of the problems with these early, extremely long cables was that well-defined Morse code signals sent from one end would arrive at the other end in a delayed and considerably distorted form, making the signals blurred and useless. The problem and solution can be understood by noting that the two coupled equations for the propagation of the voltage $u(x,t)$ and current $I(x,t)$ along the cable are

$$C\frac{\partial u}{\partial t} + Gu + \frac{\partial I}{\partial x} = 0 \qquad \text{and} \qquad L\frac{\partial I}{\partial t} + RI + \frac{\partial u}{\partial x} = 0, \tag{S1}$$

where L and R respectively denote constant values of the series inductance and resistance, while C and G denote constant values of the shunt capacity and conductance. Eliminating $I(x,t)$ between the two first order equations in Eq. S1, we obtain the second order "equation of telegraphy"

$$\frac{\partial^2 u}{\partial t^2} + (\alpha+\beta)\frac{\partial u}{\partial t} + \alpha\beta u - c^2\frac{\partial^2 u}{\partial x^2} = 0, \tag{S2}$$

where $\alpha = G/C$, $\beta = R/L$, and $c^2 = (LC)^{-1}$. If $\alpha = \beta = 0$, Eq. S2 reduces to the simple wave equation, so, under such conditions, a signal could propagate without damping and without dispersion. Although it is not possible to design such a perfect cable, it is insightful to note the following simple transformation of Eq. S2 from dependent variable $u(x,t)$ to $v(x,t)$:

$$u(x,t) = e^{-\frac{1}{2}(\alpha+\beta)t} v(x,t) \qquad \Longrightarrow \qquad \frac{\partial^2 v}{\partial t^2} - \frac{1}{4}(\alpha-\beta)^2 v - c^2\frac{\partial^2 v}{\partial x^2} = 0. \tag{S3}$$

If we can design a cable with $\alpha = \beta \neq 0$, then $v(x,t)$ will obey the simple wave equation and $u(x,t)$ will propagate with damping $e^{-\frac{1}{2}(\alpha+\beta)t}$ but without dispersion. If the damping could be tolerated by using a very sensitive signal detector, a useful non-dispersed signal should be possible. Thus, the mirror galvanometer was greatly improved by Kelvin, who patented his device in 1858. The device proved to be an excellent means of receiving messages through a long cable. Note that another way to compensate for damping effects is to place signal amplifiers at several places along the cable, as was done later with AT&T submarine telephone cables; however, vacuum tube amplification and solid state amplification had not been invented at the time of Kelvin. An important difference between tropical cyclone dynamics and the design of submarine telegraph cables is that in the tropical cyclone case we cannot adjust the coefficients in our partial differential equation; we must accept the coefficients given to us by nature. In any event, such was the modest beginning of later (1950's and 60's) submarine telephone cables and today's submarine fiber optic cables that are an important part of the world's internet. For further discussion of the equation of telegraphy, see Stratton (1941, pages 550 and 594) and Courant and Hilbert (1962, page 192).

The Klein-Gordon equation is named in honor of Oskar Klein (1894–1977) and Walter Gordon (1893–1939), who wrote separate papers in 1926 proposing the equation to describe quantum particles in the framework of special relativity. According to Pais (1986, pages 288–289), between April and September 1926, this equation was independently stated by at least six authors. Pais refers to it as the "scalar wave equation," and describes its role as "relativity without spin." For further discussion, see Tomonaga (1997) and Robinett (1997, pages 54 and 435). Gordon received his doctoral degree in 1921 from the University of Berlin, working with Max Planck, and later with Max von Laue. The publication dates (1926) of the two papers is just after classic quantum mechanics papers by Werner Heisenberg and Erwin Schrödinger, but before the classic Dirac equation, which predicted the existence of antimatter. Oskar Klein (1894–1977) grew up in Sweden and in his early career studied with Svante Arrhenius (1859–1927) in Stockholm and with Niels Bohr in Copenhagen during the period 1918–1920. Note that Arrhenius was one of the founders of the science of physical chemistry and was the recipient of the 1903 Nobel Prize in Chemistry; among atmospheric scientists, Arrhenius is perhaps best known for his 1896 paper hypothesizing that atmospheric CO_2 increases due to human activity could result in higher global temperatures (Arrhenius, 1896). In 1923 Klein accepted a position at the University of Michigan, where he worked on the problem of the anomalous Zeeman effect. In 1926 Klein was appointed docent at Lund University in Sweden, becoming one of Bohr's closest collaborators and contributing ideas on correspondence and complementarity and on the development of Heisenberg's uncertainty principle. One of Klein's interests was unified field theory, a problem which he approached by extension to a fifth dimension. These early ideas of Kaluza-Klein theory continue to be explored by the present generation of physicists. In 1926 Klein published the paper in which he determined atomic transition probabilities and introduced what later became known as the Klein-Gordon equation, the first relativistic wave equation:

$$\frac{\partial^2\psi}{\partial t^2} + \frac{m^2c^4}{\hbar^2}\psi - c^2\nabla^2\psi = 0 \quad \Longrightarrow \quad \nu^2 = (mc^2/\hbar)^2 + c^2/\lambda^2 \quad \text{(de Broglie-Einstein relation)}, \tag{S4}$$

where m is the particle mass, c is the speed of light, $\hbar$ is Planck's constant divided by 2π, ν is the frequency of the de Broglie wave and λ is its wavelength. However, this equation did not result in the correct fine structure of the hydrogen atom, leading to the introduction of the concept of spin by Pauli and Dirac. Since the Klein-Gordon equation is incompatible with spin, it is only useful for calculations involving spinless particles. Nevertheless, it was an important part of the development of quantum theory. To help interpret Eq. S4, note that the Compton wavelength is a quantum mechanical property of a particle. It was introduced by Arthur Compton in his explanation of the scattering of photons by electrons (a process known as Compton scattering). The Compton wavelength of a particle is equal to the wavelength of a photon whose energy is the same as the mass (using mass-energy equivalence) of that particle. The "reduced" Compton wavelength of a particle is given by $(\hbar/mc)$, where m is the particle mass and c is the speed of light. The value for the reduced Compton wavelength of the electron is 3.8616×10^{-13} m. Comparing the fluid mechanical equations (Eq. 11 in the main article) with the quantum mechanical equation (Eq. S4), the analogies are $f \leftrightarrow mc^2/\hbar$ and $\sqrt{g\bar{h}} \leftrightarrow c$, so the Rossby length $\sqrt{g\bar{h}}/f \sim 10^6$m is analogous to the reduced Compton length $\hbar/mc \sim 10^{-13}$m. It is remarkable that the same partial differential equation can be relevant for physical processes on such vastly different scales. For further reading on the remarkable life and legacy of Oskar Klein, see the webpage of the Oskar Klein Centre in Stockholm, Sweden.

While the telegraphy equation (Eq. S2) and the Klein-Gordon equation (Eq. S4) indicate some interesting mathematical connections with the gradient adjustment problem of atmospheric dynamics, there are also some interesting family connections. Oskar Klein was the grandfather of Dr. Clara Deser, an NCAR senior scientist who has made important contributions to understanding the earth's weather and climate, including the effects of increased atmospheric CO_2 levels due to human activity.

APPENDIX B: SOLUTION OF THE LINEAR PROBLEM VIA HANKEL TRANSFORMS

A useful method for solving the linear Klein-Gordon equation for $v(r,t)$ in Eq. 11 in the main article is based on integral transforms. The integral representation of $v(r,t)$ involves order one Bessel functions $J_1(kr)$, since $v(r,t)$ vanishes at $r=0$; note that the integral transform solution of the equation for $h'(r,t)$, on the other hand, involves order zero Bessel functions $J_0(kr)$ since the radial derivative of $h'(r,t)$ vanishes at $r=0$. It follows that the azimuthal wind $v(r,t)$ can be represented as a superposition of order one Bessel functions $J_1(kr)$, where k is the radial wavenumber. The order one Hankel transform pair for the azimuthal wind is

$$\hat{v}(k,t) = \int_0^\infty v(r,t)\, J_1(kr)\, r\, dr, \qquad \text{and} \qquad v(r,t) = \int_0^\infty \hat{v}(k,t)\, J_1(kr)\, k\, dk. \tag{S5}$$

Note that standard terminology refers to Eq. S5 as a Hankel transform pair rather than a Bessel transform pair. The order one Hankel transform of the v equation in Eq. 11 in the main article yields the ordinary differential equation

$$\begin{gathered} \frac{d^2\hat{v}}{dt^2} + \omega^2 \hat{v} = r_0 f g\bar{h} J_1(kr_0)\left[1-(1+st)e^{-st}\right] \\ \text{where} \quad \omega(k) = \left(f^2 + g\bar{h}k^2\right)^{1/2} = \left(g\bar{h}\right)^{1/2}\left(\mu^2+k^2\right)^{1/2}, \end{gathered} \tag{S6}$$

with $s = 1/t_s$ and with $\mu = f/(g\bar{h})^{1/2}$ denoting the inverse of the Rossby length. Note that the order one Hankel transform of Eq. 11 in the main article involves multiplying it by $rJ_1(kr)$ and integrating over all r, followed by two integrations by parts on the third term in Eq. 11 in the main article. See Schubert et al. (1980) for further discussion on the use of these particular integral transforms. The solution of Eq. S6, satisfying the initial conditions $\hat{v}(k,0)=0$ and $\hat{v}_t(k,0)=0$, is

$$\begin{aligned} \hat{v}(k,t) = r_0 f \Bigg\{ 1 &- \frac{\omega^2}{s^2+\omega^2}\left(\frac{3s^2+\omega^2}{s^2+\omega^2} + st\right) e^{-st} \\ &- \left(\frac{s^2(s^2-\omega^2)}{(s^2+\omega^2)^2}\right)\cos(\omega t) - \left(\frac{2s^3\omega}{(s^2+\omega^2)^2}\right)\sin(\omega t) \Bigg\} \frac{J_1(kr_0)}{\mu^2+k^2}. \end{aligned} \tag{S7}$$

Using the spectral space solution (Eq. S7) in the second entry of Eq. S5, we find that the integral representation of the solution of the linear Klein-Gordon equation for $v(r,t)$ in Eq. 11 in the main article is

$$\begin{aligned} v(r,t) &= r_0 f \int_0^\infty \left\{ 1 - \frac{\omega^2}{s^2+\omega^2}\left(\frac{3s^2+\omega^2}{s^2+\omega^2} + st\right) e^{-st} \right\} \frac{J_1(kr_0)\,J_1(kr)}{\mu^2+k^2}\, k\, dk \\ &\quad - r_0 f \int_0^\infty \left\{ \left(\frac{s^2(s^2-\omega^2)}{(s^2+\omega^2)^2}\right)\cos(\omega t) + \left(\frac{2s^3\omega}{(s^2+\omega^2)^2}\right)\sin(\omega t) \right\} \frac{J_1(kr_0)\,J_1(kr)}{\mu^2+k^2}\, k\, dk. \end{aligned} \tag{S8}$$

The second line of Eq. S8 contains the oscillatory factors $\cos(\omega t)$ and $\sin(\omega t)$, where $\omega(k) = \left(f^2 + g\bar{h}k^2\right)^{1/2}$ is the inertia-gravity frequency for wavenumber k. Since the second line involves an integral over k, it describes dispersive inertia-gravity wave packets. In contrast, the first line does not contain oscillatory factors, and thus describes the evolving non-oscillatory part of the solution. In fact, when the forcing is slow enough, the entire right hand side of Eq. S8 reduces to the balanced solution $v_b(r,t)$. To see this, note that, when $s \ll \omega$, the oscillatory terms involving $\cos(\omega t)$ and $\sin(\omega t)$ become negligible and the time dependence of the remaining part (i.e., the first line) becomes identical to the time dependence of the forcing, so that the physical space solution (Eq. S8) simplifies to

$$v_b(r,t) = r_0 f \left[1 - (1+st)\,e^{-st}\right] \int_0^\infty \frac{J_1(kr_0)\,J_1(kr)}{\mu^2+k^2}\, k\, dk, \tag{S9}$$

where the subscript 'b' indicates that this is the 'balanced' solution. A simpler form of the balanced solution (Eq. S9) can be obtained by noting that the integral can be evaluated (e.g., see page 40 of Erdélyi et al., 1954, and page 679 of Gradshteyn and Ryzhik, 1980) to obtain the second entry of Eq. 15 in the main article.

APPENDIX C: ANALOGY BETWEEN SHALLOW WATER DYNAMICS AND THE DYNAMICS OF A CONTINUOUSLY STRATIFIED FLUID

In section 2 we developed an analogy between shallow water dynamics and the dynamics of a continuously stratified fluid. The analogy was based on a comparison of the mass continuity equations for the two fluid systems. Here we discuss an alternative analogy based on the PV dynamics of the two systems. Using independent variables (r,θ,t), the potential vorticity equation for the axisymmetric, hydrostatic, primitive equation model is $\sigma(DP/Dt) = -(\partial v/\partial\theta)(\partial\dot{\theta}/\partial r) + (f+\zeta_\theta)(\partial\dot{\theta}/\partial\theta)$, where $f+\zeta_\theta$ is the isentropic absolute vorticity, $P = (f+\zeta_\theta)/\sigma$ is the potential vorticity, $\sigma = -(1/g)(\partial p/\partial\theta)$ is the pseudodensity, and $(D/Dt) = (\partial/\partial t) + u(\partial/\partial r) + \dot{\theta}(\partial/\partial\theta)$ is the material derivative. Now transform this PV equation from (r,θ,t) to (R,Θ,τ), where $\Theta = \theta$, $\tau = t$, and R is the Lagrangian coordinate defined in section 4. Note that $(\partial/\partial\Theta) \neq (\partial/\partial\theta)$ and $(\partial/\partial\tau) \neq (\partial/\partial t)$ because $(\partial/\partial\Theta)$ and $(\partial/\partial\tau)$ imply fixed R, while $(\partial/\partial\theta)$ and $(\partial/\partial t)$ imply fixed r. Since $-(\partial v/\partial\theta)(\partial\dot{\theta}/\partial r) + (f+\zeta_\theta)(\partial\dot{\theta}/\partial\theta) = (f+\zeta_\theta)(\partial\dot{\Theta}/\partial\Theta)$, the PV equation becomes $(DP/Dt) = P(\partial\dot{\Theta}/\partial\Theta)$, where $\dot{\Theta} = \dot{\theta}$ and $(D/Dt) = (\partial/\partial\tau) + \dot{\Theta}(\partial/\partial\Theta)$. Defining the potential pseudodensity by $\sigma^* = f/P$, we can easily convert the PV equation to $(D\sigma^*/Dt) = -\sigma^*(\partial\dot{\Theta}/\partial\Theta)$. Now compare the shallow water model's potential depth equation in the independent variables (R,τ) and the continuously stratified model's potential pseudodensity equation in the independent variables (R,Θ,τ). These two equations can be written as

$$\frac{\partial h^*}{\partial\tau} = -h^* S \qquad \text{and} \qquad \frac{\partial\sigma^*}{\partial\tau} = -\sigma^*\left(\frac{\partial(\sigma^*\dot{\Theta})}{\sigma^*\partial\Theta}\right). \tag{S10}$$

Because of the close correspondence of the two equations in Eq. S10, we can make the following analogies: $h^* \Leftrightarrow \sigma^*$ and $S \Leftrightarrow [\partial(\sigma^*\dot{\Theta})/\sigma^*\partial\Theta]$. Note that in regions where the fractional variation of σ^* with respect to Θ is much less than the fractional variation of $\dot{\Theta}$ with respect to Θ (e.g., in a PV tower), this analogy simplifies to $S \Leftrightarrow (\partial\dot{\Theta}/\partial\Theta)$. In the lower tropospheric core region of a tropical cyclone, $[\partial(\sigma^*\dot{\Theta})/\sigma^*\partial\Theta] > 0$, so there should be an analogous mass sink in the shallow water equations to simulate a lower tropospheric layer. In the upper tropospheric core region of a tropical cyclone, $[\partial(\sigma^*\dot{\Theta})/\sigma^*\partial\Theta] < 0$, so there should be an analogous mass source in the shallow water equations to simulate an upper tropospheric layer. Note that the analogy for S derived from Eq. S10 differs in two ways from the analogy for S derived from Eq. 6 in the main article: σ^* replaces σ and $\partial/\partial\Theta$ replaces $\partial/\partial\theta$. It could be argued that the analogy for S derived from Eq. S10 is more useful because it is based on PV dynamics rather than simply mass conservation. For further discussion of the analogies presented here, the reader is referred to the axisymmetric, hydrostatic, three-layer, hurricane models of Ooyama (1969) and DeMaria and Pickle (1988), the former of which is based on shallow water concepts (i.e., no explicit thermodynamics) and the latter of which is based on the dynamics of a compressible fluid formulated in isentropic coordinates.

APPENDIX D: A SIMPLE UPPER BOUND ON THE AZIMUTHAL WIND

To help understand which tropical depressions and tropical storms do not develop into hurricanes, consider how the analytical solution for h^* and P (Eqs. 28 and 31 in the main article) can set a simple upper bound on the azimuthal wind $v(r,t)$. Since $P = (f+\zeta)(\bar{h}/h)$, and since $h < \bar{h}$ for the cyclonic flows considered here, it follows that $\zeta < P - f$, which can be used to set an upper bound on $v(r,t)$. First, write the solution for P (Eq. 31 in the main article) as in the second half of Eq. S11, where $r_0(t)$ is the decreasing radius of the PV disk. Define the maximum $v(r,t)$ field, denoted by $v^{(\mathrm{m})}(r,t)$, as the field that would result if all of the potential vorticity were partitioned to the wind field and none to the mass field, meaning that the fluid depth remained everywhere equal to $\bar{h}$. Then, $v(r,t)$ and $v^{(\mathrm{m})}(r,t)$ are related by

$$\frac{\partial(rv)}{r\,\partial r} \le \frac{\partial(rv^{(\mathrm{m})})}{r\,\partial r} \equiv P(r,t) - f, \qquad \text{where} \quad P(r,t) = f \begin{cases} (1-\epsilon)^{(1+t/t_s)e^{-t/t_s}-1} & \text{if } \ 0 \le r \le r_0(t) \\ 1 & \text{if } \ r_0(t) < r < \infty. \end{cases} \tag{S11}$$

Integration over r of the first entry in Eq. S11 yields the simple upper bound

$$v(r,t) \le v^{(\mathrm{m})}(r,t) = \tfrac{1}{2} r_0(t)\,[P(0,t) - f] \begin{cases} r/r_0(t) & \text{if } \ 0 \le r \le r_0(t) \\ r_0(t)/r & \text{if } \ r_0(t) \le r < \infty, \end{cases} \qquad \text{where} \quad r_0(t) = \left(\frac{f}{P(0,t)}\right)^{1/2} R_0. \tag{S12}$$

The last formula in Eq. S12 can be confirmed by noting that the absolute angular momentum of fluid particles on the edge of the collapsing PV disk must be conserved, i.e., $r_0(t)v^{(\mathrm{m})}(r_0(t),t) + \frac{1}{2}fr_0^2(t) = \frac{1}{2}fR_0^2$. The ultimate (i.e., $t \to \infty$) maximum wind and radius of maximum wind, obtained from Eq. S12, are given by

$$v^{(\mathrm{m})}(r_0(\infty),\infty) = \tfrac{1}{2} f R_0 \left(\frac{\epsilon}{(1-\epsilon)^{1/2}}\right) \quad \text{and} \quad r_0(\infty) = (1-\epsilon)^{1/2} R_0. \tag{S13}$$

Table S1 lists $r_0(\infty)$ and $v^{(\mathrm{m})}(r_0(\infty), \infty)$ for selected values of ϵ. Figure S1 shows radial profiles of $v^{(\mathrm{m})}(r, \infty)$, as determined from Eq. S12, for the five values $\epsilon = 0.960,\ 0.970,\ 0.980,\ 0.990,\ 0.995$. Since $v(r,t) \leq v^{(\mathrm{m})}(r,t)$, hurricane force values of $v(r,t)$ cannot be produced if $\epsilon < 0.975$.

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FIGURES

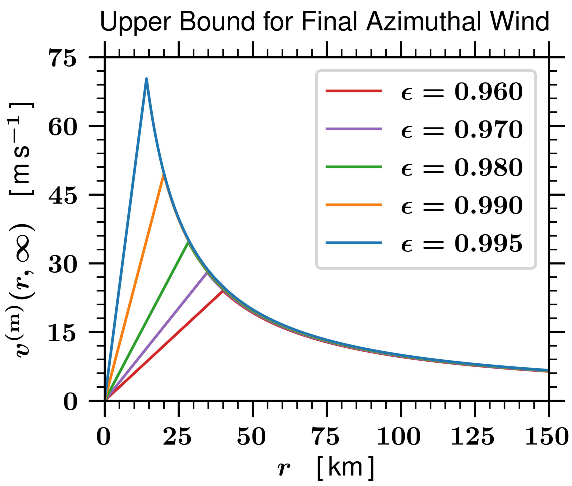

Figure S1. Radial profiles of $v^{(\mathrm{m})}(r, \infty)$, a simple upper bound for the final azimuthal wind as determined from (S12), for $R_0 = 200$ km, $f = 5 \times 10^{-5}\ \mathrm{s}^{-1}$, and five different $\epsilon = \mathcal{V}/(\pi R_0^2 \bar{h})$ values.

TABLES

Table S1. Ultimate (i.e., $t \to \infty$) tropical cyclone properties for a range of values of $\epsilon = \mathcal{V}/(\pi R_0^2 \bar{h})$, computed with the parameters $R_0 = 200$ km and $f = 5 \times 10^{-5}\ \mathrm{s}^{-1}$. The second column lists the value of the dimensionless potential vorticity $P(r,t)/f$ for $r = 0$ and in the limit $t \to \infty$, as computed from (S11). The third and fourth columns list the radius of maximum wind $r_0(t)$ and the maximum possible wind $v^{(\mathrm{m})}(r_0(t), t)$ in the limit $t \to \infty$, as computed from (S13). The last column lists the storm category, ranging from tropical depression (TD), to tropical storm (TS), to one of the five hurricane categories (C1–C5).

ϵ	$P(0,\infty)/f$	$r_0(\infty)$ (km)	$v^{(\mathrm{m})}(r_0(\infty), \infty)$ (m s^{-1})	Category
0.900	10.0	63.2	14.2	TD
0.950	20.0	44.7	21.2	TS
0.960	25.0	40.0	24.0	TS
0.970	33.3	34.6	28.0	TS
0.975	40.0	31.6	30.8	TS
0.980	50.0	28.3	34.6	C1
0.985	66.7	24.5	40.2	C1
0.990	100.0	20.0	49.5	C3
0.995	200.0	14.1	70.4	C5