A Non-dimensionalization

Here we non-dimensionalize (2.2) to obtain the PDE-ODE system (2.3). Let $[z]$ denote the unit of some variable z. In the SI unit system, we have

$$\begin{array}{rclrclrclrclrcl} [U] &=& \frac{\text{moles}}{\text{m}^2}\,, & [D_U] &=& \frac{\text{m}^2}{\text{s}}\,, & [\kappa_U] &=& \frac{1}{\text{s}}\,, & [T] &=& \text{s}\,, & [X] &=& \text{m}\,, \\ [M_j] &=& \text{moles}\,, & [\kappa_R] &=& \frac{1}{\text{s}}\,, & [\mu_c] &=& \text{moles}\,, & [\beta_{U,1}] &=& \frac{\text{m}}{\text{s}}\,, & [\beta_{U,2}] &=& \frac{1}{\text{m}\,\text{s}}\,. \end{array}$$

Letting L denote the length-scale of the domain, we introduce the dimensionless variables $u \equiv L^2U/\mu_c$, $v \equiv L^2V/\mu_c$, $t \equiv \kappa_R T$, $\mathbf{x} \equiv X/L$, $\mu_j \equiv M_j/\mu_c$, and $\eta_j \equiv H_j/\mu_c$. Then, we obtain that

$$\begin{array}{rclcrcll} \kappa_R\,\partial_t U &=& \frac{D_U}{L^2}\,\Delta_{\mathbf{x}}U - \kappa_U U\,, & & \partial_t u &=& D_u\Delta_{\mathbf{x}}u - \sigma_u u, & \mathbf{x}\in\Omega\backslash\bigcup_{j=1}^m\Omega_j\,, \\ \kappa_R\,\partial_t V &=& \frac{D_V}{L^2}\,\Delta_{\mathbf{x}}V - \kappa_V V\,, & \Leftrightarrow & \partial_t v &=& D_v\Delta_{\mathbf{x}}v - \sigma_v v, & \mathbf{x}\in\Omega\backslash\bigcup_{j=1}^m\Omega_j\,, \end{array}$$

where we have defined the dimensionless effective diffusivities D_u and D_v and degradation rates σ_u and σ_v by

$$D_u \equiv \frac{D_U}{L^2\kappa_R}\,, \qquad \sigma_u \equiv \frac{\kappa_U}{\kappa_R}\,, \qquad D_v \equiv \frac{D_V}{L^2\kappa_R}\,, \qquad \sigma_v \equiv \frac{\kappa_V}{\kappa_R}\,.$$

Since we assume that the common radius, denoted by L_c, of the cells is much smaller than the domain length-scale L, we introduce $\varepsilon \ll 1$ by $\varepsilon = L_c/L \ll 1$.

Next, by non-dimensionalizing the Robin boundary conditions in (2.2), we obtain

$$\begin{array}{rclcrcll} \frac{D_U}{L}\frac{\mu_c}{L^2}\,\partial_{n_{\mathbf{x}}}u &=& \beta_{U,1}\frac{\mu_c}{L^2}u - \beta_{U,2}\,\mu_c\mu_j\,, & & \varepsilon D_u\partial_{n_{\mathbf{x}}}u &=& d_1^u u - d_2^u\mu_j, & \mathbf{x}\in\partial\Omega_j\,, \\ \frac{D_V}{L}\frac{\mu_c}{L^2}\,\partial_{n_{\mathbf{x}}}v &=& \beta_{V,1}\frac{\mu_c}{L^2}v - \beta_{V,2}\,\mu_c\eta j\,, & \Leftrightarrow & \varepsilon D_v\partial_{n_{\mathbf{x}}}v &=& d_1^v v - d_2^v\eta_j, & \mathbf{x}\in\partial\Omega_j\,, \end{array}$$

where we have defined d_1^u, d_2^u, d_1^v, and d_2^v by

$$d_1^u \equiv \frac{\beta_{U,1}}{\kappa_R L}\varepsilon\,, \qquad d_2^u \equiv \frac{\beta_{U,2}L}{\kappa_R}\varepsilon\,, \qquad d_2^v \equiv \frac{\beta_{V,1}}{\kappa_R L}\varepsilon\,, \qquad d_2^v \equiv \frac{\beta_{V,2}L}{\kappa_R}\varepsilon\,.$$

Here, in order that there is an $\mathcal{O}(1)$ exchange across the cell membranes we have assumed that $\frac{\beta_{U,1}}{\kappa_R L}$, $\frac{\beta_{U,2}L}{\kappa_R}$, $\frac{\beta_{V,1}}{\kappa_R L}$ and $\frac{\beta_{V,2}L}{\kappa_R}$ are all $\mathcal{O}\left(\varepsilon^{-1}\right)$.

Lastly, we non-dimensionalize the intracellular reaction kinetics in (2.2) by

$$\begin{array}{rcl} \kappa_R\mu_c\,\frac{d}{dt}\mu_j &=& \kappa_R\mu_c\,f(\mu_j,\eta_j) + \int_{\partial\Omega_j}\left(\beta_{U,1}\frac{\mu_c L}{L^2}\,u - \beta_{U,2}\mu_c L\,\mu_j\right)\,dS_{\mathbf{x}}\,, \\ \kappa_R\mu_c\,\frac{d}{dt}\eta_j &=& \kappa_R\mu_c\,g(\mu_j,\eta_j) + \int_{\partial\Omega_j}\left(\beta_{V,1}\frac{\mu_c L}{L^2}\,v - \beta_{V,2}\mu_c L\,\eta_j\right)\,dS_{\mathbf{x}}\,, \end{array}$$

which yields the dimensionless intracellular reactions

$$\frac{d\mu_j}{dt} = f(\mu_j,\eta_j) + \frac{1}{\varepsilon}\int_{\partial\Omega_j}(d_1^u\,u - d_2^u\,\mu_j)\,dS_{\mathbf{x}}\,, \qquad \frac{d\eta_j}{dt} = g(\mu_j,\eta_j) + \frac{1}{\varepsilon}\int_{\partial\Omega_j}(d_1^v\,v - d_2^v\,\eta_j)\,dS_{\mathbf{x}}\,,$$

for each $j\in\{1,...,m\}$. This completes the derivation of (2.3).

B Reduced-wave Green's function for the unit disk

When Ω is the unit disk, the reduced-wave Green's function $G_\omega(\mathbf{x};\xi)$ and its regular part, satisfying (2.9) can be determined analytically using separation of variables as (see equations (6.10) and (6.11) of [20])

$$G_\omega(\mathbf{x};\xi) = \frac{1}{2\pi}K_0\left(\omega|\mathbf{x}-\xi|\right) - \frac{1}{2\pi}\sum_{n=0}^{\infty}\beta_n\cos\left(n(\psi-\psi_0)\right)\frac{K_n'(\omega)}{I_n'(\omega)}I_n\left(\omega|\mathbf{x}|\right)I_n\left(\omega|\xi|\right)\,, \tag{B.1a}$$

$$R_\omega(\xi) = \frac{1}{2\pi}\left(\log 2 - \gamma_e - \log\omega\right) - \frac{1}{2\pi}\sum_{n=0}^{\infty}\beta_n\frac{K_n'(\omega)}{I_n'(\omega)}\left[I_n\left(\omega|\xi|\right)\right]^2\,, \tag{B.1b}$$

where $\gamma_e \approx 0.5772$ is Euler's constant, and $I_n(z)$ and $K_n(z)$ are the modified Bessel functions of the first and second kind of order n, respectively. In (B.1), $\beta_0 \equiv 1$, $\beta_n \equiv 2$ for $n \geq 1$, while $\mathbf{x} \equiv |\mathbf{x}|(\cos\psi, \sin\psi)^T$, and $\xi \equiv |\xi|(\cos\psi_0, \sin\psi_0)^T$.

For a ring pattern where the cell centers $\mathbf{x}_k$ for $k \in \{1, \ldots, m\}$, are equidistantly spaced on a ring of radius r concentric within the unit disk, as in (4.1), all the Green's matrices used in the steady-state and linear stability analysis are circulant and symmetric matrices. As a result, each such matrix spectrum is available analytically.

Following Appendix A of [50], for an $m \times m$ circulant matrix $\mathcal{A}$, with possibly complex-valued matrix entries, its complex-valued eigenvectors $\tilde{\mathbf{v}}_j$ and eigenvalues α_j are $\alpha_j = \sum_{k=1}^{m} \mathcal{A}_{1k}\omega_j^{k-1}$ and $\tilde{\mathbf{v}}_j = \left(1, Z_j, \ldots, Z_j^{m-1}\right)^T$, for $j \in \{1, \ldots, m\}$. Here $Z_j \equiv \exp\left(\frac{2\pi i(j-1)}{m}\right)$ and $\mathcal{A}_{1k}$, for $k \in \{1, \ldots, m\}$, are the elements of the first row of $\mathcal{A}$. Since $\mathcal{A}$ is also a symmetric matrix, we have $\mathcal{A}_{1,j} = \mathcal{A}_{1,m+2-j}$, for $j \in \{2, \ldots, \lceil m/2 \rceil\}$, where the ceiling function $\lceil x \rceil$ is defined as the smallest integer not less than x. Therefore, $\alpha_j = \alpha_{m+2-j}$, for $j \in \{2, \ldots, \lceil m/2 \rceil\}$, so that there are $m-1$ eigenvalues with a multiplicity of two when m is odd, and $m-2$ such eigenvalues when m is even. As a result, we conclude that $\frac{1}{2}\left[\tilde{\mathbf{v}}_j + \tilde{\mathbf{v}}_{m+2-j}\right]$ and $\frac{1}{2i}\left[\tilde{\mathbf{v}}_j - \tilde{\mathbf{v}}_{m+2-j}\right]$ are two independent real-valued eigenvectors of $\mathcal{A}$, corresponding to the eigenvalues of multiplicity two. In summary, the matrix spectrum of a circulant and symmetric matrix $\mathcal{A}$, where the eigenvectors have been normalized by $\mathbf{v}_j^T \mathbf{v}_j = 1$, is

$$\alpha_j = \sum_{k=1}^{m} \mathcal{A}_{1k} \cos\left(\theta_j (k-1)\right), \quad j \in \{1, \ldots, m\}; \qquad \theta_j \equiv \frac{2\pi(j-1)}{m}; \qquad \mathbf{v}_1 = \frac{1}{\sqrt{m}}\mathbf{e}, \tag{B.2a}$$

$$\mathbf{v}_j = \sqrt{\frac{2}{m}}\left(1, \cos\left(\theta_j\right), \ldots, \cos\left(\theta_j(m-1)\right)\right)^T, \quad \mathbf{v}_{m+2-j} = \sqrt{\frac{2}{m}}\left(0, \sin\left(\theta_j\right), \ldots, \sin\left(\theta_j(m-1)\right)\right)^T \tag{B.2b}$$

for $j \in \{2, \ldots, \lceil m/2 \rceil\}$, where $\theta_j \equiv 2\pi(j-1)/m$. When m is even, there is an additional normalized eigenvector of multiplicity one given by $\mathbf{v}_{m/2+1} = m^{-1/2}(1, -1, 1, \ldots, -1)^T$.