# Supplementary Appendix

## A Conventions

For a connection with torsion whose connection coefficients are $\Gamma^\rho_{\mu\nu}$ the Riemann tensor $R_{\mu\nu\sigma}{}^\rho$ and torsion tensor $T^\rho{}_{\mu\nu}$ are given by

$$
\begin{aligned}
\left[\nabla_\mu, \nabla_\nu\right] X_\sigma &= R_{\mu\nu\sigma}{}^\rho X_\rho - T^\rho{}_{\mu\nu}\nabla_\rho X_\sigma \,, &\text{(A.1)}\\
\left[\nabla_\mu, \nabla_\nu\right] X^\rho &= -R_{\mu\nu\sigma}{}^\rho X^\sigma - T^\sigma{}_{\mu\nu}\nabla_\sigma X^\rho \,, &\text{(A.2)}
\end{aligned}
$$

which implies

$$
\begin{aligned}
R_{\mu\nu\sigma}{}^\rho &\equiv -\partial_\mu \Gamma^\rho_{\nu\sigma} + \partial_\nu \Gamma^\rho_{\mu\sigma} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \,, &\text{(A.3)}\\
T^\rho{}_{\mu\nu} &\equiv 2\Gamma^\rho_{[\mu\nu]} \,. &\text{(A.4)}
\end{aligned}
$$

These obey the following Bianchi identities

$$
\begin{aligned}
R_{[\mu\nu\sigma]}{}^\rho &= T^\lambda{}_{[\mu\nu} T^\rho{}_{\sigma]\lambda} - \nabla_{[\mu} T^\rho{}_{\nu\sigma]} \,, &\text{(A.5)}\\
\nabla_{[\lambda} R_{\mu\nu]\sigma}{}^\kappa &= T^\rho{}_{[\lambda\mu} R_{\nu]\rho\sigma}{}^\kappa \,. &\text{(A.6)}
\end{aligned}
$$

We define the Ricci tensor as

$$R_{\mu\nu} \equiv R_{\mu\rho\nu}{}^\rho \,. \quad \text{(A.7)}$$

We will assume that the connection is such that

$$\Gamma^\rho_{\mu\rho} = \partial_\mu \log M \,, \quad \text{(A.8)}$$

where $M$ is the integration measure, so that

$$R_{\mu\nu\rho}{}^\rho = 0 \,. \quad \text{(A.9)}$$

## B Details for Trautman condition computation

In Section 3.4 it is left to show that $h^{\mu[\gamma}\check{R}_{\mu(\nu\sigma)}{}^{\rho]} = 0$. This expression can be shown to be Galilean boost invariant. We prove $h^{\mu[\gamma}\check{R}_{\mu(\nu\sigma)}{}^{\rho]} = 0$ by showing that all the projections with $v^\nu v^\sigma$, $v^\nu h^{\sigma\lambda}$ and $h^{\nu\kappa}h^{\sigma\lambda}$ give zero. We will use that

$$\check{\nabla}_\rho v^\mu = -h^{\mu\nu} K_{\rho\nu} \,, \quad \text{(B.1)}$$

where $K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_v h_{\mu\nu}$ with $\mathcal{L}_v$ the Lie derivative along $v^\mu$. Using the definition of the Riemann tensor as well as (B.1) it follows that

$$v^\nu v^\sigma \check{R}_{\mu\nu\sigma}{}^\rho = -h^{\rho\lambda} v^\kappa \check{\nabla}_\kappa K_{\mu\lambda} \,. \quad \text{(B.2)}$$

Hence we find $h^{\mu[\gamma}\check{R}_{\mu(\nu\sigma)}{}^{\rho]} v^\nu v^\sigma = 0$ which holds since $h^{\mu\nu}$ is covariantly constant and we also used the fact that $K_{\mu\nu}$ is symmetric. We next turn to the projection of $h^{\mu[\gamma}\check{R}_{\mu(\nu\sigma)}{}^{\rho]} = 0$ with $h^{\nu\kappa}h^{\sigma\lambda}$. To this end define

$$X^{\alpha\beta\mu\nu} = h^{\alpha\rho} h^{\beta\sigma} \check{R}_{\rho\sigma\kappa}{}^\nu h^{\mu\kappa} \,. \quad \text{(B.3)}$$

The tensor $X^{\alpha\beta\mu\nu}$ is antisymmetric in its first two and last two indices[1]. It also obeys $X^{[\alpha\beta\mu]\nu} = 0$. Adding and subtracting $X^{[\alpha\beta\mu]\nu} = 0$, $X^{[\alpha\beta\nu]\mu} = 0$, $X^{[\beta\mu\nu]\alpha} = 0$ and $X^{[\mu\nu\alpha]\beta} = 0$ appropriately we obtain $X^{\alpha\beta\mu\nu} = X^{\mu\nu\alpha\beta}$. This can be used to show that $X^{\mu(\nu\alpha)\beta} - X^{\beta(\nu\alpha)\mu} = 0$ which is equivalent to $h^{\rho[\alpha}\check{R}_{\rho(\sigma\kappa)}{}^{\nu]}h^{\beta\sigma}h^{\mu\kappa} = 0$.

Finally we need to show that $h^{\rho[\alpha}\check{R}_{\rho(\sigma\kappa)}{}^{\nu]}v^{\sigma}h^{\mu\kappa} = 0$. To this end define

$$Y^{\alpha\mu\nu} = h^{\rho\alpha}\check{R}_{\rho\sigma\kappa}{}^{\nu}v^{\sigma}h^{\kappa\mu}\,. \tag{B.4}$$

In terms of this new object we have

$$4h^{\rho[\alpha}\check{R}_{\rho(\sigma\kappa)}{}^{\nu]}v^{\sigma}h^{\mu\kappa} = Y^{\alpha\mu\nu} - Y^{\nu\mu\alpha} + 2h^{\rho[\alpha}\check{R}_{\rho\kappa\sigma}{}^{\nu]}v^{\sigma}h^{\mu\kappa}\,. \tag{B.5}$$

Using $\check{R}_{[\rho\sigma\kappa]}{}^{\nu} = 0$ we can show that

$$Y^{\alpha\mu\nu} - Y^{\mu\alpha\nu} + h^{\rho\alpha}\check{R}_{\kappa\rho\sigma}{}^{\nu}v^{\sigma}h^{\mu\kappa} = 0\,. \tag{B.6}$$

Cyclically permuting the indices on this last equation leads to two more equations. Adding and subtracting these off (B.6) leads to

$$2Y^{\alpha\mu\nu} = -h^{\rho\alpha}\check{R}_{\kappa\rho\sigma}{}^{\nu}v^{\sigma}h^{\mu\kappa} - h^{\rho\nu}\check{R}_{\kappa\rho\sigma}{}^{\mu}v^{\sigma}h^{\kappa\alpha} + h^{\rho\mu}\check{R}_{\kappa\rho\sigma}{}^{\alpha}v^{\sigma}h^{\kappa\nu}\,. \tag{B.7}$$

This tells us that

$$Y^{\alpha\mu\nu} - Y^{\nu\mu\alpha} = -h^{\rho\nu}\check{R}_{\kappa\rho\sigma}{}^{\mu}v^{\sigma}h^{\kappa\alpha}\,. \tag{B.8}$$

This in turn can be used to obtain

$$4h^{\rho[\alpha}\check{R}_{\rho(\sigma\kappa)}{}^{\nu]}v^{\sigma}h^{\mu\kappa} = -h^{\rho\nu}\check{R}_{\kappa\rho\sigma}{}^{\mu}v^{\sigma}h^{\kappa\alpha} + h^{\rho\alpha}\check{R}_{\rho\kappa\sigma}{}^{\nu}v^{\sigma}h^{\kappa\mu} - h^{\rho\nu}\check{R}_{\rho\kappa\sigma}{}^{\alpha}v^{\sigma}h^{\kappa\mu}\,. \tag{B.9}$$

Finally, using

$$\check{R}_{\kappa\rho\sigma}{}^{\mu}v^{\sigma} = h^{\mu\gamma}\left(\check{\nabla}_{\kappa}K_{\rho\gamma} - \check{\nabla}_{\rho}K_{\kappa\gamma}\right)\,, \tag{B.10}$$

which follows from the definition of the Riemann tensor and (B.1), we obtain the result that

$$h^{\rho[\alpha}\check{R}_{\rho(\sigma\kappa)}{}^{\nu]}v^{\sigma}h^{\mu\kappa} = 0\,. \tag{B.11}$$

[1]Antisymmetry in the last two indices follows from $[\check{\nabla}_{\rho}\,, \check{\nabla}_{\sigma}]h^{\mu\nu} = 0$ which leads to $\check{R}_{\rho\sigma\kappa}{}^{(\nu}h^{\mu)\kappa} = 0$.