Supplementary Material

Appendix A: Brief background of the model eqs(1) and (2)

(a) Spruce budworm model [3, 4, 15, 45]

The Spruce budworm infestation model provides a real example of occasional outbreak of infestation in Spruce forests in North America, which in particular defoliate balsam fir trees. The model provides a good example to understand the oscillatory dynamics of interacting ecosystem of forest trees and insects. The main features of this model are:

(i) Leaves of the trees supply food and coverage for the budworm (to hide from birds),

(ii) Birds feed on budworm (if exist in large number) and other sources of food. So, birds population are assumed to be constant.

The model rate equation for the budworms population (N) considered here has the dimensionless from, eq.(1), viz.,

$$\begin{aligned} \frac{d}{d\tau}N(\tau) &= N(\tau) - \frac{N^2(\tau)}{K} - F\,N^2(\tau)/\left(N^2(\tau)+B^2\right) \\ &\equiv T_1 - T_2 - T_3 \end{aligned} \tag{A.1}$$

Here, $\tau = rt$ is the normalised time, with r the (linear) birth rate of budworms, k is the saturation population density related to density of food (foliage) on the trees in the absence of birds, F is the predation population and B is the measure of threshold for the budworm population. The function $f(N) = F\,N^2(\tau)/\left(N^2(\tau)+B^2\right)$ (called sigmoidal curve) represents the predation rate generated by birds subject to a maximum value, which satisfies the following conditions:

1. If N is small, $f(N)$ is close to zero as the birds will choose other types of prey,
2. $f(N)$ grows as the budworms become very large, and
3. $f(N)$ reaches some maximum value, if N is very large.

Consideration of the first term (where $T_{2,3} = 0$) in (A.1) leads to $\frac{dN}{d\tau} = N$, i.e $N(t) \propto e^{rt}$ which represents the growth of budworms at low density in absence of predators (so, death rate by normal causes is almost zero). The second term in $(A.1)$, $T_2 = \frac{N^2}{K}$ represents (premature) death rate due to finite resources. The competition between the two terms T_1 and T_2 (in absence of third term, $T_3 = 0$) forms what is known as basic logistic equation, $\frac{dN}{d\tau} = N(1 - \frac{N}{K})$, whose solution, $N(t) = c_1 K/(c_1 + Ke^{-rt})$ with c_1 is constant, is called the logistic function. The general eq.(A.1) in its dimensionless form with all terms $T_{1,2,3}$ present has three dimensionless parameters, K, F, B. Analytical and computational investigation of the bistable behaviour according to (A.1), in both steady and transient regions are given in details in [3, 4, 15, 45].

(b) The Thomas reaction model [4, 16], [45]-[47]

Enzymes are types of proteins that work as catalysts in chemical reactions in biological systems. They react effectively with definite components, called substrates, at very low concentrations. An example is, the haemoglobin substance in red blood cells that is composed of enzyme and oxygen together it forms a substrate. Most basic functions of biological membranes is to define a boundary within or between cells. Celluar processes depend on membrane ability to separate different areas of components, while allowing for regulated biological transport processes (such as, activators or inhibitors in a reaction) to occur. An artificial enzyme membranes can be a model for biological membranes having bound enzymes, as well, provide a good model to study certain enzyme- staining reactions. Specifically, in diffusion cells [16], the artificial enzyme membrane separates into two compartments containing solutions of substrates (product and effector). Both solutions enter the membrane as "inputs" and the output solution flux is then analysed. In the Thomas reaction mechanism model [4], the enzyme membrane involves two substrates, oxygen and uric acid of concentrations v and u, respectively. In dimensionless form, the rate equations are given as

[4],

$$u^{\cdot} = a - u - lR(u,v) \tag{A.2a}$$
$$v^{\cdot} = \gamma(b - v) - lR(u,v) \tag{A.2b}$$

with,

$$R(u,v) = \frac{uv}{1 + u + ku^2} \tag{A.2c}$$

The constants a, b, γ, l, k are real and positive: a and γb are the constant supplied rates for u, v, respectively. The terms $(-u)$ in (A.2a) and $(-\gamma v)$ in (A.2b) describe linear damping processes of both concentrations. Both concentrations u, v are used in the reaction at a rate $lR(u,v)$, where the form of $R(u,v)$ in (A.2c) exhibits substrate inhibition, with (k) is a measure of intensity of inhibition. Analytical and computational investigations of the model eqs.(A.2) have been given in detail in both transient and steady state regimes regarding the bistable behaviour in [4, 14].

Appendix B: Euler's Computational method for fractional differential equations

The fractional Euler's numerical method [28] to solve the FDE,

$$D^{(\alpha)}y(t) = f(t, y(t)), \;\; y(0) = y_o, \;\; 0 < \alpha \leq 1, t > 0$$

is summarised as follows:

(1) Subdivide the interval $t \in (0, a)$ into N-sub-intervals of equal width $h = a/N$ which gives with $t_i = ih, i = 0, 1 \ldots, N$ with $t_o = 0$.

(2) For $i = 1, 2, \ldots, N$, the program evaluate the formula $w_i = w_{i-1} + h^{\alpha} f(t_{i-1}, w_i)/\Gamma(\alpha+1)$, in the Caputo's derivative case, and $w_i = w_{i-1} + e^{h\alpha/(1-\alpha)} f(t_{i-1}, w_i)/(1-\alpha)$, in the Caputo- Fabrizio case.

(3) A set of points $(t_i, w_i), i = 0, 1, .., N$, is generated, where $y(t_i) = w_i$ with a varied accuracy $\epsilon = 10^{-4}$ to 10^{-3}.

As in [48], the accuracy differs according to the α values. In general, the truncated error is of order $(D^{(2\alpha)}y(t))(\xi_i).h^{2\alpha}(\Gamma(2\alpha+1))$. This was neglected for small step size. Odibat et al. [48], also, developed Trapezoidal method for fractional differential equations (as well as Taylor's formula). The error according to Euler's method was enough in our case and can be evaluated in a shorter evaluation-time.
The sample code of programmes of the numerical solutions of the FDEs, eq.(3),(4) are given by:
http://www.kau.edu.sa/GetFile.aspx?id=302674&fn=model1.pdf
http://www.kau.edu.sa/GetFile.aspx?id=302675&fn=model2.pdf