Supplementary Material

Gravitational influence of Saturn's rings on its moons

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Contents:

1. Annotated Mathematica computer code

Mathematica computer code

Ellipsoidal gravity definitions:

GM is a magnitude for the gravitational force, where M the mass interior to the ellipsoid,
a the equatorial radius,
b the polar semi-axis,
ee is measure of eccentricity: $ee^2 = a^2 - b^2$,
ω the rotation rate about the polar axis,
u is the ‘distance’ of the equipotential line from the origin,
β is the parametric latitude,
RR is cylindrical radius,
ZZ is cylindrical z-coord.

Cylindrical coordinates RR and ZZ are obtained from ellipsoidal coordinates u and β using:

$$RR = \sqrt{u^2 + ee^2}\ \mathrm{Cos}[\beta];$$

$$ZZ = u\ \mathrm{Sin}[\beta];$$

$$z = \frac{ee}{u}$$

Conversion from cylindrical into ellipsoidal coordinates:

This is not needed for calculations here, but the solution is used to produce the functions beta[] and uu[] below. Note that there are 16 solutions accounting for all symmetries and antisymmetries; only 1 is needed.

```
In[71]:= ClearAll["Global`*"]
thetsfm = Quiet[Solve[{R1 == Sqrt[u^2 + ee^2] Cos[β], Z1 == u Sin[β]}, {u, β}]];
```

Helper functions used in final solution:

In[57]:=
```
beta[RR_, ZZ_, EE_] =
  ArcCos[Sqrt[(EE^2 + RR^2 + ZZ^2 - Sqrt[EE^4 + RR^4 + ZZ^4 + 2 EE^2 ZZ^2 + 2 RR^2 ZZ^2 - 2 EE^2 RR^2]) / (2 EE^2)]];
uu[RR_, ZZ_, EE_] = ZZ / Sin[beta[RR, ZZ, EE]];
Quad[z_] = 1/(2 z^3) ((1 + 3/z^2) ArcTan[z] - 3/z); (* function used for quadrupole *)
Um[u_, a_, b_, GM_] = GM/Sqrt[a^2 - b^2] ArcTan[Sqrt[a^2 - b^2]/u]; (* mass term *)
Uq[u_, β_, ω_, a_, b_] = ω^2/2 (a^2 b^3)/u^3 Quad[Sqrt[a^2-b^2]/u]/Quad[Sqrt[a^2-b^2]/b] (Sin[β]^2 - 1/3);
(* quadrupole term *)
Ur[u_, β_, a_, b_, ω_] = ω^2/2 (u^2 + a^2 - b^2) Cos[β]^2; (* = ω^2/2 R^2: centrifugal term *)
Utotal[u_, β_, GM_, a_, b_, ω_ ] = Um[u, a, b, GM] + Uq[u, β, ω, a, b] + Ur[u, β, a, b, ω];
```

Equipotentials for central ellipsoid combined with plane as described in Shinbrot “Gravitational influence of Saturn’s rings on its moons”:

Only one quadrant is plotted: this is associated with a choice of symmetric, antisymmetric solutions mentioned above. Other quadrants can easily be provided, but doing so is computationally redundant.

In[63]:=
```
SetOptions[Manipulator, Appearance -> "Labeled"]; (* optional *)
MaxOuterDisk = 100;(* outer radius of plane, R_plane *)
gravfactor = 1 / 2; (* factor by which density of plane < density of moon *)

(* following is used by Mathematica to allow changes in parameters with sliders *)
Manipulate[
 theee = Sqrt[themajor^2 - theminor^2];
  (* following is plot of planar mass: *)p00 = Graphics[{Darker[Cyan], Opacity[0.9],
     {Disk[{0, 0}, {themajor, theminor}], Disk[{0, 0}, {MaxOuterDisk, diskthickness}]},
     White, Disk[{0, 0}, {negratio, diskthickness}]},
    PlotRange → {{-plotmin - plotradius, -plotmin}, {0, plotradius}}];

  (* following is definition of sum of net mass,
 quadrupole mass and centrifugal acceleration: *)SumFun[XX_, ZZ_,
    theGMFlat_, negratio_, themajor_, theminor_, theee_, theGM_, theomega_] =
```

```
Utotal[uu[√(XX² + 0²), ZZ, √(MaxOuterDisk² - diskthickness²)],
   beta[√(XX² + 0²), ZZ, √(MaxOuterDisk² - diskthickness²)],
   theGMFlat, MaxOuterDisk, diskthickness, 0]

  - Utotal[uu[√(XX² + 0²), ZZ, √((negratio themajor)² - diskthickness²)],
   beta[√(XX² + 0²), ZZ, √((negratio themajor)² - diskthickness²)],
   theGMFlat / (MaxOuterDisk² / (negratio themajor)²)
   , negratio themajor, diskthickness, 0]

  + Utotal[uu[√(XX² + 0²), ZZ, theee],
   beta[√(XX² + 0²), ZZ, theee], theGM, themajor, theminor, theomega];

 (* following is plot of equipotentials calculated above: *)
p01 = ContourPlot[SumFun[XX, ZZ, theGMFlat, negratio, themajor,
   theminor, theee, theGM, theomega] , {XX, -plotmin - plotradius, -plotmin},
  {ZZ, 0, plotradius}, Contours → Range[mincontour, maxcontour, numcontours]
  , ContourShading → None];

 (* following displays both equipotentials and surrounding plane: *)
Show[p01, p00],

 (* following are parameters that can be changed using sliders: *)
{{theGM, 10 000, "Moon Gravity"}, 0, 50 000},
{{theGMFlat, gravfactor theGM diskthickness (MaxOuterDisk / theminor)² / themajor,
  Style["Disk Gravity (default is 1/2 moon)", {Blue, 12, Bold}]}, 0, 10 000 000},
Style["                 (same density ⇒ M_disk = (diskthickness/themajor)((MaxOuterDisk/theminor)²)M_moon",
 {Blue, Italic}],

{{diskthickness, 0.02, "Disk thickness"}, 0.02, 1},
{{theminor, 0.999, "Minor axis of spheroid (must be <1)" }, 0, 0.999},
{{themajor, 1, "Major axis of spheroid"}, theminor, 2},
{{negratio, 2.5,
  Style["Ratio of excised radius to spheroid radius", {Bold, Darker[Red]}]}, 1, 100},
{{theomega, 15, Style["Rotation Rate, ω", {Bold, Darker[Red]}]}, 0, 60},
Delimiter,
{{plotmin, 0, "Minimum radius plotted"}, 0, 100},
{{plotradius, 6, "Range of radii plotted"}, 0.4, 100},
```

```
{{mincontour, 33, "Minimum contour plotted"}, 0, 10000},
{{numcontours, 100, Style["Separation between contours", Bold]}, 1, 1500},
{{maxcontour, 50000, "Maximum contour plotted"}, 10000, 250000}
]
```

2. Effects of parent gravitational body

In this section we briefly present results of varying the strength and location of an external, distant, gravitational body. These results supplement Fig. 6, which concerns perturbations to equipotentials due to a parent gravitational body in a moon's equatorial plane (Saturn in the case of Pan). Only the lowest order terms of the tidal potential are considered, and the parent body is assumed to be distant enough that its gravitational fieldlines can be treated as parallel. The field then drops off as $1/R_{orbital}{}^2$, where $R_{orbital}$ is the distance between centers of mass of the parent and the moon, as shown in Eq. [2] in the main text.

The net force on a particle on the surface of a moon then depends on relative magnitude and direction between the parent body's gravitational field (Eq. [2]) and the gradient of the moon's equipotentials (Eq. [1]). That is, if the parent body's net force vector aligns with the vector of the moon embedded in a plane, one will get different perturbed equipotentials than if the two vectors are in different directions.

In Fig. A1, we plot the two extremes in direction of the parent body with respect to the moon and surrounding plane. First, in panel (a) we plot equipotentials for the case in which the parent body is in the surrounding plane, so tides are parallel to the plane (denoted $tide_{||}$), and second in panel (b) we plot the case in which the parent body is located perpendicular to the surrounding plane (denoted $tide_{\perp}$). In both cases, we assume that the moon is tidally locked, so that its orientation toward the parent body is unchanging, and in both sets of plots we indicate the tidal direction (toward Saturn) by red arrows. "No tide" applies to a free satellite embedded in a gravitational plane, and tidal amplitudes are expressed as fractions of a moon's gravity at the equator of the unperturbed satellite (i.e the $tide = 0$ case). So for example $tide_{||(\perp)} = 14\%$ means that an acceleration with magnitude of 14% of a satellite's gravity at the unperturbed equator is applied in direction parallel (||) or perpendicular (⊥) to the surrounding plane. In all cases, we take the moon to rotate slowly ($\omega = 5.15$), and to be embedded in a plane that reaches the moon ($R_{excised} = R_{moon}$). As shown in Fig. 4(a) of the main text, these values produce a single outward cusp. The plane's density and size is as in the left panel of Fig. 3(a) of the main text: $\rho_{plane} = \frac{1}{2}\rho_{moon}$, $R_{plane} = 10^2 R_{polar}$, and plane thickness $0.02R_{polar}$.

Fig. A1(a) shows that the perturbation by a parent body (Eq. [2]) produces only a quantitative change in equipotentials, simply elongating the moon's ravioli shape toward the parent body. Fig. 6 in the main text uses $tide_{||} = 8\%$: a value chosen to roughly correspond to Pan's apparent distortion.

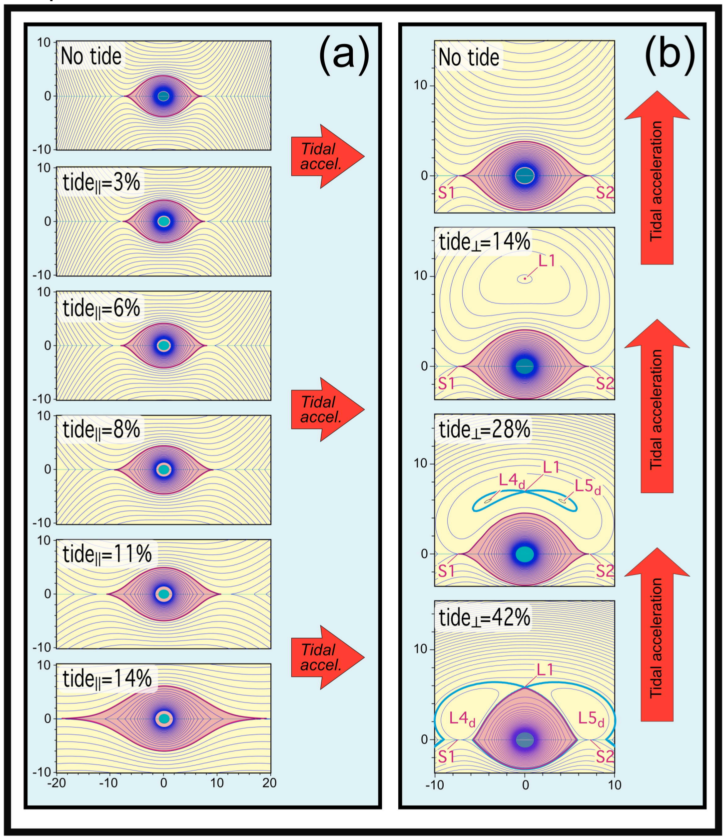
Fig. A1(b) by contrast shows that if the two vectors are perpendicular, qualitative bifurcations arise. Without a parent body ("No tide"), the usual Lagrange points are supplemented with two saddle points, denoted S1 and S2.

We make two orientational remarks. First, the plots of Fig. A1 are all 2D cross-sectional views of a 3D problem, so it should be borne in mind that elliptic points could equally be cross sections of deformed ellipsoids or of tori: the choice depends on details outside of the cross-sectional plane.

Second, the saddle points, S1 and S2, are in the surrounding massive plane, and Fig. A1(b) considers the parent body to be perpendicular to that plane. If the parent is in the plane, one obtains Fig. A1(a). Thus the results of Fig. A1(b) are likely academic and would not apply to Saturn and its disk in its current state. Nevertheless, Saturn's disk inclination with respect to the ecliptic is 27°, and so the same analysis as presented here could be used to describe the Sun as a parent gravitational body perturbing equipotentials surrounding Saturn. Indeed, Uranus' disks are nearly perpendicular to the ecliptic, so it is possible that equipotentials of Uranus' irregular[**] moons could be deformed by the sun as suggested by Fig. A1(b). In

[**] Nesvorný, D., Vokrouhlický, D. & Morbidelli, A., "Capture of irregular satellites during planetary encounters," *Astronomical Journal* 133 (2007) 1962

short, scenarios in which the orientations of moons, disks, and external bodies are misaligned may be speculative, but are not implausible.



scenario in the Supplementary Material in which "the parent body is located perpendicular to the surrounding plane". I don't see how that can possibly result from a coherent orbiting system, and the manuscript contains no diagram to clarify

Figure A1 – Effects of gravitation by parent body: Cross sections of gravitational equipotentials using Eqs. [1] plus Eq. [2] in the main text to include effects a parent body on a rotating moon in a massive disk. Parameters used are defined in the text. (a) Equipotentials where the parent body is in the plane defining the moon's rotation and its surrounding disk (as is Pan). (b) Equipotentials where the parent body is perpendicular to that plane. Lagrange points are denoted by L, and S_1 & S_2 are saddle points not present without effects of the planar mass. As described in the text to this Appendix, a parent body in the plane produces only elongation of the equatorial cusp, whereas a parent body perpendicular to the plane produces a pitchfork bifurcation (between $tide_{\perp} = 14\%$ and 28%) followed by a change of allegiance of the point L1 (between $tide_{\perp} = 28\%$ and 42%) as the parent body's gravity increases.

Bearing this in mind, as the parent body's gravitation grows from zero to to about 14% of the moon's unperturbed equatorial magnitude, Fig. A1(b) shows that an elliptic Lagrange point approaches the moon. This point is in a plane connecting the moon and parent, but is perpendicular to the plane of the disk surrounding the moon. We denote this point L1 to agree with the usual nomenclature in the absence of a disk, but note that in the presence of a massive disk L1 is elliptic, rather than hyperbolic as it would otherwise be.

As the parent's gravitation grows to about 28%, L1 returns to its hyperbolic state following a Hamiltonian pitchfork bifurcation. To conserve topological index, this bifurcation necessarily produces a pair of elliptic points, which we denote $L4_d$ and $L5_d$. The bifurcation is accompanied by a polar bulge in equipotentials surrounding the moon. Unlike the usual Lagrange points L4 and L5, the points $L4_d$ and $L5_d$ make no connection below (in the orientation shown) the moon, and to record this distinction we indicate the presence of a disk with the subscript "d". At larger parent gravitation yet, the point L1 changes allegiance (Shinbrot, 1996), and its homoclinic connection becomes heteroclinic, making new connections to S1 and S2. This state persists as the parent gravitation grows to dominate over the disk gravity. At this point L1 more closely approaches the moon and $L4_d$ and $L5_d$ recede to become the more usual L4 and L5.

In summary, we find that a parent body in the plane surrounding a moon (as with Saturn and its moons) produces only minor elongation in a moon's ravioli-shaped equipotentials, while a parent body located perpendicular to this plane (as perhaps with the Sun and Uranus) produces qualitative bifurcations in Lagrange points, and a noticeable polar bulge facing the parent body.