Appendix: Derivation of the exact RG flow equation (24)

The Wetterich equation (5) can be written as

$$\partial_t \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr} \Big\{ G_k \, \partial_t R_k \Big\} \,, \tag{80}$$

where $t = \ln k$ and

$$G_k = \Big[\Gamma_k^{(2)}[\phi] + R_k \Big]^{-1} \tag{81}$$

is the inverse matrix of $\Gamma_k^{(2)}[\phi] + R_k$. The matrix elements of $\Gamma_k^{(2)}[\phi]$ for the scalar model considered here are

$$\left(\Gamma_k^{(2)} \right) (\mathbf{q}, \mathbf{q}') = \frac{\delta^2 \Gamma_k[\phi]}{\delta\phi(-\mathbf{q}) \, \delta\phi(\mathbf{q}')} \tag{82}$$

in accordance with (8). The RG flow equation for $\Gamma_k^{(2)}(\mathbf{p}_1, \mathbf{p}_2)$ is obtained from (80) by performing certain functional derivatives to obtain $\Gamma_k^{(2)}(\mathbf{p}_1, \mathbf{p}_2)$ from Γ_k, following the definition (11), i. e.,

$$\partial_t \Gamma_k^{(2)}(\mathbf{p}_1, \mathbf{p}_2) = \frac{1}{2} \mathrm{Tr} \left\{ \frac{\delta^2 G_k}{\delta\phi(\mathbf{p}_1)\delta\phi(\mathbf{p}_2)} \, \partial_t R_k \right\} . \tag{83}$$

The derivative on the right hand side of (83) is further deciphered as follows. According to the general formula for the derivative on an inverse matrix, i. e., Eq. (96) in [10], applied to G_k given by (81), we have

$$\frac{\delta G_k}{\delta\phi(\mathbf{p})} = -G_k \, \frac{\delta \Gamma_k^{(2)}}{\delta\phi(\mathbf{p})} \, G_k \, . \tag{84}$$

Applying this repeatedly for $\mathbf{p} = \mathbf{p}_2$ and then for $\mathbf{p} = \mathbf{p}_1$, we obtain

$$\begin{aligned} \frac{\delta^2 G_k}{\delta\phi(\mathbf{p}_1)\delta\phi(\mathbf{p}_2)} \quad &= \quad G_k \, \frac{\delta \Gamma_k^{(2)}}{\delta\phi(\mathbf{p}_1)} \, G_k \frac{\delta \Gamma_k^{(2)}}{\delta\phi(\mathbf{p}_2)} \, G_k - G_k \frac{\delta^2 \Gamma_k^{(2)}}{\delta\phi(\mathbf{p}_1)\delta\phi(\mathbf{p}_2)} G_k \\ &+ \; G_k \, \frac{\delta \Gamma_k^{(2)}}{\delta\phi(\mathbf{p}_2)} \, G_k \frac{\delta \Gamma_k^{(2)}}{\delta\phi(\mathbf{p}_1)} \, G_k \, . \end{aligned} \tag{85}$$

The RG flow equation for $\Gamma_k^{(2)}(\mathbf{p}_1, \mathbf{p}_2)$ of the desired form is obtained by inserting (85) into (83). This treatment is straightforwardly generalized to obtain such RG flow equation for the n-point function $\Gamma_k^{(n)}(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$ from

$$\partial_t \Gamma_k^{(n)}(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n) = \frac{1}{2} \Omega^{\frac{n}{2}-1} \, \mathrm{Tr} \left\{ \frac{\delta^n G_k}{\delta\phi(\mathbf{p}_1)\delta\phi(\mathbf{p}_2) \cdots \delta\phi(\mathbf{p}_n)} \, \partial_t R_k \right\} , \tag{86}$$

applying the derivative with respect to $\phi(\mathbf{p}_i)$ sequentially for $i = n, n-1, \ldots, 1$. At each step, the derivative operator acts according to the chain rule. It finally produces a series of terms, corresponding to all possible combinations of that how the derivatives with respect to specific $\phi(\mathbf{p}_i)$ act on $\Gamma_k^{(2)}$ in a specific position of a chain. However, if several such derivatives act on the same $\Gamma_k^{(2)}$, then they are ordered in accordance with the rule that the derivative with respect to $\phi(\mathbf{p}_l)$ is performed before the derivative with respect to $\phi(\mathbf{p}_j)$ if $l > j$ (as we have defined at the beginning). A new $\Gamma_k^{(2)}$-derivative term of the lowest order is generated and the sign is changed when the derivative operator acts on G_k

according to (84). Therefore, the sign of a specific contribution with M $\Gamma_k^{(2)}$-derivative terms is given by $(-1)^M$. The order of a specific $\Gamma_k^{(2)}$-derivative term is increased by one, if the derivative operator acts on it. As a result, we have

$$\partial_t \Gamma_k^{(n)}(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n) = \frac{1}{2}\Omega^{\frac{n}{2}-1}\,\mathrm{Tr}\left\{ \sum_{M=1}^{n} (-1)^M \sum_{\substack{m_1, m_2, \ldots, m_M \\ m_1+\ldots+m_M=n}} \sum_{\{j_\ell(i)\}^n_{m_1,\ldots,m_M}} \right.$$

$$\left. G_k \prod_{i=1}^{M} \left(\frac{\delta^{m_i}\Gamma_k^{(2)}}{\delta\phi\Big(\mathbf{p}_{j_1(i)}\Big)\delta\phi\Big(\mathbf{p}_{j_2(i)}\Big)\cdots\delta\phi\Big(\mathbf{p}_{j_{m_i}(i)}\Big)} G_k \right) \partial_t R_k \right\}, \tag{87}$$

where the meaning of $\{j_\ell(i)\}^n_{m_1,\ldots,m_M}$ and $j_\ell(i)$ is already explained in the text below Eq. (24). Our ordering rule means that $j_1(i) < j_2(i) < \ldots < j_{m_i}(i)$ holds for any given i.

Eq. (87) contains products of matrices. According to (82), the elements of the $\Gamma_k^{(2)}$-derivative matrices are

$$\begin{aligned}
&\left(\frac{\delta^{m_i}\Gamma_k^{(2)}}{\delta\phi\Big(\mathbf{p}_{j_1(i)}\Big)\delta\phi\Big(\mathbf{p}_{j_2(i)}\Big)\cdots\delta\phi\Big(\mathbf{p}_{j_{m_i}(i)}\Big)} \right)\Big(\mathbf{q}, \mathbf{q}'\Big) \\
&= \frac{\delta^{2+m_i}\Gamma_k}{\delta\phi(-\mathbf{q})\delta\phi(\mathbf{q}')\,\delta\phi\Big(\mathbf{p}_{j_1(i)}\Big)\delta\phi\Big(\mathbf{p}_{j_2(i)}\Big)\cdots\delta\phi\Big(\mathbf{p}_{j_{m_i}(i)}\Big)} \\
&= \Omega^{-\frac{m_i}{2}}\,\Gamma_k^{(2+m_i)}\Big(\mathbf{p}_{j_1(i)}, \mathbf{p}_{j_2(i)}, \ldots, \mathbf{p}_{j_{m_i}(i)}, -\mathbf{q}, \mathbf{q}'\Big),
\end{aligned} \tag{88}$$

noting that the arguments of the n-point functions can always be exchanged.

Further on, up to the end of this Appendix, we consider a homogeneous field $\phi = const.$ In this case, the matrix G_k is diagonal [10], whereas the n-point function is nonvanishing only for zero sum of wave vectors, on which it depends (generalizing the consideration for the 3-point function in Appendix A of [10]), i. e.,

$$G_k(\mathbf{q}, \mathbf{q}') = \delta_{\mathbf{q},\mathbf{q}'}\,G_k(\mathbf{q}) \tag{89}$$

$$\Gamma_k^{(2+m_i)}\Big(\mathbf{p}_{j_1(i)}, \mathbf{p}_{j_2(i)}, \ldots, \mathbf{p}_{j_{m_i}(i)}, -\mathbf{q}, \mathbf{q}'\Big) \neq 0 \quad \text{only if} \quad \mathbf{P}_i - \mathbf{q} + \mathbf{q}' = \mathbf{0}, \tag{90}$$

where $\mathbf{P}_i$ is defined by (28).

Further on, we calculate the matrix elements for the products of matrices in (87) at $\phi = const$, starting with

$$\begin{aligned}
&\left(G_k \frac{\delta^{m_1}\Gamma_k^{(2)}}{\delta\phi\Big(\mathbf{p}_{j_1(1)}\Big)\delta\phi\Big(\mathbf{p}_{j_2(1)}\Big)\cdots\delta\phi\Big(\mathbf{p}_{j_{m_1}(1)}\Big)} G_k \right)\Big(\mathbf{q}, \mathbf{q}'\Big) \\
&= \Omega^{-\frac{m_1}{2}}\,G_k(\mathbf{q})\,\Gamma_k^{(2+m_1)}\Big(\mathbf{p}_{j_1(1)}, \mathbf{p}_{j_2(1)}, \ldots, \mathbf{p}_{j_{m_1}(1)}, -\mathbf{q}, \mathbf{q}'\Big)\,G_k(\mathbf{q}'),
\end{aligned} \tag{91}$$

which holds according to (88) and (89). To make such calculations for a matrix product with two $\Gamma_k^{(2)}$-derivative matrices, a matrix with elements (91) is multiplied by a matrix with elements

$$\begin{aligned}
&\left(\frac{\delta^{m_2}\Gamma_k^{(2)}}{\delta\phi\Big(\mathbf{p}_{j_1(2)}\Big)\delta\phi\Big(\mathbf{p}_{j_2(2)}\Big)\cdots\delta\phi\Big(\mathbf{p}_{j_{m_2}(2)}\Big)} G_k \right)\Big(\mathbf{q}, \mathbf{q}'\Big) \\
&= \Omega^{-\frac{m_2}{2}}\,\Gamma_k^{(2+m_2)}\Big(\mathbf{p}_{j_1(2)}, \mathbf{p}_{j_2(2)}, \ldots, \mathbf{p}_{j_{m_2}(2)}, -\mathbf{q}, \mathbf{q}'\Big)\,G_k(\mathbf{q}').
\end{aligned} \tag{92}$$

Applying the matrix-product rule, we set $\mathbf{q}' = \mathbf{p}$ in (91) and $\mathbf{q} = \mathbf{p}$ in (92) and sum up over $\mathbf{p}$ within $p < \Lambda$. According to condition (90), the only nonzero contribution appears at $\mathbf{p} = \mathbf{q} - \mathbf{P}_1 = \mathbf{P}_2 + \mathbf{q}'$, which is possible only if $\mathbf{P}_1 + \mathbf{P}_2 - \mathbf{q} + \mathbf{q}' = \mathbf{0}$ holds. Thus, we have

$$\begin{aligned}
&\left[G_k \prod_{i=1}^{2} \left(\frac{\delta^{m_i} \Gamma_k^{(2)}}{\delta\phi\Big(\mathbf{p}_{j_1(i)}\Big)\delta\phi\Big(\mathbf{p}_{j_2(i)}\Big)\cdots\delta\phi\Big(\mathbf{p}_{j_{m_i}(i)}\Big)} G_k \right) \right] \Big(\mathbf{q}, \mathbf{q}'\Big) \\
&= \Omega^{-\frac{m_1+m_2}{2}} G_k(\mathbf{q})\, \Gamma_k^{(2+m_1)} \Big(\mathbf{p}_{j_1(1)}, \mathbf{p}_{j_2(1)}, \ldots, \mathbf{p}_{j_{m_1}(1)}, -\mathbf{q}, \mathbf{q} - \mathbf{P}_1\Big)\, \hat{G}_k(\mathbf{q} - \mathbf{P}_1) \\
&\times \Gamma_k^{(2+m_2)} \Big(\mathbf{p}_{j_1(2)}, \mathbf{p}_{j_2(2)}, \ldots, \mathbf{p}_{j_{m_2}(2)}, -\mathbf{P}_2 - \mathbf{q}', \mathbf{q}'\Big)\, G_k(\mathbf{q}')\, \delta_{\mathbf{P}_1+\mathbf{P}_2-\mathbf{q}+\mathbf{q}',\mathbf{0}} \,,
\end{aligned} \tag{93}$$

where $\hat{G}_k$ includes the cut-off function, as defined in (25), which appears due to the condition $p < \Lambda$.

This procedure can be continued to obtain expressions for products with any number of $\Gamma_k^{(2)}$-derivative matrices, recursively multiplying (93) by corresponding matrices on the right hand side. In particular, we have

$$\begin{aligned}
&\left[G_k \prod_{i=1}^{3} \left(\frac{\delta^{m_i} \Gamma_k^{(2)}}{\delta\phi\Big(\mathbf{p}_{j_1(i)}\Big)\delta\phi\Big(\mathbf{p}_{j_2(i)}\Big)\cdots\delta\phi\Big(\mathbf{p}_{j_{m_i}(i)}\Big)} G_k \right) \right] \Big(\mathbf{q}, \mathbf{q}'\Big) \\
&= \Omega^{-\frac{m_1+m_2+m_3}{2}} G_k(\mathbf{q})\, \Gamma_k^{(2+m_1)} \Big(\mathbf{p}_{j_1(1)}, \mathbf{p}_{j_2(1)}, \ldots, \mathbf{p}_{j_{m_1}(1)}, -\mathbf{q}, \mathbf{q} - \mathbf{P}_1\Big)\, \hat{G}_k(\mathbf{q} - \mathbf{P}_1) \\
&\times \Gamma_k^{(2+m_2)} \Big(\mathbf{p}_{j_1(2)}, \mathbf{p}_{j_2(2)}, \ldots, \mathbf{p}_{j_{m_2}(2)}, -\mathbf{P}_2 - \mathbf{P}_3 - \mathbf{q}', \mathbf{P}_3 + \mathbf{q}'\Big)\, \hat{G}_k(\mathbf{P}_3 + \mathbf{q}') \\
&\times \Gamma_k^{(2+m_3)} \Big(\mathbf{p}_{j_1(3)}, \mathbf{p}_{j_2(3)}, \ldots, \mathbf{p}_{j_{m_3}(3)}, -\mathbf{P}_3 - \mathbf{q}', \mathbf{q}'\Big)\, G_k(\mathbf{q}')\, \delta_{\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3-\mathbf{q}+\mathbf{q}',\mathbf{0}}
\end{aligned} \tag{94}$$

and

$$\begin{aligned}
&\left[G_k \prod_{i=1}^{4} \left(\frac{\delta^{m_i} \Gamma_k^{(2)}}{\delta\phi\Big(\mathbf{p}_{j_1(i)}\Big)\delta\phi\Big(\mathbf{p}_{j_2(i)}\Big)\cdots\delta\phi\Big(\mathbf{p}_{j_{m_i}(i)}\Big)} G_k \right) \right] \Big(\mathbf{q}, \mathbf{q}'\Big) \\
&= \Omega^{-\frac{m_1+m_2+m_3+m_4}{2}} G_k(\mathbf{q})\, \Gamma_k^{(2+m_1)} \Big(\mathbf{p}_{j_1(1)}, \mathbf{p}_{j_2(1)}, \ldots, \mathbf{p}_{j_{m_1}(1)}, -\mathbf{q}, \mathbf{q} - \mathbf{P}_1\Big)\, \hat{G}_k(\mathbf{q} - \mathbf{P}_1) \\
&\times \Gamma_k^{(2+m_2)} \Big(\mathbf{p}_{j_1(2)}, \mathbf{p}_{j_2(2)}, \ldots, \mathbf{p}_{j_{m_2}(2)}, -\mathbf{P}_2 - \mathbf{P}_3 - \mathbf{P}_4 - \mathbf{q}', \mathbf{P}_3 + \mathbf{P}_4 + \mathbf{q}'\Big) \\
&\times \hat{G}_k(\mathbf{P}_3 + \mathbf{P}_4 + \mathbf{q}') \\
&\times \Gamma_k^{(2+m_3)} \Big(\mathbf{p}_{j_1(3)}, \mathbf{p}_{j_2(3)}, \ldots, \mathbf{p}_{j_{m_3}(3)}, -\mathbf{P}_3 - \mathbf{P}_4 - \mathbf{q}', \mathbf{P}_4 + \mathbf{q}'\Big)\, \hat{G}_k(\mathbf{P}_4 + \mathbf{q}') \\
&\times \Gamma_k^{(2+m_4)} \Big(\mathbf{p}_{j_1(4)}, \mathbf{p}_{j_2(4)}, \ldots, \mathbf{p}_{j_{m_4}(4)}, -\mathbf{P}_4 - \mathbf{q}', \mathbf{q}'\Big)\, G_k(\mathbf{q}')\, \delta_{\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3+\mathbf{P}_4-\mathbf{q}+\mathbf{q}',\mathbf{0}} \,.
\end{aligned} \tag{95}$$

The rules of evolving these matrix-product elements are already evident from these calculations. The products with M n-point functions, evaluated at $m_1 + \cdots + m_M = n$ and $\mathbf{q}' = \mathbf{q}$, represent specific contributions to the trace in (87) at the given M. These are identified with the corresponding contributions to (24) when $\Omega^{-1}\mathrm{Tr} \to \int_q$ (a standard procedure). Taking into account (30) and (90), these contributions are nonvanishing only at $\sum_{\ell=1}^{n} \mathbf{p}_\ell = \mathbf{0}$, and the function arguments in these product expressions are precisely consistent with (26), (27) and (29) inserted into (24). In particular, $\mathbf{P}_1 = \mathbf{0}$ holds at $M = 1$ according to (30) and, therefore, (29) gives $\grave{\mathbf{Q}}_1 = -\mathbf{q}$, $\mathbf{Q}_1 = \mathbf{q}$ for $M = 1$ in agreement with (91) at $\mathbf{q}' = \mathbf{q}$. The term $G_k(\mathbf{q}')$ in (91)–(95) is replaced by $\hat{G}_k(\mathbf{q}')$, since $q' < \Lambda$. It makes the final form of the RG flow equation exactly such as written in (24).