Appendix 1: Summary Table

Table 1 presents a summary of results for various distributions. As before, let Q_1 and Q_3 denote the first and third quartiles and let O_i denote the corresponding octiles of the standard normal distribution. We note that for the distributions considered, $T_M < T_Q < K$.

Appendix 2: Details for Log-normal Distribution

We provide some derivations on comparing the performance measures for the log-normal distribution. Recall that classical measures for log-normal (see table 1) are:

$$\mu' = e^{\mu+\sigma^2/2}, \qquad M' = e^{\mu}, \qquad (\sigma')^2 = \big(e^{\sigma^2} - 1\big)e^{2\mu+\sigma^2},$$
$$S' = \big(e^{\sigma^2} + 2\big)\sqrt{e^{\sigma^2} - 1}, \qquad K' = e^{4\sigma^2} + 2e^{3\sigma^2} + 3e^{2\sigma^2} - 6$$

To compute MAD-based measures, we consider the following integral:

$$I(z) = \int_0^z t f(t)\, dt = \frac{1}{\sigma\sqrt{2\pi}} \int_0^z e^{-(\log t - \mu)^2/2\sigma^2}\, dt, \qquad z > 0$$

To evaluate this integral, we consider the following change of variables: $u = (\log t - \mu)/\sigma$ giving us $dt = \sigma e^{\mu+\sigma u}\, du$. If we define $z^* = (\log z - \mu)/\sigma$, then we can we-write the integral in equation () as follows:

$$I(z) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{z^*} e^{-u^2/2} \sigma e^{\mu+\sigma u}\, du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z^*} e^{-(u^2-2\sigma u+\sigma^2)/2} e^{\mu+\sigma^2/2}\, du$$
$$= \mu' \int_{-\infty}^{z^*} \left[\frac{1}{\sqrt{2\pi}}\, e^{-(u-\sigma)^2/2}\right] du$$

The term in the bracket is the density function of a normal random variable with mean σ and unit variance. With the simple change of variable $v = u - \sigma$, we can re-write the above integral as

$$I(z) = \mu' \int_{-\infty}^{z^*-\sigma} \frac{1}{\sqrt{2\pi}}\, e^{-v^2/2}\, dv = \mu' \Phi(z^* - \sigma) = \mu' \Phi\left(\frac{\log z - \mu}{\sigma} - \sigma\right)$$

Next, let us compare MAD-based and quantile-based measures. The MAD-based measures for log-normal distribution (Table 1) are

$$H' = \mu'\big(2\Phi(\sigma) - 1\big), \qquad A'_M = \frac{e^{\sigma^2/2} - 1}{e^{\sigma^2/2}\big(2\Phi(\sigma) - 1\big)},$$
$$T'_M = \frac{-1 - 2\Phi(\sigma) + 2\Phi(\sigma - Q_1) + 2\Phi(\sigma - Q_3)}{2\Phi(\sigma) - 1} \tag{1}$$

Table 1: Summary Comparison of Distributions

Measure	**Classical**	**MAD-based**	**Quantile-based**
		Uniform Distribution (symmetric)	
Deviation	$\sigma = \dfrac{(b-a)}{2\sqrt{3}}$	$H = \dfrac{(b-a)}{4}$	$H_Q = \dfrac{(b-a)}{2}$
$H < \sigma < H_Q$			
Skewness	$S = 0$	$A_M = 0$	$A_Q = 0$
Kurtosis	$K = 1.8$	$T_M = 0.5$	$T_Q = 1$
$T_M < T_Q < K$			
		Normal Distribution $\mathrm{N}(\mu, \sigma^2)$ **(symmetric)**	
Deviation	σ	$H = \sqrt{\dfrac{2}{\pi}}\,\sigma \approx 0.80\,\sigma$	$H_Q = Q_3\,\sigma \approx 0.68\,\sigma$
$H_Q < H < \sigma$			
Skewness	$S = 0$	$A_M = 0$	$A_Q = 0$
Kurtosis	$K = 3$	$T_M = -1 + 2e^{-Q_3^2/2}$	$T_Q = \dfrac{O_7 - O_5}{Q_3}$
$T_M < T_Q < K$		≈ 0.59	≈ 1.23
		Log-Normal Distribution (μ, σ^2) **(asymmetric)**	
Deviation	$\sqrt{(e^{\sigma^2}-1)e^{2\mu+\sigma^2}}$	$e^{\mu+\sigma^2/2}\,(2\Phi(\sigma)-1)$	$e^{\mu}\,\dfrac{(e^{\sigma Q_3} - e^{-\sigma Q_3})}{2}$
$H'_Q < H' < \sigma'$			
Skewness	$(e^{\sigma^2}+2)\sqrt{e^{\sigma^2}-1}$	$\dfrac{(e^{\sigma^2/2}-1)}{e^{\sigma^2/2}\,(2\Phi(\sigma)-1)}$	$-1 + \dfrac{2e^{\sigma Q_3}}{(e^{\sigma Q_3}+1)}$
$0 < A'_M, A'_Q < S'$	(unbounded)		
Kurtosis	$e^{4\sigma^2} + 2e^{3\sigma^2}$ $+3e^{2\sigma^2} - 6$	$\dfrac{N_M(\sigma, Q_1, Q_3)}{2\Phi(\sigma)-1}$	$\dfrac{N_Q(\sigma, O_5, O_7)}{(e^{\sigma Q_3} - e^{-\sigma Q_3})}$
$T'_M < T'_Q < K'$	(unbounded)		(unbounded)

$$N_M(\sigma, Q_1, Q_3) = -1 - 2\Phi(\sigma) + 2\Phi(\sigma - Q_1) + 2\Phi(\sigma - Q_3)$$
$$N_Q(\sigma, O_5, O_7) = (e^{-\sigma O_5} - e^{-\sigma O_7}) + (e^{\sigma O_7} - e^{\sigma O_5})$$

Table 1. Summary Comparison of Distributions (continued)

Measure	**Classical**	**MAD-based**	**Quantile-based**
Exponential Distribution with rate λ (asymmetric)			
Deviation	$\sigma = \frac{1}{\lambda}$	$H = \frac{\log 2}{\lambda} \approx 0.69\sigma$	$H_Q = \frac{\log 3}{2\lambda} \approx 0.55\sigma$
$H < H_Q < \sigma$			
Skewness	$S = 2$	$A_M = -1 + \frac{1}{\log 2}$	$A_Q = -1 + \frac{2\log 2}{\log 3}$
$A_Q < A_M < S$		≈ 0.44	≈ 0.26
Kurtosis	$K = 9$	$T_M = 3 - \frac{3\log 3}{2\log 2}$	$T_Q = 1 + \frac{\log(7/5)}{\log 3}$
$T_M < T_Q < K$		≈ 0.62	≈ 1.31
Laplace Distribution with location μ and scale b (symmetric)			
Deviation	$\sigma = b\sqrt{2} \approx 1.41b$	$H = b$	$H_Q = b\log 2 \approx 0.69b$
$H_Q < H < \sigma$			
Skewness	$S = 0$	$A_M = 0$	$A_Q = 0$
Kurtosis	$K = 6$	$T_M = \log 2 \approx 0.69$	$T_Q = \frac{\log 3}{\log 2} \approx 1.58$
$T_M < T_Q < K$			
Pareto Distribution with shape α and scale β (asymmetric)			
Deviation	$\sigma = \frac{\beta}{(\alpha-1)}\sqrt{\frac{\alpha}{\alpha-2}}$	$H = \frac{\beta\alpha(\sqrt[\alpha]{2}-1)}{(\alpha-1)}$	$H_Q = \frac{\beta(\sqrt[\alpha]{4}-\sqrt[\alpha]{4/3})}{2}$
$H_Q < H < \sigma$	$(\alpha > 2)$	$(\alpha > 1)$	
Skewness	$S = \frac{2(1+\alpha)}{(\alpha-3)}\sqrt{\frac{\alpha-2}{\alpha}}$	$A_M = -1 + \frac{\sqrt[\alpha]{2}}{\alpha(\sqrt[\alpha]{2}-1)}$	$A_Q = -1 + \frac{2(\sqrt[\alpha]{4}-\sqrt[\alpha]{2})}{\sqrt[\alpha]{4}-\sqrt[\alpha]{4/3}}$
$A_Q < A_M < S$	$(\alpha > 3)$		
Kurtosis	$K = \frac{6(\alpha^3+\alpha^2-6\alpha-2)}{\alpha(\alpha-3)(\alpha-4)}$	$T_M = \frac{N_M(\alpha)}{2(\sqrt[\alpha]{2}-1)}$	$T_Q = \frac{N_Q(\alpha)}{\sqrt[\alpha]{4}-\sqrt[\alpha]{4/3}}$
$T_M < T_Q < K$	$(\alpha > 4)$		

$$N_M(\alpha) = 3\sqrt[\alpha]{4/3} + \sqrt[\alpha]{4} - 2\sqrt[\alpha]{2} - 2$$
$$N_Q(\alpha) = (\sqrt[\alpha]{8/5} - \sqrt[\alpha]{8/7}) + (\sqrt[\alpha]{8} - \sqrt[\alpha]{8/3})$$

whereas the quantile-based measures (table 1) are:

$$H'_Q = e^{\mu}\ \frac{(e^{\sigma Q_3} - e^{-\sigma Q_3})}{2}, \qquad A'_Q = -1 + \frac{2e^{\sigma Q_3}}{e^{\sigma Q_3}+1}, \\ T'_Q = \frac{(e^{-\sigma O_5} - e^{-\sigma O_7}) + (e^{\sigma O_7} - e^{\sigma O_5})}{e^{\sigma Q_3} - e^{-\sigma Q_3}} \tag{2}$$

To compare MAD-based and quantile-based measures, we will use the following bound on complementary error function

$$\text{erfc}(x)\ \leq\ e^{-x^2} \quad \text{where} \quad \text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}\,dt \quad \text{and} \quad x > 0$$

From this bound and the equation $H' = 2\mu'\ \text{erf}(\sigma/\sqrt{2})$ we immediately obtain $H' \geq 2e^{\mu}(e^{\sigma^2/2} - 1)$. On the other hand, since $Q_1 = -Q_3$ and $Q_3 < 1$ we have for $\sigma > \sqrt{2}$:

$$H'_Q = e^{\mu}\ \frac{(e^{\sigma Q_3} - e^{-\sigma Q_3})}{2} < e^{\mu}\frac{(e^{\sigma Q_3} - 1)}{2} < 2e^{\mu}(e^{\sigma} - 1) < 2e^{\mu}(e^{\sigma^2/2} - 1) \leq H'$$

Since $H' \leq \sigma'$ we immediately obtain $H'_Q < H' \leq \sigma'$.

Next, we consider the skewness. It is easy to show that for large σ, the classical skewness $S' > 1$. In fact, if we define $a = \exp(\sigma^2)$ then we can re-write $S' = (a+2)\sqrt{a-1}$. It is clear that S' increases with a and is unbounded. Since the MAD skewness $A'_M \leq 1$ and quantile skewness $A'_Q \leq 1$, we can solve for $S^* = 1$ and (numerically) find $\sigma \approx 0.133$. For $\sigma > 0.133$ we have have $S' > 1$ and, therefore, both $A'_M < S'$ and $A'_Q < S'$.

Next, we consider the quantile-based kurtosis T'_Q. From equation (2) we have

$$T'_Q = e^{\sigma(O_7 - Q_3)}\left[\frac{1 - e^{-\sigma(O_7 - O_5)} + e^{-\sigma(O_7 + \sigma O_5)} - e^{-2\sigma O_7}}{1 - e^{-2\sigma Q_3}}\right] > e^{\sigma(O_7 - Q_3)} \tag{3}$$

Since $(O_7 - Q_3) > 0$ it follows that $T'_Q > 1$ and $T'_Q \mapsto \infty$ as $\sigma \mapsto \infty$. Therefore, for log-normal distributions, the quantile-based kurtosis T'_Q is unbounded. By contrast, the MAD-based kurtosis T'_M is always in the range $[0, 1]$ and therefore, $T'_M < T'_Q$. Similarly, since $(O_7 - Q_3) < 1$, from equation (3) we can show

$$T'_Q < 2e^{\sigma(O_7 - Q_3)} < 2e^{\sigma}$$

On the other hand, for the classical kurtosis K' we have

$$K' = e^{4\sigma^2}(1 + 2e^{-\sigma^2} + 3e^{-2\sigma^2} - 6e^{-4\sigma^2}) > e^{4\sigma^2} > 2e^{\sigma} > T'_Q$$

Therefore, for large σ, $K' > 2e^{\sigma} > T'_Q$.

Let us establish some asymptotic expressions for MAD-based skewness and kurtosis for $\sigma \mapsto 0$ and for $\sigma \mapsto \infty$. We start with $\sigma \mapsto 0$. Since the derivative of the cumulative distribution function $\Phi(\cdot)$ is the density function $f(\cdot)$, we have $\partial\Phi(y)/\partial y = e^{-y^2/2}/\sqrt{2\pi}$. Therefore, for $\sigma \mapsto 0$ we have

$$\frac{\partial\Phi(\sigma - Q_1)}{\partial\sigma} \mapsto \frac{e^{-Q_1^2/2}}{\sqrt{2\pi}}, \qquad \frac{\partial\Phi(\sigma - Q_3)}{\partial\sigma} \mapsto \frac{e^{-Q_3^2/2}}{\sqrt{2\pi}}, \qquad \frac{\partial\Phi(\sigma)}{\partial\sigma} \mapsto \frac{1}{\sqrt{2\pi}}$$

Noting that $Q_1 = -Q_3$ and $\sigma^* \mapsto e^{\mu}$ and applying the l'Hopital rule to MAD-based measures in equation (1) we find

$$H'/\sigma' \mapsto \sqrt{2/\pi}, \quad A'_M \mapsto 0, \quad T'_M \mapsto -1 + 2e^{-Q_3^2/2} \approx 0.59$$

Applying the l'Hopital rule for the quantile-based measures in equation (2) we find:

$$H'_Q/\sigma' \mapsto Q_3 \approx 0.68, \qquad A'_Q \mapsto 0, \qquad T'_Q \mapsto (O_7 - O_5)/Q_3 \approx 1.23$$

In other words, for $\sigma \mapsto 0$, both MAD-based and quantile-based measures to the corresponding values for normal distribution.

Next, we consider the case $\sigma \mapsto \infty$. In this case, $\Phi(\sigma) \mapsto 1$. It is easy to show that in this case,

$$H'/\sigma' \mapsto 1, \qquad A'_M \mapsto 1, \qquad T'_M \mapsto 1$$

For the quantile measures, it easy to show $H'_Q \mapsto \infty$ An application of the l'Hopital's rule to A'_Q in equation (2) shows that $A'_Q \mapsto 1$. Finally, it follows from equation (3) that $T'_Q \mapsto \infty$ and is unbounded. Therefore, for $\sigma \mapsto \infty$ we have

$$H^*_Q \mapsto 0, \qquad A^*_Q \mapsto 1, \qquad T^*_Q \mapsto \infty$$

Appendix 3: Details for Laplace Distribution

We consider the following integral

$$J(z) = \int_z^{\infty} x f(x)\, dx = \frac{1}{2b}\int_z^{\infty} x e^{|-x|/b}\, dx$$

To evaluate this integral, we first consider the case $z \geq 0$. With a change of variable $u = x/b$, we obtain

$$\begin{aligned} J(z) &= \frac{1}{2b}\int_z^{\infty} x e^{-x/b}\, dx = \frac{b}{2}\int_{z/b}^{\infty} u e^{-u}\, du = \frac{b}{2}\int_0^{\infty} u e^{-u}\, du - \frac{b}{2}\int_0^{z/b} u e^{-u}\, du \\ &= \frac{b}{2}\Gamma(2) - \frac{b}{2}\left[-(u+1)e^{-u}\right]\Big|_0^{z/b} = \frac{(b+z)}{2}\, e^{-z/b} \end{aligned}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Note that in particular, $J(0) = b/2$.

Next, we consider the case $z \leq 0$. Again we consider a change of variable $u = x/b$ and obtain

$$\begin{aligned} J(z) &= \frac{1}{2b}\int_z^{\infty} x e^{-|x|/b}\,dx = \frac{1}{2b}\int_z^{0} x e^{x/b}\,dx + \frac{1}{2b}\int_0^{\infty} x e^{-x/b}\,dx \\ &= \frac{b}{2}\int_{z/b}^{0} u e^{u}\,du + \frac{b}{2}\int_0^{\infty} x e^{-x/b}\,dx = \frac{b}{2}\left[(u-1)e^{u}\right]\Big|_{z/b}^{0} + \frac{b}{2} \\ &= \frac{(b-z)}{2}\,e^{z/b} \end{aligned}$$

Appendix 4: Details for Pareto Distribution

This distribution has infinite mean μ for $\alpha \leq 1$, undefined variance σ^2 for $\alpha \leq 2$, undefined skewness S for $\alpha \leq 3$ and undefined (excess) kurtosis K for $\alpha \leq 4$. These measures are

$$\mu = \frac{\alpha\beta}{\alpha-1}, \qquad \sigma^2 = \frac{\beta^2\alpha}{(\alpha-1)^2(\alpha-2)}, \qquad S = \frac{2(1+\alpha)}{\alpha-3}\sqrt{\frac{\alpha-2}{\alpha}},$$
$$K = \frac{6(\alpha^3+\alpha^2-6\alpha-2)}{\alpha(\alpha-3)(\alpha-4)}$$

Recall that MAD-based measures for Pareto are

$$H = \frac{\beta\alpha(\sqrt[\alpha]{2}-1)}{\alpha-1}, \qquad A_M = -1 + \frac{\sqrt[\alpha]{2}}{\alpha(\sqrt[\alpha]{2}-1)},$$
$$T_M = \frac{3\sqrt[\alpha]{4/3} + \sqrt[\alpha]{4} - 2\sqrt[\alpha]{2} - 2}{2(\sqrt[\alpha]{2}-1)}$$

whereas quantile-based measures are

$$H_Q = \frac{\beta(\sqrt[\alpha]{4} - \sqrt[\alpha]{4/3})}{2}, \qquad A_Q = -1 + \frac{2(\sqrt[\alpha]{4} - \sqrt[\alpha]{2})}{\sqrt[\alpha]{4} - \sqrt[\alpha]{4/3}},$$
$$T_Q = \frac{(\sqrt[\alpha]{8/5} - \sqrt[\alpha]{8/7}) + (\sqrt[\alpha]{8} - \sqrt[\alpha]{8/3})}{\sqrt[\alpha]{4} - \sqrt[\alpha]{4/3}}$$

We start with comparing deviation measures σ, H and H_Q. Since all these measures are proportional to β, we can take $\beta = 1$ and obtain:

$$H_Q < \frac{\sqrt[\alpha]{4}-1}{2} = \frac{(\sqrt[\alpha]{2}-1)(\sqrt[\alpha]{2}+1)}{2} = \frac{(\alpha-1)H}{\alpha}\cdot\frac{(\sqrt[\alpha]{2}+1)}{2} < H$$

Since $H < \sigma$ we have $H_Q < H < \sigma$. Next, we consider the skewness. The median is decreasing with α and for $\alpha > 1$ we have

$$A_Q < -1 + \frac{2(\sqrt[\alpha]{4} - \sqrt[\alpha]{2})}{\sqrt[\alpha]{4} - \sqrt[\alpha]{4/3}} < 0.5$$

Using the inequality $e^t \leq 1/(1-t)$ for $t \in [0.1)$ and substituting $t = \log 2/\alpha$ we have $\log 2/\alpha \geq 1 - 1/2^{1/\alpha}$ and obtain

$$A_M = -1 + \frac{\sqrt[\alpha]{2}}{\sqrt[\alpha]{2} - 1)} \geq -1 + \frac{1}{\log 2} > 1/35$$

Therefore, $A_Q < A_M$. Note that for the classical skewness with $\alpha > 3$

$$S = 2\left(1 + \frac{4}{\alpha - 3}\right)\sqrt{1 - \frac{2}{\alpha}} > 2\sqrt{1 - \frac{2}{\alpha}} > \frac{2}{\sqrt{3}} > 1$$

Since $A_M \leq 1$ we conclude $A_Q < A_M < S$. For $\alpha \mapsto \infty$, a simple application of the L'Hopital's rule gives $\sigma \mapsto \beta$ and $S \mapsto 2$ as $\alpha \mapsto \infty$.

Finally, let us consider kurtosis. On the one hand,

$$T_Q < \frac{(\sqrt[\alpha]{2} - 1) + (\sqrt[\alpha]{8} - \sqrt[\alpha]{4})}{\sqrt[\alpha]{4} - 1} = \frac{(\sqrt[\alpha]{2} + 1)^2 - 2\sqrt[\alpha]{2}}{\sqrt[\alpha]{2} + 1} < \sqrt[\alpha]{2} + 1 < 3$$

On the other hand. since $O_7 > Q_3$ and $O_3 > Q_1$ we have

$$T_Q > \frac{\sqrt[\alpha]{4} + \sqrt[\alpha]{4/3}}{\sqrt[\alpha]{4} - \sqrt[\alpha]{4/3}} = 1 + \frac{2\sqrt[\alpha]{4/3}}{\sqrt[\alpha]{4} - \sqrt[\alpha]{4/3}} > 1 + \frac{2\sqrt[\alpha]{4/3}}{\sqrt[\alpha]{4}} = 1 + \frac{2}{\sqrt[\alpha]{3}} > 1$$

Since MAD-based kurtosis $0 < T_M < 1$ we immediately obtain $T_M < T_Q$. On the other hand, for classical kurtosis with $\alpha > 4$, we have

$$K = 6\left(1 + \frac{8(\alpha - 2)^2 + 12(\alpha - 3) + 6}{\alpha(\alpha - 3)(\alpha - 4)}\right) > 6$$

Therefore $T_M < T_Q < K$. By the L'Hopital's rule, it is easy to show that $K \mapsto 6$ as $\alpha \mapsto \infty$.

Let us examine the asymptotic behavior of quantile-based and MAD-based measures for $\alpha \mapsto 1$ and $\alpha \mapsto \infty$. As before, assume $\beta = 1$. We start with $\alpha \mapsto 1$. From equations () and () we immediately obtain the following asymptotic expressions for Pareto distribution as $\alpha \mapsto 1$:

$$\begin{cases} H & \mapsto & \infty \\ A_M & \mapsto & 1 \\ T_M & \mapsto & 1 \end{cases} \quad \text{and} \quad \begin{cases} H_Q & \mapsto & 4/3 \\ A_Q & \mapsto & 0.5 \\ T_Q & \mapsto & \dfrac{(8/5 - 8/7) + (8 - 8/3)}{(4 - 4/3)} \approx 2.17 \end{cases}$$

Next, we consider the limiting case when $\alpha \mapsto \infty$ Since for $\alpha > 1$ we have $0 < \log 2/\alpha < 1$. Let us consider the following inequality $1 + t < e^t < 1/(1-t)$ for $t < 1$. Substituting $t = (\log 2)/\alpha$ we have

$$\log 2 < \alpha(\sqrt[\alpha]{2} - 1) < \frac{\log 2}{1 - (\log 2)/\alpha}$$

Therefore, for large α we have $\alpha(\sqrt[\alpha]{2}-1) \mapsto \log 2$. A simple application of the l'Hopital rule gives us $H/\sigma \mapsto \log 2$. Substituting results from the previous equation to expressions in equations () and () we obtain the following asymptotic expressions for Pareto distribution as $\alpha \mapsto \infty$

$$\begin{cases} H \mapsto & \dfrac{\log 2}{\alpha - 1} \mapsto 0 \\ A_M \mapsto & -1 + \dfrac{1}{\log 2} \approx 0.44 \\ T_M \mapsto & 3 - \dfrac{3\log 3}{2\log 2} \approx 0.62 \end{cases} \quad \text{and} \quad \begin{cases} H_Q & \mapsto \dfrac{\log 3}{2(\alpha - 1)} \mapsto 0 \\ A_Q & \mapsto -1 + \dfrac{2\log 2}{\log 3} \approx 0.26 \\ T_Q & \mapsto 1 + \dfrac{\log(7/5)}{\log 3} \approx 1.31 \end{cases}$$