ELEMENTARY MATHEMATICS FROM AN ADVANCED STANDPOINT

Appendix 1. Knowledge test

Project:

Identifying Mathematics teachers' competency to look at elementary Mathematics from an advanced standpoint

Questionnaire (1 hour)

INSTRUCTIONS

1. ANSWER ALL QUESTIONS (21 in total).
2. All questions carry equal marks.
3. You should mark only ONE answer for each multiple-choice question. If you mark more than one answer, you will receive NO MARKS for that multiple-choice question.
4. No marks will be deducted for wrong answers.

Q1: Determine the differentiability of the function $f(x) = x^2 sin\ \frac{1}{x^2}$

a) The function $f(x)$ is differentiable anywhere as both x^2 and the sine function are differentiable for any *x.*

b) The function $f(x)$ is differentiable anywhere as x^2 is differentiable for any x, and the sine function is bounded.

c) The function $f(x)$ is differentiable anywhere as the left limit and the right limit at $x = 0$ are the same.

d) The function $f(x)$ is differentiable anywhere except at $x = 0$ as $\frac{1}{x^2}$ is undefined at $x = 0$.

Q2: Determine the value of $\sqrt{(-4)^2}$

a) -4 because $\sqrt{(-4)^2} = (-4)^{2\times\frac{1}{2}} = (-4)^1 = -4$.

b) 4 because $\sqrt{(-4)^2} = \sqrt{16} = 4$.

c) It can be either 4 or -4 depending on the choice of sign on taking the square root.

d) Both are correct as $x = -4$ and $x = 4$ are the roots of the quadratic equation $x^2 = 16$.

Q3: To solve the simultaneous linear equations

$$\{3x + 2y - 1 = 0 \cdots\cdots (i) \qquad x - y + 3 = 0 \cdots\cdots (ii)$$

we get the solution (-1, 2). As both equations share the same zero right-hand side, we can combine equations (i) and (ii) by equating the sides and come up with equation (iii). That is, equating the R.H.S.,

$$3x + 2y - 1 = 0 = x - y + 3$$

giving

$$3x + 2y - 1 = x - y + 3$$

$$2x + 3y - 4 = 0 \cdots\cdots \text{(iii)}$$

Explain why none of the solutions of equation (iii) except (-1, 2) satisfies equation (i) and (ii) given that equation (iii) is derived from equations (i) and (ii).

a) Combining the two equations (i) and (ii) giving the third equation(iii) results in two unknowns, which does not reduce to a linear equation with one unknown. Hence, there will be many solutions to (iii) that distort the original design of the simultaneous equations.

b) Solving simultaneous equations involves some choices of solutions. By trial and error, the solution (-1, 2) satisfies (iii), and satisfies (i) and (ii) simultaneously. So, (-1, 2) will be the solution of (i) and (ii).

c) Making (i) and (ii) equal objectively imposes a new condition on the system of simultaneous equations, hence creating an extraneous solution to the system.

d) Making 0 = 0 is a kind of circular argument, or tautology. Solving a system of simultaneous equations requires a legitimate and independent condition. Hence (iii) cannot give a unique solution.

Q4: A car travels an average speed of 60 km/h from Tai Po to Yuen Long and 90 km/h on the return trip. What is the car's average speed of travel?

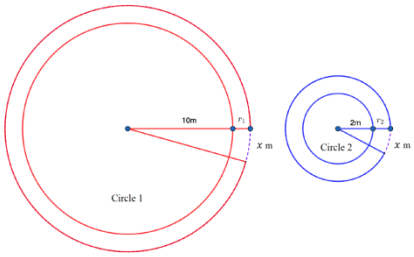
a) 75 km/h.

b) 72 km/h.

c) Cannot be determined as the distance between two places is not given.

d) Cannot be determined as the travel times of the two trips are not given.

Q5: Below are two circles with radius of 10 m and 2 m in red and blue, respectively. Both circles are then enlarged by adding x m to their circumferences.



Which of the following statements is correct?

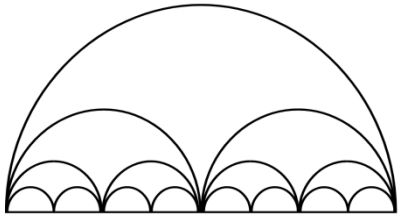
a) r_1 is greater than r_2.

b) r_1 is smaller than r_2.

c) r_1 is equal to r_2.

d) The comparison between r_1 and r_2 depends on the value of x.

Q6: In the following figure, the outermost semi-circle has unit radius. First, we divide in half the diameter of the outermost semi-circle and draw two smaller semi-circles on each half. Then, we repeat the process in the same manner to divide the original diameter into 4 equal pieces and draw a semi-circle on each of them, and so on. Determine which of the following statements is true after repeating the process for *n* steps.



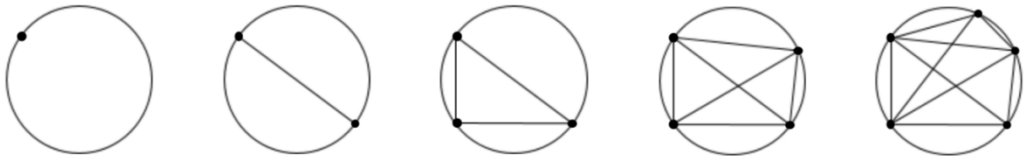
a) The sum of the circumferences of the semi-circles at the *n*-th step is smaller than the circumference of the outermost semi-circle.

b) The sum of the circumferences of the semi-circles at the *n*-th step is greater than the circumference of the outermost semi-circle.

c) The sum of the circumferences of the semi-circles at the *n*-th step is equal to the diameter of the outermost semi-circle.

d) The sum of the circumferences of the semi-circles at the *n*-th step is equal to the circumference of the outermost semi-circle.

Q7: The figures below show 5 circles, the first with one point on the boundary, the second with two, and so on. All the possible chords drawn by joining these boundary points have been drawn and they divide the circle into regions. How many regions can we get if we add one more point on the circle and construct the chords by joining this point to the existing points on the circle?



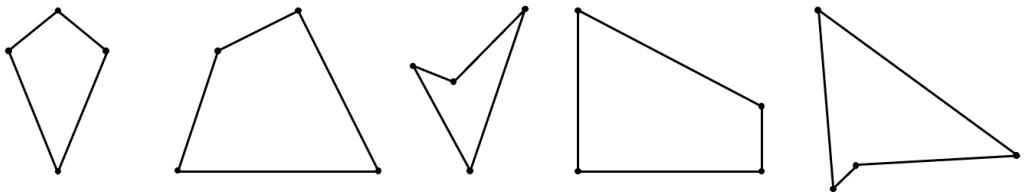
a) The total number of regions is 28.

b) The total number of regions is 31.

c) The total number of regions is 32.

d) The total number of regions is 34.

Q8: Which of the following plane figures can be used to tessellate the plane?



(a) (b) (c) (d) (e)

a) None of them

b) All of them

c) Only (a), (b) and (d)

d) Only (a) and (d)

Q9: You are playing a game. The game host shows you three boxes. One box contains money, the other two boxes are empty. You need to choose one of the boxes. If the chosen box contains money, you win and get the money. Suppose that you pick box 1. Instead of opening the box you have picked, the game host opens another box and shows you that the opened box is empty. Then he gives you a chance to change your decision. Your options are

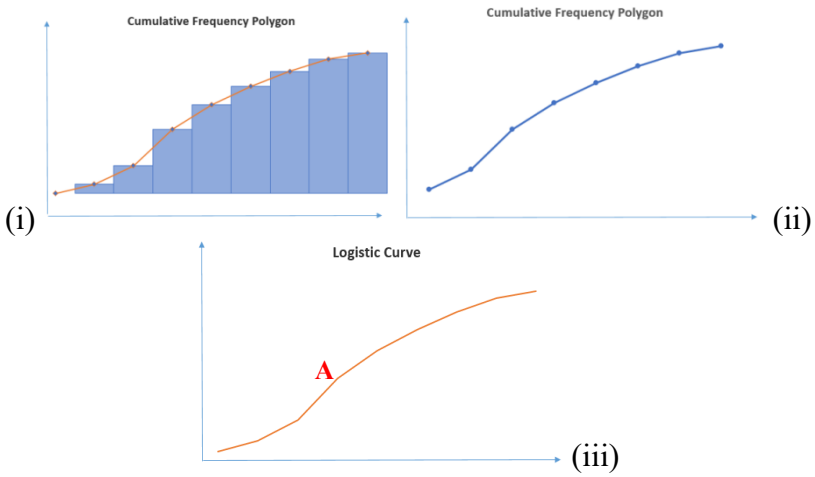
a) Switch, because the probability of winning increases from 1/3 to a larger value.

b) Stay, because opening any box will not affect the probability of your original choice containing the money and opening an empty box is only a trick of this game.

c) Switch or stay. It does not matter because the host always opens an empty box, so the probabilities of staying or switching are both 1/2.

d) Switch or stay. It does not matter because the probability of winning is still 1/3.

Q10: The logistic curve (figure iii) is an estimation graph for the cumulative frequency polygons (figure i, and figure ii) of a set of data.



Point A is the steepest point (where the slope of the curve is the greatest) of the logistic curve, which is a continuous model of the corresponding cumulative frequency polygon of a set of data. Point A is likely to be the

(1) Mean,

(2) Median, and/or

(3) Mode

of the set of data. Which of the following is true?

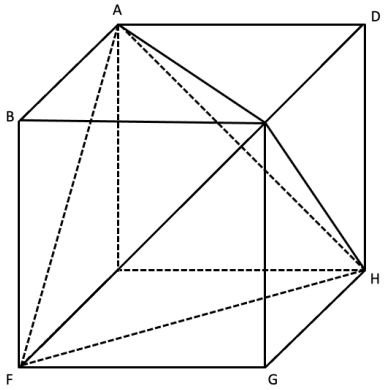
a) (2) only

b) (3) only

c) (1) and (3) only

d) (1), (2) and (3)

Q11: In the figure below, the side of the cube is *a* cm. Find the volume of the tetrahedron CAFH.



a) $\frac{1}{3}a^3\ cm^3$

b) $a^3\ cm^3$

c) $\frac{3}{2}a^3\ cm^3$

d) $3a^3\ cm^3$

Q12: A student used integration by parts to integrate $f(x) = \frac{1}{x}$ as follows:

$$\int \frac{1}{x}dx = \frac{1}{x}(x) - \int x\left(-\frac{1}{x^2}\right)dx = 1 + \int \frac{1}{x}dx \quad \Rightarrow \quad 0 = 1$$

Please comment the above calculation.

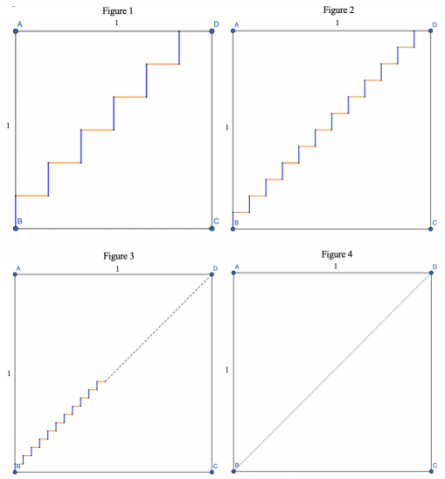
a) It is invalid to cancel out $\int \frac{1}{x} dx$ at both sides.

b) Integration by parts does not apply to certain functions such as $f(x) = \frac{1}{x}$.

c) An indefinite integral should not be regarded as a single function. The absurd result is due to the differences of the constant term in the integrals on L.H.S. and R.H.S.

d) Integration by parts must be used with the change of variables in applying to certain functions when applying it to an indefinite integral.

Q13:



Consider the steps shown in the figures, which are composed of horizontal and vertical straight lines of identical length connecting two opposite vertices of a unit square. Imagine that there are infinitely many such steps. As a teacher, how would you evaluate and explain to your students the total length L of these steps?

a) The infinitely many total horizontal and vertical lengths are respectively 1, hence $L = 2$.

b) The triangle inequality tells us that the length must be larger than 1 and less than 2, hence $1 < L < 2$.

c) When there are infinitely many steps, the steps become a straight line, hence the diagonal of the square with length $\sqrt{2}$. Hence $L = \sqrt{2}$.

d) The sum of horizontal lengths is 1 and the sum of vertical lengths is also 1. Hence $L = 2$.

Q14: Consider the following statements:

Statement 1: A and B are independent

Statement 2: A and B are conditionally independent, given C

Student X argues Statement 1 implies Statement 2, while Student Y claims Statement 2 implies Statement Comment on the validity of both students' arguments.

a) Student X is correct while Student Y is incorrect. (The independence of A and B is a sufficient condition of the conditional independence of A and B given C.)

b) Student X is incorrect while Student Y is correct. (The independence of A and B is a necessary condition of the conditional independence of A and B given C.)

c) Both students are correct. (The independence of A and B is equivalent to the conditional independence of A and B given C.)

d) Both students are incorrect. (The independence of A and B is neither a necessary condition nor a necessary condition of the conditional independence of A and B given C, and vice versa.)

Q15: Each vector has direction and magnitude, where the directions of λa, where λ is a scalar, and a are the same. Students query the direction of **0**. Similarly, the direction of **0** is the same as the direction of $\boldsymbol{i}$ and $\boldsymbol{j}$ (since $0 = 0i = 0j$). Does that mean the directions of $\boldsymbol{i}$ and $\boldsymbol{j}$ are the same? As teacher, which one of the following do you think is the best description of the characteristic of **0** vector?

a) Direction depends on the existence of magnitude, vector **0** has no magnitude, we cannot write **0** in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$.

b) Vector **0** is directionally undefined, or the direction of **0** is not well defined. Hence the equality $0 = 0i = 0j$ does not imply that $i = j$.

c) Vector 0 possesses infinite many directions. We cannot say definitely that $0 = 0i = 0j$**.**

d) $\boldsymbol{i}, \boldsymbol{j}$ are unit vectors. $v = \alpha i + \beta j$ is defined for scalars α, β where they are not both 0 and **0** vector cannot be expressed as $0\boldsymbol{i}$ or $0\boldsymbol{j}$ explicitly.

Q16: The general (or standard) form of quadratic equations, assuming $x \neq 0$, is given by

$$ax^2 + bx + c = 0, \quad a \neq 0 \ldots\ldots (1)$$

Divide the whole equation by x^2 and change (1) to another quadratic equation in variable y by the substitution (i.e. $y = \frac{1}{x}$),

$$cy^2 + by + a = 0, \quad c \neq 0 \ \ldots\ldots (2)$$

If α and β are the roots of equation (1) and if $\alpha'$and β' are the roots of equation (2), then

a) $\alpha + \beta = \frac{1}{\alpha'} + \frac{1}{\beta'},\ \alpha\beta = \frac{1}{\alpha'\beta'}$

b) $\alpha + \beta = \alpha' + \beta', \alpha\beta = \alpha'\beta'$

c) $\alpha + \beta = -(\alpha' + \beta'),\ \alpha\beta = \alpha'\beta'$

d) $\alpha + \beta = \alpha' + \beta',\ \alpha\beta = -\alpha'\beta$

Q17: The following is a harmonic sequence

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \ldots\ldots\ldots + \frac{1}{n}$$

where *n* is an integer.

Determine if *S* is convergent or divergent.

Student A's argument

$$S = 1 + 0.5 + 0.\dot{3}\dot{3} + 0.25 + 0.2 + \ldots\ldots\ldots$$

The additional amount is getting smaller. Thus, student A concludes that the sum of the harmonic sequence is finite and less than 10.

Student B's argument

Student B tries $S_{20}S_{20} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{20} = 3.5977 \ldots$

and concludes that it is a relatively a small value. The sum of *S* is finite and it will be greater than 10 but less than 100.

Student C's argument

Even though the term $\frac{1}{1{,}000{,}000} = 0.0000001$ is a very small number, the sum of S will only be greater than 100 but less than 1000.

Student D's argument

Even though the terms are relatively small, the sum will still be a very large number. It will approach infinity.

Which student's comment is correct?

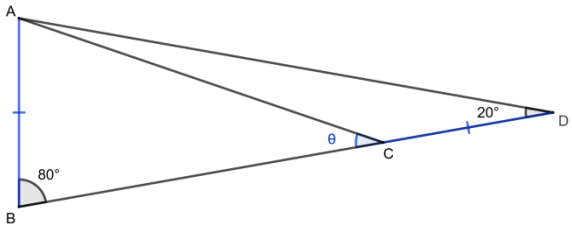
a) Student A

b) Student B

c) Student C

d) Student D

Q18:



In the above figure, C is a point on BD such that AB = CD. Find the value of θ.

a) 30°

b) 35°

c) 40°

d) 45°

Q19: Consider the division algorithm

If $a \neq 0$ and b are integers,

then there exist a unique pair of integers q and r

such that $b = aq + r$ $with$ $0 \leq r < a$

Which of following is correct?

a) $4.5 \div 2 = 2 \ldots 0.5$.

b) The division algorithm still works when the variables are rational numbers.

c) The division algorithm still works when the variables are real numbers.

d) The division algorithm becomes trivial as $r = 0$ when the variables are real numbers.

Q20: Miss Wong conducts a survey of the pupils in her school to test the hypothesis *Left-handed pupils play more different kinds of sport than right-handed pupils do*.

Select two potential questions that Miss Wong could have included in the design of her questionnaire.

(i) Left-handed tennis players have won more *Grand Slam* titles in the last decade, do you agree?

(ii) Are you a member of the school swimming club, soccer team, table tennis group, etc.?

(iii) Are you a cheer group member or leader?

(iv) How many times a year on average have you been injured during sport training or competitions?

a) (i) and (iv)

b) (ii) and (iii)

c) (iii) and (iv)

d) (ii) and (iv)

Q21: Tom has a rule in his mind. The rule determines how three numbers (i.e., a triple) are organized. The rule is "any ascending numbers from 6 to 12." Ivy does not know Tom's rule and wants to discover it. She has a hypothesis that the rule is "any ascending numbers from 1 to 10." She can propose some triples to Tom. Tom will say "Yes" for triples that are consistent with his rule, and "No" for triples that are inconsistent with his rule.

Which of the following triples will lead to a conclusive falsification of Ivy's hypothesis?

a) 7-8-9

b) 8-9-10

c) 9-10-11

d) 10-11-9