Supplementary file

Appendix

Appendix 1:

According to the optimal control theory, the optimal profit function of the supply chain system at time t is $J_s^{C*}=e^{-\rho t}V_s^{C*}$,Where $V_s^C(g^C(t),x^C(t))$, for any $g^C(t)\geq 0$, $x^C(t)\geq 0$ HJB equation is satisfied, namely:

$$\rho V_s^C(g^C(t),x^C(t))=\max_{n^C(t),a_m^C(t),a_r^C(t)}[(2p\gamma-\varepsilon V_{sg}^{C'})g^C(t)+(2p\eta+\theta V_{sg}^{C'}-\beta V_{sx}^{C'})x^C(t)+p(b_1+b_2+\tau(a_m^C(t)+a_r^C(t)))-\frac{1}{2}\kappa_1 n^{C^2}(t)-\frac{1}{2}\kappa_2 a_m^{C^2}(t)-\frac{1}{2}\kappa_3 a_r^{C^2}(t)+V_{sg}^{C'}\sigma(a_m^C(t)+a_r^C(t))+V_{sx}^{C'}\alpha n^C(t)] \tag{A1}$$

Then calculate the first derivative of $n^C(t)$, $a_m^C(t)$, $a_r^C(t)$ on both sides of equation (A1), then the optimal emission reduction effort input and the optimal low-carbon publicity effort input at time t is as follows:

$$n^C(t)=\frac{\alpha V_{sx}^{C'}}{\kappa_1} \tag{A2}$$

$$a_m^C(t)=\frac{p\tau+\sigma V_{sg}^{C'}}{\kappa_2} \tag{A3}$$

$$a_r^C(t)=\frac{p\tau+\sigma V_{sg}^{C'}}{\kappa_3} \tag{A4}$$

By observing the HJB equation(A1), we can find that $V_s^C(g^C(t),x^C(t))$ are about the linear analytic formula of $g^C(t)$ and $x^C(t)$,We can assume that:

$$V_s^C(g^C(t),x^C(t))=m_1^C g^C(t)+m_2^C x^C(t)+m_3^C \tag{A5}$$

Where m_1^C, m_2^C, m_3^C are constants.

Substitute equation (A5) into equation (A1). By comparing the coefficients of similar items at both ends of the equation, we can get the following constraint equations about the parameters of the optimal profit function(m_i^C ,i=1、2、3):

$$\rho m_1^C = 2p\gamma - \varepsilon m_1^C \tag{A6}$$

$$\rho m_2^C = 2p\eta + \theta m_1^C - \beta m_2^C \tag{A7}$$

$$\begin{aligned}\rho m_3^C = {} & p(b_1 + b_2 + \tau(a_m^C(t) + a_r^C(t))) - \frac{1}{2}\kappa_1 n^{C^2}(t) - \frac{1}{2}\kappa_2 a_m^{C^2}(t) \\ & - \frac{1}{2}\kappa_3 a_r^{C^2}(t)) + m_1^C \sigma(a_m^C(t) + a_r^C(t)) + m_2^C \alpha n^C(t)\end{aligned} \tag{A8}$$

Solve equations (A6) and (A7) to obtain the expression of parameter m_i^C （i=1、2）of the optimal profit function:

$$m_1^{C*} = \frac{2p\gamma}{\rho + \varepsilon} \tag{A9}$$

$$m_2^{C*} = \frac{2p[\eta(\rho+\varepsilon) + \theta\gamma]}{(\rho+\varepsilon) + (\rho+\beta)} \tag{A10}$$

Where $V_{sg}^{C'} = m_1^{C*}, V_{sx}^{C'} = m_2^{C*}$,thus substitute m_i^C (i=1, 2) into formula (A2), (A3) and (A4) to obtain the manufacturer's optimal low-carbon emission reduction effort and optimal low-carbon advertising effort at time t under centralized decision, as well as the retailer's optimal low-carbon advertising effort , as shown in formula (11), (12) and (13) respectively. Then, the formula (11), (12), (13), (A9) and (A10) are introduced into formula (A8) to get the expression of m_3^C :

$$\begin{aligned}m_3^{C*} = {} & \frac{1}{\rho}[p(b_1 + b_2 + \tau(a_m^{C*} + a_r^{C*})) - \frac{1}{2}\kappa_1 n^{C*2} - \frac{1}{2}\kappa_2 a_m^{C*2} \\ & - \frac{1}{2}\kappa_3 a_r^{C*} + m_1^{C*}\sigma(a_m^{C*} + a_r^{C*}(t)) + m_2^{C*}\alpha n^{C*}(t)]\end{aligned} \tag{A11}$$

Substituting equations (11), (12) and (13) into equations (1) and (2) for solution, it is obtained that the optimal trajectories of low-carbon emission reduction level and low-carbon goodwill of products are formula (14) and (15) of proposition 1, respectively. When time t tends to infinity, the stable values of the optimal trajectories of low-carbon emission reduction level and low-carbon goodwill of products are formula (16) and (17) of proposition 1, respectively. Observe the optimal trajectories (14) and (15), then we can get the following simplified optimal trajectories:

$$x^{C*}(t) = C_1 e^{-\rho t} + x_{\infty}^{C*} \tag{A12}$$

$$g^{C*}(t) = C_2 e^{-\rho t} + g_{\infty}^{C*} \tag{A13}$$

By substituting $m_1^{C*}, m_2^{C*}, m_3^{C*}$ $x^{C*}(t), g^{C*}(t), n^{C*}, a_m^{C*}, a_r^{C*}$ into equation (A5), the present value of the profits of the supply chain system under centralized decision is obtained as equation (20) in proposition 1.

Appendix 2:

According to the optimal control theory, the optimal profit function of the manufacturer and retailer in the low-carbon advertising competition at time t is $J_m^{D*} = e^{-\rho t} V_m^{D*}$ 、 $J_r^{D*} = e^{-\rho t} V_r^{D*}$, where $V_m^D(g^D(t), x^D(t)), V_r^D(g^D(t), x^D(t))$,the HJB equation is satisfied for any $g^D(t) \geq 0, x^D(t) \geq 0$, namely

$$\begin{aligned}
&\rho V_m^D(g^D(t), x^D(t)) = \max_{n^D(t), a_m^D(t)} [((p+w)\gamma - \varepsilon V_{mg}^{D'}) g^D(t) + ((p+w)\eta + \\
&\theta V_{mg}^{D'} - \beta V_{mx}^{D'}) x^D(t) + p(b_1 + s(a_m^D(t) - a_r^D(t)) + \tau a_m^D(t)) + w(b_2 + s(a_r^D(t) \\
&- a_m^D(t)) + \tau a_r^D(t)) - \frac{1}{2}\kappa_1 n^{D^2}(t) - \frac{1}{2}\kappa_2 a_m^{D^2}(t) + V_{mg}^{D'} \sigma(a_m^D(t) + a_r^D(t)) + \\
&V_{mx}^{D'} \alpha n^D(t)]
\end{aligned} \tag{A14}$$

$$\begin{aligned}
&\rho V_r^D(g^D(t), x^D(t)) = \max_{a_r^D(t)} [((p-w)\gamma - \varepsilon V_{rg}^{D'}) g^D(t) + ((p-w)\eta + \\
&\theta V_{rg}^{D'} - \beta V_{rx}^{D'}) x^D(t) + (p-w)(b_2 + s(a_r^D(t) - a_m^D(t)) + \tau a_r^D(t)) - \\
&\frac{1}{2}\kappa_3 a_r{}^2(t) + V_{rg}^{D'} \sigma(a_m^D(t) + a_r^D(t)) + V_{rx}^{D'} \alpha n^D(t)]
\end{aligned} \tag{A15}$$

Calculate the first derivative of $a_r^D(t)$ on both sides of equation (A15) to obtain equation (A16):

$$a_r^D(t)=\frac{(p-w)(s+\tau)+\sigma V_{rg}^{D'}}{\kappa_3} \tag{A16}$$

Substitute $a_r^D(t)$ into equation (A14), and calculate the first derivative of $n^D(t), a_m^D(t)$ for the equation brought in, then the manufacturer's optimal emission reduction efforts and optimal advertising efforts at time t are respectively:

$$n^D(t)=\frac{\alpha V_{mx}^{D'}}{\kappa_1} \tag{A17}$$

$$a_m^D(t)=\frac{p(s+\tau)-ws+\sigma V_{mg}^{D'}}{\kappa_2} \tag{A18}$$

By observing HJB equations (A14) and (A15), we can find that $V_m^D(g^D(t),x^D(t)), V_r^D(g^D(t),x^D(t))$ is a linear analytic expression of $g^D(t), x^D(t)$ and $g^D(t), x^D(t)$,we can assume that:

$$V_m^D(g^D(t),x^D(t))=h_1^D g^D(t)+h_2^D x^D(t)+h_3^D \tag{A19}$$

$$V_r^D(g^D(t),x^D(t))=f_1^D g^D(t)+f_2^D x^D(t)+f_3^D \tag{A20}$$

Where $h_1^D, h_2^D, h_3^D; f_1^D, f_2^D, f_3^D$ are constants.Substitute equations (A19) and (A20) into equations (A14) and (A15) respectively. By comparing the coefficients of similar items at both ends of the equation, we can get the following constraint equations about the parameter h_i^D, f_i^D （i=1、2、3）of the optimal profit function:

$$\rho h_1^D=(p+w)\gamma-\varepsilon h_1^D \tag{A21}$$

$$\rho h_2^D=(p+w)\eta+\theta h_1^D-\beta h_2^D \tag{A22}$$

$$\begin{aligned}\rho h_3^D &= p(b_1+s(a_m^D(t)-a_r^D(t))+\tau a_m^D(t))+w(b_2+s(a_r^D(t)-a_m^D(t))+\tau a_r^D(t)) \\ &-\frac{1}{2}\kappa_1 n^{D^2}(t)-\frac{1}{2}\kappa_2 a_m^{D^2}(t)+h_1^D\sigma(a_m^D(t)+a_r^D(t))+n_2^D\alpha n^D(t)\end{aligned} \tag{A23}$$

$$\rho f_1^D=(p-w)\gamma-\varepsilon f_1^D \tag{A24}$$

$$\rho f_2^D=(p-w)\eta+\theta f_1^D-\beta f_2^D \tag{A25}$$

$$\rho f_3^D = (p-w)(b_2 + s(a_r^D(t) - a_m^D(t)) + \tau a_r^D(t)) - \frac{1}{2}\kappa_3 a_r^{D^2}(t) + f_1^D \sigma(a_m^D(t) + a_r^D(t)) + f_2^D \alpha n^D(t) \tag{A26}$$

Then , the parameter equations of the formula are combined in the same way, and the expressions of the parameters h_i^D, f_i^D (i=1, 2) of the optimal profit function are obtained as follows:

$$h_1^{D*} = \frac{(p+w)\gamma}{(\rho+\varepsilon)} \tag{A27}$$

$$h_2^{D*} = \frac{(p+w)[(\rho+\varepsilon)\eta+\theta\gamma]}{(\rho+\varepsilon)(\rho+\beta)} \tag{A28}$$

$$f_1^{D*} = \frac{(p-w)\gamma}{(\rho+\varepsilon)} \tag{A29}$$

$$f_2^{D*} = \frac{(p-w)[(\rho+\varepsilon)\eta+\theta\gamma]}{(\rho+\varepsilon)(\rho+\beta)} \tag{A30}$$

Where $V_{mg}^{D'} = h_1^{D*}, V_{mx}^{D'} = h_2^{D*}; V_{rg}^{D'} = f_1^{D*}, V_{rx}^{D'} = f_2^{D*}$ .thus substitute h_i^{D*}, f_i^{D*} (i=1, 2) into formula (A16), (A17) and (A18) to get the equilibrium decision $n^{D*}, a_m^{D*}, a_r^{D*}$ of the manufacturer's optimal low-carbon emission reduction effort level and the optimal low-carbon advertising effort level, and the retailer's optimal low-carbon advertising effort level at time t under the low-carbon advertising competition.

Then substitute $h_1^{D*}, h_2^{D*}, f_1^{D*}, f_2^{D*}; n^{D*}, a_m^{D*}, a_r^{D*}$ into equations (A23) and (A26) to get the expression of parameter h_3^D, f_3^D of the optimal profit function:

$$h_3^{D*} = \frac{1}{\rho}[p(b_1 + +s(a_m^{D*} - a_r^{D*}) + \tau a_m^{D*}) - w(b_2 + s(a_r^{D*} - a_m^{D*}) + \tau a_r^{D*}) - \frac{1}{2}\kappa_1 n^{D*2} - \frac{1}{2}\kappa_2 a_m^{D*2} + h_1^{D*}\sigma(a_m^{D*} + a_r^{D*}(t)) + h_2^{D*}\alpha n^{D*}(t)] \tag{A31}$$

$$f_3^{D*} = \frac{1}{\rho}[(p-w)(b_2 + s(a_r^{D*} - a_m^{D*}) + \tau a_r^{D*}) - \frac{1}{2}\kappa_3 a_r^{D*} + f_1^{D*}\sigma(a_m^{D*} + a_r^{D*}(t)) + f_2^{D*}\alpha n^{D*}(t)] \tag{A32}$$

Substituting Equations (37), (38) and (39) into Equations (1) and (2) to solve the problem, the optimal trajectories of product emission reduction and low-carbon goodwill are respectively equations (40) and (41) of proposition 2. Then we set $D_1 = (x_0 - x_\infty^{D*}), D_2 = (g_0 - g_\infty^{D*})$ to obtain the following simplified formula:

$$x^{D*}(t) = D_1 e^{-\rho t} + x_\infty^{D*} \tag{A33}$$

$$g^{D*}(t) = D_2 e^{-\rho t} + g_\infty^{D*} \tag{A34}$$

Then,substitute $h_1^{D*}, h_2^{D*}, h_3^{D*}; f_1^{D*}, f_2^{D*}, f_3^{D*}\ n^{D*}, a_m^{D*}, a_r^{D*}$,equations(A33) and (A34) into equations (A19) and (A20) to obtain the profit present value of the manufacturer, retailer and supply chain under the low-carbon advertising competition in proposition 2.The propositional proofs in the contract model are the same, so we omit them.

The model proof process in Section 4 is similar to that in Section 3.