1 APPENDIX

1.1 Kac's Chessboard and Changing Switches

The Kac model, expressed in the 'chessboard' format is:

$$K_K(x,t) = \sum_R N(R)(m\epsilon)^R(1-m\epsilon)^{\mathcal{N}-R} \tag{S1}$$

where $\mathcal{N} = t/\epsilon$ is the total number of steps in the path from the origin to the position x at time t. The coefficients $N(R)$ are the same for the FCM and the Kac model. The reason for the weight $(1-m\epsilon)^{\mathcal{N}-R}$ in the sum is to compensate for the fact that in terms of 'Persist' and 'Switch', the term $(P+\epsilon m\sigma_x)$ codes for a branching process that grows over time. As an operator on a two component column vector, for any positive initial condition, say $x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, powers $(P+\epsilon m\sigma_x)^n$ ultimately count the number of paths. Operating on the initial condition will increase the norm of the column vector. To avoid the increase and place the model in the domain of probability, the coefficient of P, taken to be $(1-\epsilon m)$ normalizes the process. So in terms of the binary expansion a path is actually described by:

$$(P \veebar S)^n \tag{S2}$$

where $\veebar$ is an exclusive OR. A particle either persists or switches at each step, but does not do both. That is, for a single particle

$$(P \wedge S) = \perp \tag{S3}$$

and a single path cannot both Persist and Switch at a single step. An ensemble of paths or a branching process can sensibly do both but in the Kac model we choose the former and normalize the process appropriately so that the end result is a probabilistic model.

If the switch is changed to $S_M = -i\sigma_y$ we can see that this takes us out of a probabilistic interpretation immediately. If we sequentially list powers of the switch acting on x_0 we get

$$\{x_0, S_M x_0 \ldots S_M^4 x_0 \ldots\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ldots\} \tag{S4}$$

This makes little direct sense when thinking probabilistically of a Kac path. On the other hand the KMM model is coding for special *pairs* of paths; namely Fraternal twins. If we use the right twin to code for the pair we notice that the right twin has S_M for the switch because it allows for the switch to change photon path direction at the outside corners, but *not* change direction at inside corners. We do not easily see the consequences of this in the above sequence because the other path is not present to exchange at switches. Our initial condition obscures what is really happening . If we put in a more suitable initial condition, say $x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ that starts with two paths, positively oriented to the right and left respectively, we get:

$$\{x_1, S_M x_1 \ldots S_M^4 x_1 \ldots\} = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ldots\} \tag{S5}$$

We can see that this is indeed a rotation (First, second, third and fourth quadrant null lines respectively.) Rotations may be made 'small' even though reflections cannot be. So in this case:

$$(P \wedge S_M) = \perp \tag{S6}$$

That is, in the sum over paths we can simultaneously have Persist *and* Switch. A way to see this is to write S_M to look like a continuous function of t, for example:[1]

[1] Here we are choosing $m = \frac{\pi}{2}$ so the switching conveniently occurs at integers.

$$S_M = S_q(\lfloor t \rfloor) \text{ where } S_q(t) = \begin{pmatrix} \cos\left(\frac{\pi}{2}t\right) & \sin\left(\frac{\pi}{2}t\right) \\ -\sin\left(\frac{\pi}{2}t\right) & \cos\left(\frac{\pi}{2}t\right) \end{pmatrix}. \tag{S7}$$

S_q is a smooth function of t but $S_q(\lfloor t \rfloor)$ gives powers of the switch.

$$S_q(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_q(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S_q(2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \ldots \tag{S8}$$

While $(P + \epsilon m S_K)^n$ *required* a probabilistic normalization for P to make sense because P and S_K were mutually exclusive on an actual particle path, here $(P + \epsilon m S_M)^n$, for small epsilon can be replaced by $S_q(\epsilon)$ without need of the normalization term because $(P + \epsilon m S_q(\epsilon))$ makes sense in terms of $(P \wedge S)$ as a small rotation. *Ultimately, causal areas may be created by rotating their opposing boundaries into one another, rather than 'growing' them from the events.*

A contact can be made with classical statistical mechanics in that, for example, the character of solutions of both the Kac and KMM models changes as the scale of observation approaches the mean free path represented by the characteristic scale m. Once you get close to this scale you start to escape the time averaging of observations on coarser scales and begin to 'see' the emergence of Newton's first law in the context of space-time or spacetime respectively.

In the Kac model Newton's first law is reflected in the unbroken path of a particle with velocity ± 1 on fine scales. In the KMM the law of inertia is reflected in rectangular causal areas between Events, with Events lying on a single timelike ray. This is manifest as a rotation because the boundaries of the rectangles have opposite orientation (See the colourings of area boundaries in fig:KFSwitch).

On very large scales the Kac model is a thermodynamic clock that measures time through an exponential decay to equilibrium eqn:decay.

On very large scales the KMM model would depend on what can be detected. If mass and hence momentum cannot be detected we could only expect that the reaction to boosts would respond to spacetime, but the marking by Events and the resulting aspects of wave propagation would be absent. The clock aspect of a particle would be lost due to the undetectability of the Events that mark the worldline. If p and m *can* be detected the result would more closely resemble a stopwatch and for small p we would expect the propagator to look like the non-relativistic version rather than a thermodynamic clock, as the propagator measures time modulo a fundamental wavelength.