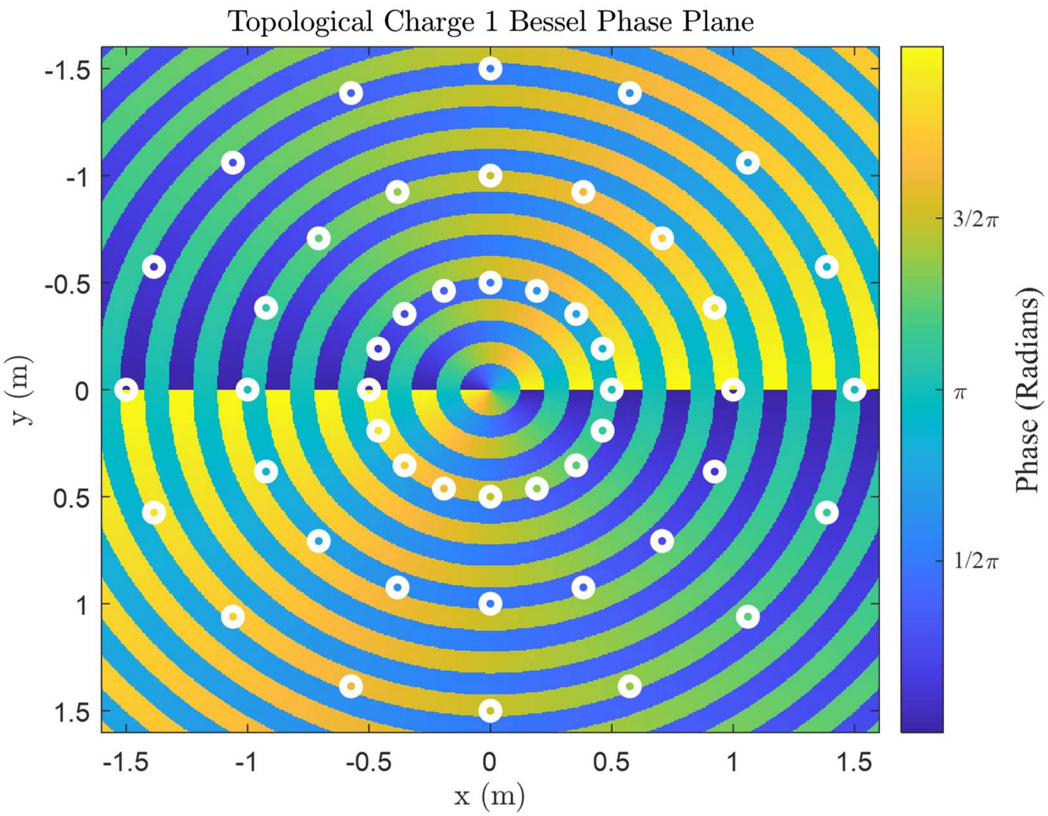
Supplemental Information

1. Array Configurations

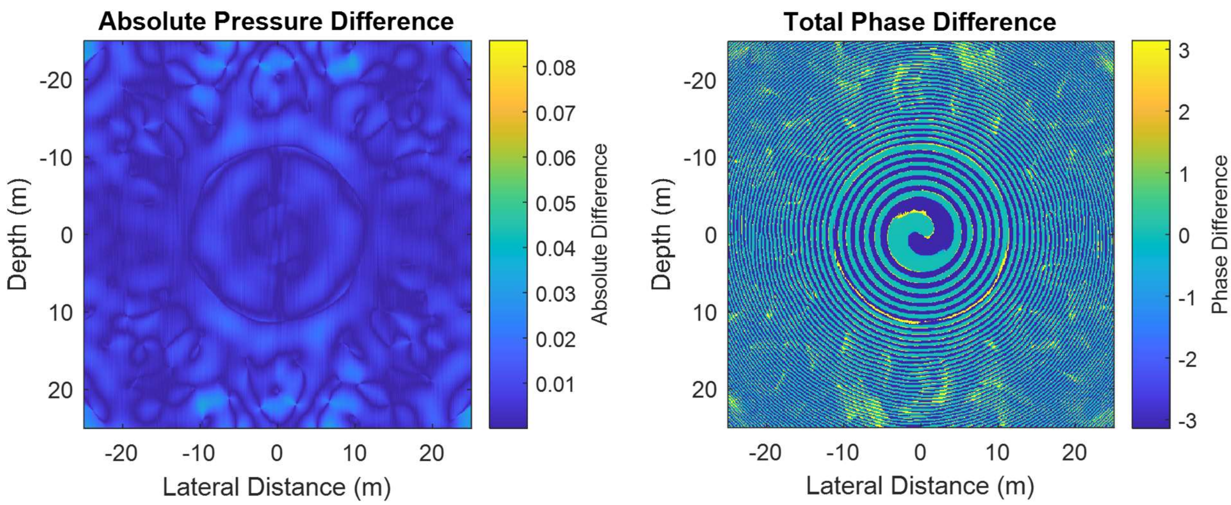
The array phase relationships simulated for validation cases in the main body of the text are depicted pictorially in Supplemental Figure 1.



Supplemental Figure 1: The phase relationships for a hypothetical vortex array transmitting topological charge l = +1

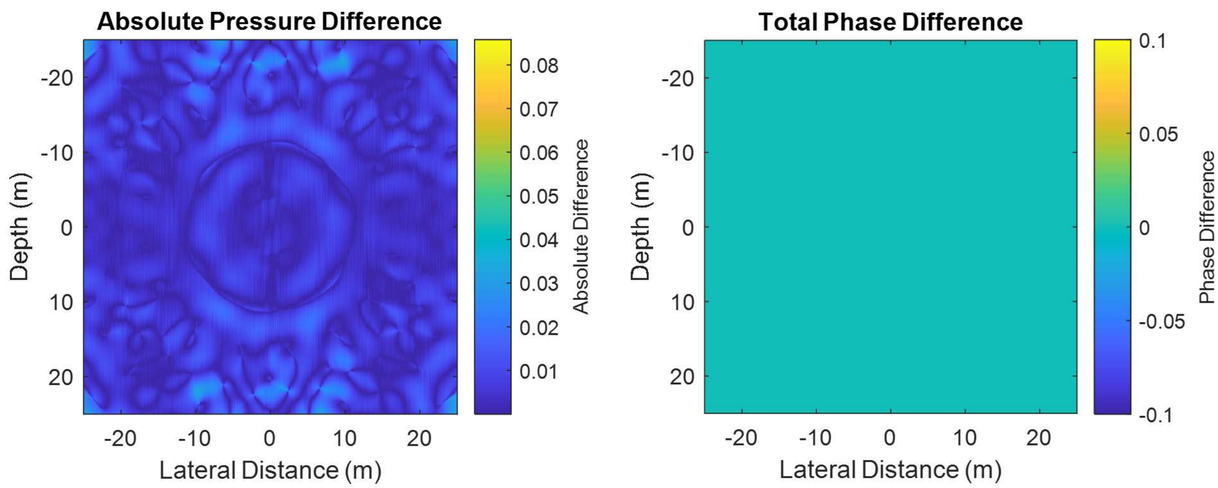
2. Error analysis on wavefront propagating plots

The primary applications of the range v. depth plots like those in Fig. 2 are to observe the central singularity, and track the general propagation of this feature in the inhomogeneous environment. Because the assumptions are fundamentally different, the numerical comparison of the range vs. depth field would have little meaning. Furthermore, because the vortex itself is of primary interest, the fine-resolution behavior of the side lobes is also of little concern. The performance of the plots like those in Figure 3, however, are of greater concern, as these are the plots which allow researchers to perform analysis on the health of the vortex. Supplemental Figure 2 shows a plot of the error in both absolute pressure and phase for the two plots.



Supplemental Figure 2 The absolute difference between the plots depicted in Figure 3 of the main body of the text.

Close inspection of Figure 3 reveals that there is a difference in the precession of the vortex between the two plots. This was caused by a difference in indexing during the simulations and leads to the phase errors seen above. When this is corrected for, the phase error disappears, as can be seen in Supplemental Figure 3.



Supplemental Figure 3: The error between the plots depicted in Figure 3 when the difference in precession of the vortex is corrected for.

In both cases the maximum absolute pressure difference is .062 and the mean absolute error is .0062. This represents strong agreement between the two computational methods.

3. K-Space filtering and the 2D Fast Fourier Transform

K-Space analysis is achieved using the 2D Fast Fourier Transform (FFT). This process transforms the spatially distributed pressure information into the wavenumber domain (also known as the K-space). This was done in MATLAB, which employs the following fundamental equations.

$$Y_{p+1,q+1} = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} e^{-i2\pi jp/m} \, e^{-i2\pi kq/m} X_{j+1,k+1} \quad (1)$$

Here Y represents the transformed n-by-m matrix in the wavenumber domain and X the original n-by-m matrix of spatially distributed complex pressure values. The indices *p, q, j,* and *k* are all shifted by 1 to account for MATLAB's indexing convention which starts at 1. Filtering operations were performed using a unit step mask which is of the same dimensionality as the transformed matrix Y and has a value of 1 for those values which are to be retained, and 0 everywhere else. The mask is applied by pixelwise multiplication of the matrix Y and the mask function.

4. Vortex Beams BELLHOP Tutorial

A step-by-step implementation of the methods are presented here for the purpose of replicating and expanding upon this study. This is written with the assumption that the user is familiar with BELLHOP. BELLHOP is available open source as a part of the ocean acoustics library[1]. Example files are provided as a reference, though the .env files are premade for the scenarios presented in the main body of the text. To generate new cases, new .env files need to be made. The key pieces to modeling vortex beams using the 2D ray tracing code are proper accounting of the source positions and where receivers must be simulated. This tutorial will focus primarily on setting the proper source and receiver depths and ranges for any arbitrary scenario you may wish to simulate.

A. Range vs Depth Plots.

Source locations for the range vs depth plots should be set as follows. For a circular array centered at the origin, the x and y positions for an array consisting of *M* concentricings of *N* elements are given by equation 2.

[1] http://oalib.hlsresearch.com/Rays/

$$x_n = r_m \cos(\frac{2\pi n}{N})$$

$$y_n = r_m \sin(\frac{2\pi n}{N}) \quad (2)$$

The depths of the elements are then given as $y_n + Array\ Depth$. These will be the source locations for the range vs. depth plots. For simplicity, the example files use only one source per .env file, though more can be simulated as long as the proper depth and phase are well accounted for. Receiver locations are set by simply selecting the bounds in range and depth and determining the desired resolution. The coherent transmission loss mode should be used, and when BELLHOP is run, a .shd file will be produced with the same name as the corresponding .env file. The 'get_pressures' function provided will read the output file, and extract the calculated pressure field in the form of a matrix of complex numbers. By adding the appropriate phase shift for each transducer and summing for the fields produced by each transducer, the range v. depth plots are replicated. The features of the vortex will be apparent in both pressure and TL. For shorter ranges, plotting pressure over a small dynamic range will work to visualize the features (0 to 1 was used for this work). For longer propagation ranges TL will work better at capturing the dynamic scale as was used in Fan et al and Kelly and Shi.

B. Wavefront Plane Plots.

For the wavefront plane, the source positions are determined using Equation 2, and the .env files are generated with the source depths again located at $y_n + Array\ Depth$. The receiver ranges and depths are determined by the range and dimensions of the plane you want to simulate. For a given wavefront plane bounded in depth by an arbitrary z_{min} & z_{max}, the receiver depths will always be set by those bounds and the number of receivers should be set as $N_{rd} = \Delta z/desired\ resolution.$ The receiver ranges should be bounded by the range of the desired plane from the source array (r as depicted in Figure 2) and the max receiver range value calculated using equation 3 (r' as depicted in Figure 2).

$$r'_{max} = \sqrt{r^2 + (x_n + \frac{L}{2})^2} \tag{3}$$

Here *L* represents the length of the x dimension of the wavefront plane. Similarly to the receiver depths, receiver ranges should be set as $N_{rr} = (r - r'_{max}/desired\ resolution.$ BELLHOP should again be run using the coherent transmission loss method, and the pressures extracted using "get_pressures." The results will be an *M x N* matrix of the pressure field. This matrix must be interpolated onto the full grid of the desired imaging plane, the process of which eliminates any possible rounding or truncation errors as well as exploits the symmetry in the r' values. Applying the appropriate phase shift to the pressure grid and coherently summing the fields for each transducer yields the desired result.