Appendix:

Assuming that is the shear wave spectrum at location , the spectrum after the shear wave has traveled a distance of is given as follows:

|  |  |
| --- | --- |
|  | (1) |

where and represent geometric spreading and shear wave attenuation within the medium, respectively. The attenuation coefficient () varies linearly with frequency *(f)* and is defined as follows (Bernard et al., 2016):

|  |  |
| --- | --- |
|  | (2) |

The shear wave spectrum can be fitted to a gamma distribution as follows (Bernard et al., 2016):

|  |  |
| --- | --- |
| , | (3) |

where and represent the shape and rate parameters, respectively, which characterize a Gamma distribution; *A* is the amplitude. In the standard frequency shift method (Bernard et al., 2016), the shape parameter, , is assumed to be constant as the shear waves propagate, but the spatial variation of the attenuation coefficient changes the rate parameter. Therefore, can be expressed as follows:

|  |  |
| --- | --- |
|  | (4) |

where is the slope of the line. Calculating the rate parameter over a range of lateral positions, is estimated by minimizing the following optimization function in least squared sense:

|  |  |
| --- | --- |
|  | (5) |

Conversely, the shear wave spectrum (*S(f)*) can be represented by a Gaussian distribution as follows (Parker and Baddour, 2014):

(6)

Here *fs*, and *σs*represent the centroid frequency and standard deviation of the shear wave spectrum, respectively. After traveling *Δx*, the shear wave spectrum can be represented with a new Gaussian distribution (*V(f)*) with a centroid frequency of *fv* and a standard deviation of *σv* or alternatively as a function of *S(f)* and *H(f)* as follows:

(7)

(8)

Furthermore, we can rearrange the equation as follows:

(9)

Mathematically, a relationship between *S(f)* and *V(f)* can only be established if both spectra have a common standard deviation (*σv*= *σs*). Hence, the centroid frequencies is the only varying parameter for the Gaussian distribution. Assuming that *σv*is equal to *σs*, *Eq. 5* is expanded as follows:

(10)

Simplifying further:

(11)

This can be rewritten as:

(12)

(13)

where:

(14)

and:

(15)

Since the shear wave spectra are normalized by *S(f)* amplitude, the termin *Eq.13* is ignored. The attenuation coefficient is estimated from *Eq. 14* by minimizing *Eq. 5* in the least squares sense with multiple lateral positions using the estimated *fs* and *fv.*