frontiers

Supplementary Material

Continue from Eq. 90. Integrating by parts for $\left(\rho u\right)$ and $\left(\rho v\right)$ results in:

$$\begin{aligned}&\int_0^h \frac{\partial \rho}{\partial t}dz + h\left[\frac{\partial(\rho U)_2}{\partial x}+\frac{\partial(\rho V)_2}{\partial y}\right]-\frac{\partial}{\partial x}\int_0^h z\left(\rho\frac{\partial u}{\partial z}+u\frac{\partial \rho}{\partial z}\right)dz\\&-\frac{\partial}{\partial y}\int_0^h z\left(\rho\frac{\partial v}{\partial z}+v\frac{\partial \rho}{\partial z}\right)dz+\left(\rho w\right)_0^h=0\end{aligned} \tag{S1}$$

The expressions of velocity gradients and velocities are:

$$\begin{aligned}\frac{\partial u}{\partial z}=&\left(\frac{z}{\eta_{EM}(\gamma,T)}-\frac{f_5(h)}{\eta_{EM}(\gamma,T)\,f_{10}(h)}\right)\frac{\partial p}{\partial x}\\&+\frac{U_2-U_1-f_6(h)-f_8(h)}{\eta_{EM}(\gamma,T)\,f_{10}(h)}+f_1(z)+f_3(z)\end{aligned} \tag{S2}$$

$$\begin{aligned}\frac{\partial v}{\partial z}=&\left(\frac{z}{\eta_{EM}(\gamma,T)}-\frac{f_5(h)}{\eta_{EM}(\gamma,T)\,f_{10}(h)}\right)\frac{\partial p}{\partial y}\\&+\frac{V_2-V_1-f_7(h)-f_9(h)}{\eta_{EM}(\gamma,T)\,f_{10}(h)}+f_2(z)+f_4(z)\end{aligned} \tag{S3}$$

$$\begin{aligned}u=&\ f_{11}(z)\frac{\partial p}{\partial x}+\frac{U_2-U_1-f_6(h)-f_8(h)}{f_{10}(h)}\ f_{10}\left(z\right)\\&+f_6\left(z\right)+f_8\left(z\right)+U_1\end{aligned} \tag{S4}$$

$$\begin{aligned}v=&\ f_{11}(z)\frac{\partial p}{\partial y}+\frac{V_2-V_1-f_7(h)-f_9(h)}{f_{10}(h)}\ f_{10}\left(z\right)\\&+f_7\left(z\right)+f_9\left(z\right)+V_1\end{aligned} \tag{S5}$$

Substitute Eq. (S2)~(S5) into Eq. (S1), one could get:

$$\begin{aligned}&-\frac{\partial}{\partial x}\int_0^h z\left(\rho\frac{\partial u}{\partial z}+u\frac{\partial \rho}{\partial z}\right)dz\\
&=-\frac{\partial}{\partial x}\left(\begin{aligned}&\frac{\partial p}{\partial x}\int_0^h z\rho\left(\frac{z}{\eta_{EM}(\gamma,T)}-\frac{f_5(h)}{\eta_{EM}(\gamma,T)\,f_{10}(h)}\right)dz\\&+\frac{U_2-U_1-f_6(h)-f_8(h)}{f_{10}(h)}\int_0^h\frac{z\rho}{\eta_{EM}(\gamma,T)}dz+\int_0^h z\rho\,f_1(z)dz+\int_0^h z\rho\,f_3(z)dz\end{aligned}\right)\\
&-\frac{\partial}{\partial x}\left(\begin{aligned}&\frac{\partial p}{\partial x}\int_0^h z\frac{\partial \rho}{\partial z}\,f_{11}(z)dz+\frac{U_2-U_1-f_6(h)-f_8(h)}{f_{10}(h)}\int_0^h z\frac{\partial \rho}{\partial z}\,f_{10}(z)dz\\&+\int_0^h z\frac{\partial \rho}{\partial z}\,f_6(z)dz+\int_0^h z\frac{\partial \rho}{\partial z}\,f_8(z)dz+U_1\int_0^h z\frac{\partial \rho}{\partial z}dz\end{aligned}\right)\end{aligned} \tag{S6}$$

$$
\begin{aligned}
&-\frac{\partial}{\partial y}\int_0^h z\left(\rho\frac{\partial v}{\partial z}+v\frac{\partial \rho}{\partial z}\right)dz \\
&=-\frac{\partial}{\partial y}\left(\begin{aligned}&\frac{\partial p}{\partial y}\int_0^h z\rho\left(\frac{z}{\eta_{EM}(\gamma,T)}-\frac{f_5(h)}{\eta_{EM}(\gamma,T)\,f_{10}(h)}\right)dz\\&+\frac{V_2-V_1-f_7(h)-f_9(h)}{f_{10}(h)}\int_0^h\frac{z\rho}{\eta_{EM}(\gamma,T)}dz+\int_0^h z\rho f_2(z)dz+\int_0^h z\rho f_4(z)dz\end{aligned}\right) \\
&-\frac{\partial}{\partial y}\left(\begin{aligned}&\frac{\partial p}{\partial y}\int_0^h z\frac{\partial \rho}{\partial z}f_{11}(z)dz+\frac{V_2-V_1-f_7(h)-f_9(h)}{f_{10}(h)}\int_0^h z\frac{\partial \rho}{\partial z}f_{10}(z)dz\\&+\int_0^h z\frac{\partial \rho}{\partial z}f_7(z)dz+\int_0^h z\frac{\partial \rho}{\partial z}f_9(z)dz+V_1\int_0^h z\frac{\partial \rho}{\partial z}dz\end{aligned}\right)
\end{aligned}
\tag{S7}
$$

Introducing F and G functions (same as Eq. 92), then Eqs. (S6) and (S7) can be written as:

$$
\begin{aligned}
&-\frac{\partial}{\partial x}\int_0^h z\left(\rho\frac{\partial u}{\partial z}+u\frac{\partial \rho}{\partial z}\right)dz \\
&=-\frac{\partial}{\partial x}\left(\left[F_2+G_1\right]\frac{\partial p}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{(U_2-U_1)(F_3+G_2)}{F_0}+U_1G_3\right) \\
&-\frac{\partial}{\partial x}(F_4+F_5)-\frac{\partial}{\partial x}(G_4+G_5)
\end{aligned}
\tag{S8}
$$

$$
\begin{aligned}
&-\frac{\partial}{\partial y}\int_0^h z\left(\rho\frac{\partial v}{\partial z}+v\frac{\partial \rho}{\partial z}\right)dz \\
&=-\frac{\partial}{\partial y}\left(\left[F_2+G_1\right]\frac{\partial p}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{(V_2-V_1)(F_3+G_2)}{F_0}+V_1G_3\right) \\
&-\frac{\partial}{\partial y}(F_6+F_7)-\frac{\partial}{\partial y}(G_6+G_7)
\end{aligned}
\tag{S9}
$$

Substitute Eq. (S8) and Eq. (S9) into Eq. (S1) results in generalized MEMT-field Reynolds equation, Eq. 91, which is repeated here:

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left(\left[F_2+G_1\right]\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\left[F_2+G_1\right]\frac{\partial p}{\partial y}\right)=h\left[\frac{\partial(\rho U)_2}{\partial x}+\frac{\partial(\rho V)_2}{\partial y}\right] \\
&-\frac{\partial}{\partial x}\left(\frac{(U_2-U_1)(F_3+G_2)}{F_0}+U_1G_3\right)-\frac{\partial}{\partial y}\left(\frac{(V_2-V_1)(F_3+G_2)}{F_0}+V_1G_3\right) \\
&-\frac{\partial}{\partial x}(F_4+G_4)-\frac{\partial}{\partial y}(F_6+G_6)-\frac{\partial}{\partial x}(F_5+G_5)-\frac{\partial}{\partial y}(F_7+G_7) \\
&+\int_0^h\frac{\partial \rho}{\partial t}dz+(\rho W)_2-(\rho W)_1
\end{aligned}
\tag{S10}
$$

In most engineering applications, the boundary surface velocities in the *x* and *y* directions are constants, which leads to:

$$\frac{\partial\left(U_2-U_1\right)}{\partial x}=\frac{\partial\left(V_2-V_1\right)}{\partial y}=0 \tag{S11}$$

In the *z* direction, when setting $W_1=0$, the following is accepted:

$$\frac{\partial h}{\partial t}\approx W_2-W_1-U_2\frac{\partial h}{\partial x}-V_2\frac{\partial h}{\partial y} \tag{S12}$$

As a result, Eq. (S10) can be written as:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\left[F_2+G_1\right]\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\left[F_2+G_1\right]\frac{\partial p}{\partial y}\right)=U_2\frac{\partial\left(\rho h\right)}{\partial x}+V_2\frac{\partial\left(\rho h\right)}{\partial y}\\
&-\left(U_2-U_1\right)\frac{\partial}{\partial x}\left(\frac{F_3+G_2}{F_0}\right)-U_1\frac{\partial G_3}{\partial x}-\left(V_2-V_1\right)\frac{\partial}{\partial y}\left(\frac{F_3+G_2}{F_0}\right)-V_1\frac{\partial G_3}{\partial y}\\
&-\frac{\partial}{\partial x}\left(F_4+G_4\right)-\frac{\partial}{\partial y}\left(F_6+G_6\right)-\frac{\partial}{\partial x}\left(F_5+G_5\right)-\frac{\partial}{\partial y}\left(F_7+G_7\right)\\
&+\int_0^h\frac{\partial\rho}{\partial t}dz+\rho\frac{\partial h}{\partial t}
\end{aligned} \tag{S13}$$

For liquid lubricant, density variation across the film can be neglected, i.e. $\partial\rho/\partial z=0$. Thus, all G functions are zero, hence generalized MEMT-field Reynolds equation can be reduced to Eq. 93, which is repeated here:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(F_2\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(F_2\frac{\partial p}{\partial y}\right)=U_2\frac{\partial\left(\rho h\right)}{\partial x}+V_2\frac{\partial\left(\rho h\right)}{\partial y}\\
&-\left(U_2-U_1\right)\frac{\partial}{\partial x}\left(\frac{F_3}{F_0}\right)-\left(V_2-V_1\right)\frac{\partial}{\partial y}\left(\frac{F_3}{F_0}\right)\\
&-\frac{\partial}{\partial x}F_4-\frac{\partial}{\partial y}F_6-\frac{\partial}{\partial x}F_5-\frac{\partial}{\partial y}F_7+\frac{\partial\left(\rho h\right)}{\partial t}
\end{aligned} \tag{S14}$$