

Appendix

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APPENDIX A: BASIC CONCEPTS

This section is dedicated to introduce basic concepts used in the main text.

0.1 MHD waves in plasma

The expression of the MHD waves can be derived from the MHD equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \quad (1)$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \nabla p - \frac{1}{\mu_0} ((\nabla \times \mathbf{B}) \times \mathbf{B}) = 0 \quad (2)$$

$$-\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{V} \times \mathbf{B}) = 0 \quad (3)$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\rho^\Gamma} \right) = 0 \quad (4)$$

here, ρ is the plasma density, $\mathbf{V}$ the plasma velocity, p the plasma pressure, $\mathbf{B}$ the magnetic field vector, μ_0 the vacuum magnetic permeability and Γ the poly-tropic index. The linearization of these equations can be done by writing the time evolving variables in equilibrium and perturbation components ($A = A_0 + \tilde{A}$), neglecting terms multiplying perturbation components. In addition, the perturbation components are expressed assuming an evolution as $\tilde{A} = A e^{i(\vec{r} \cdot \vec{k} - \omega t)}$, with $\vec{k}$ the angular wave vector and ω the wave angular velocity (perturbation symbols is dropped):

$$\omega \mathbf{B} + \mathbf{k} \times (\mathbf{V} \times \mathbf{B}_0) = 0 \quad (5)$$

$$-\omega \rho_0 \mathbf{V} + \mathbf{k} p - \frac{(\mathbf{k} \times \mathbf{B}) \times \mathbf{B}_0}{\mu_0} = 0 \quad (6)$$

$$-\omega \rho + \rho_0 \mathbf{k} \cdot \mathbf{V} = 0 \quad (7)$$

$$-\omega \left(\frac{p}{\rho_0} - \frac{\Gamma p}{\rho_0} \right) = 0 \quad (8)$$

If $\omega \neq 0$, it follows that:

$$\rho = \rho_0 \frac{\mathbf{k} \cdot \mathbf{V}}{\omega} \quad (9)$$

$$p = \Gamma p_0 \frac{\mathbf{k} \cdot \mathbf{V}}{\omega} \quad (10)$$

$$\mathbf{B} = \frac{(\mathbf{k} \cdot \mathbf{V})\mathbf{B}_0 - (\mathbf{k} \cdot \mathbf{B}_0)\mathbf{V}}{\omega} \quad (11)$$

replacing these perturbed variables in the equation 6:

$$[\omega^2 - (\mathbf{k} \cdot \mathbf{B}_0)^2 / (\mu_0 \rho_0)] \mathbf{V} = \left[\frac{(\Gamma p_0) / \rho_0 + (B_0^2) / (\mu_0 \rho_0)}{k} - \frac{(\mathbf{k} \cdot \mathbf{B}_0)}{\mu_0 \rho_0} \mathbf{B}_0 \right] (\mathbf{k} \cdot \mathbf{V}) - \frac{(\mathbf{k} \cdot \mathbf{B}_0)(\mathbf{V} \cdot \mathbf{B}_0)}{\mu_0 \rho_0} \mathbf{k} \quad (12)$$

it is assumed, with out losing generality, that B_0 is aligned with the z-axis and the angular wave vector lies in the x-z plane. The angle θ is defined between B_0 and the angular wave vector. Thus, equation 12 reduces to an eigenvalue equation problem:

$$\begin{pmatrix} \omega^2 - k^2 V_A^2 - k^2 V_S^2 \sin^2 \theta & 0 & -k^2 V_S^2 \sin \theta \cos \theta \\ 0 & \omega^2 - k^2 V_A^2 \cos^2 \theta & 0 \\ -k^2 V_S^2 \sin \theta \cos \theta & 0 & \omega^2 - k^2 V_S^2 \cos^2 \theta \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (13)$$

here, V_A is the Alfvén velocity and V_S the sound speed, defined as:

$$V_A = \sqrt{\frac{B_0^2}{\mu_0 \rho_0}} \quad (14)$$

$$V_S = \sqrt{\frac{\Gamma p_0}{\rho_0}} \quad (15)$$

the wave dispersion relation is calculated setting the matrix determinant to zero:

$$(\omega^2 - k^2 V_A^2 \cos^2 \theta) [\omega^4 - \omega^2 k^2 (V_A^2 + V_S^2) + k^4 V_A^2 V_S^2 \cos^2 \theta] = 0 \quad (16)$$

For an infinite homogeneous plasma there are three different types of stable waves [7]:

- The shear Alfvén wave: $\omega = k V_A \cos \theta$. Motion perpendicular to the magnetic field and there is no perturbation of the plasma density / pressure.
- Slow magnetosonic wave: $\omega = k V_-$. Motion along to the magnetic field and there is a perturbation of the plasma density / pressure.
- Fast magnetosonic wave: $\omega = k V_+$. Motion perpendicular or parallel to the magnetic field and there is a perturbation of the plasma density / pressure.

0.2 Shear Alfvén waves in fusion devices

The magnetic field inside fusion devices shows a periodicity that can be expressed with respect to the safety factor, thus the wave angular velocity can be written as:

$$\omega = \frac{n - m q}{R} V_A \quad (17)$$

This dispersion relation defines the Alfvén continuum, that is to say, the Alfvén continuum indicates the frequency of the shear Alfvén waves vs. the plasma radius. Nevertheless, there is a periodic variation of the magnetic field intensity along the magnetic field line, leading to a modification of the Alfvén speed and shear Alfvén wave angular velocity. The consequence is the generation of gaps in the Alfvén continuum induced by the destructive interaction of counter propagating Alfvén waves. Several examples of Alfvén continuum gaps in tokamaks and stellarators are discussed in this article

0.3 Alfvén eigenmodes

The Alfvén Eigenmodes (AE) are instabilities triggered inside the Alfvén continuum gaps by periodic perturbations with the same or a fraction of the natural frequency of the plasma. The driver of the perturbation can be the energetic particles in fusion devices if the EP velocity is similar or a fraction of the MHD group velocity, leading to the following resonance condition:

$$\Omega = \omega - n\omega_{\zeta} - l\omega_{\theta} = 0 \quad (18)$$

with ω the AE frequency, n the AE toroidal number, l an arbitrary integer, ω_{θ} the EP orbit poloidal frequency and ω_{ζ} the EP orbit toroidal frequency. Here, ω_{θ} and ω_{ζ} are function of the invariants: energy, magnetic moment and toroidal angular momentum.

Different families of AEs can be triggered in fusion plasma linked to the toroidal and helical magnetic field periodicity, magnetic field shear, sound wave couplings, between others. Table 1 shows several examples.

Type	Name	Origin
RSAE	Reverse shear	Null magnetic shear
BAE	Beta	Compressibility
GAE	Global	Continuum minima
TAE	Toroidal	m and $m + 1$ wave coupling
EAE	Ellipticity	m and $m + 2$ wave coupling
NAE	Noncircularity	m and $m + 3$ (or higher) wave coupling
HAE	Helicity	n and $n +$ device period

Table 1. Alfvén eigenmode acronym (first column), name (second column) and origin (third column). Here, m is the poloidal AE mode number.

APPENDIX B: NUMERICAL MODEL

This appendix is dedicated to show the derivation of the gyro-fluid model included in the FAR3d code.

0.4 Reduced MHD equations

Using the full resistive MHD equations as starting point, the reduced resistive MHD equations are derived:

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mathbf{J} \times \mathbf{B} \quad (19)$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p = -\Gamma p \nabla \cdot \mathbf{v} \quad (20)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (21)$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J} \quad (22)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (23)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (24)$$

The FAR3d model uses the stellarator expansion to simplify the analysis of the stability problem. This leads to a reduced MHD model which effectively removes the fast compressional Alfvén wave dynamics. Following this approximation, the magnetic field can be expressed as an ordering of terms with the uniform toroidal field (B_ζ) as the dominant component, verifying the relation $B/B_\zeta \approx \epsilon \ll 1$, with B higher order terms of the magnetic field expansion [2, 3, 4]. Thus, the FAR3d model implicitly assumes equilibria with high aspect ratio. Another condition for the application of the stellarator expansion is the equilibria should have a thermal β value similar to the inverse aspect ratio, that is to say, the variables can be split into equilibrium + perturbation components with the perturbation smaller than the equilibrium component.

From Ohm's law, small plasma resistivity and slow field variation (slower than the Alfvén time) are assumed

$$\mathbf{E} \times \mathbf{B} + (\mathbf{v} \times \mathbf{B}) \times \mathbf{B} \simeq 0 \Rightarrow \mathbf{v}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \quad (25)$$

If we assume high aspect ratio and low thermal plasma β , that is to say $\beta \sim \epsilon \ll 1$, then:

$$\mathbf{v}_\perp = -\frac{\nabla \Phi \times B_\zeta \mathbf{e}^\zeta}{B^2} \quad (26)$$

with Φ the electrostatic potential. Equilibrium Boozer coordinates are used. The generalized radial coordinate ρ is defined such that the toroidal flux is $B_0 \rho^2/2$, where B_0 is the averaged magnetic field strength at the magnetic axis. The reciprocal base vectors are $(\nabla \rho, \rho \nabla \theta, R_0 \nabla \zeta)$, where R_0 is the major radius. The Jacobian of the transformation is:

$$\sqrt{g} = \frac{B_0}{R_0} \frac{J - \iota I}{B^2} \quad (27)$$

Thus, at the lowest order with $\tilde{B}_\zeta = 0$ and $B_{0,\zeta} = J/R_0$, the perpendicular components of the velocity field are:

$$\mathbf{v}_\perp^\rho = -\frac{1}{B_0} \frac{1}{\rho} \frac{\partial \Phi}{\partial \theta} \quad (28)$$

$$\mathbf{v}_\perp^\theta = \frac{1}{B_0} \frac{\partial \Phi}{\partial \rho} \quad (29)$$

The toroidal magnetic field perturbation is set to zero to eliminate the propagation of the fast magnetosonic waves across the magnetic field, thus the perturbed magnetic field is:

$$\tilde{\mathbf{B}}_\perp \simeq \nabla \zeta \times \nabla \tilde{\psi} \quad (30)$$

thus:

$$\tilde{\mathbf{B}}_\perp^\rho = -\frac{1}{\sqrt{g}} \frac{1}{R_0} \frac{1}{\rho} \frac{\partial \tilde{\psi}}{\partial \theta} \quad (31)$$

$$\tilde{\mathbf{B}}_\perp^\theta = \frac{1}{\sqrt{g}} \frac{1}{R_0} \frac{\partial \tilde{\psi}}{\partial \rho} \quad (32)$$

corresponding to the parallel potential vector:

$$A_{||} = -\frac{\psi}{R_0} \mathbf{e}^\zeta \quad (33)$$

Consequently, in the direction parallel to the magnetic field, the electric field is:

$$\mathbf{E} \cdot \mathbf{b} = E_{||} = -\mathbf{b} \cdot \frac{\partial A}{\partial t} - \mathbf{b} \cdot \nabla \Phi = \eta \mathbf{J} \cdot \mathbf{b} - \mathbf{V} \times \mathbf{B} \cdot \mathbf{b} = \eta \mathbf{J}_{||} \quad (34)$$

Considering the ordering of the different perturbed current components:

$$\tilde{J}^\rho \propto \tilde{J}^\theta \propto \epsilon \tilde{J}^\zeta \quad (35)$$

small terms are neglected, thus:

$$\mathbf{B} \cdot \mathbf{J} = B_\zeta J^\zeta = \frac{J}{R_0} \tilde{J}^\zeta \quad (36)$$

with the toroidal current perturbation:

$$\tilde{J}^\zeta = \frac{1}{R_0 \mu_0 \rho} \left[\frac{\partial}{\partial \rho} \left(-\frac{g_{\rho\theta}}{\sqrt{g}} \frac{\partial \psi}{\partial \theta} + \rho \frac{g_{\theta\theta}}{\sqrt{g}} \frac{\partial \psi}{\partial \rho} \right) - \frac{\partial}{\partial \theta} \left(-\frac{g_{\rho\rho}}{\sqrt{g}} \frac{1}{\rho} \frac{\partial \psi}{\partial \theta} + \frac{g_{\rho\theta}}{\sqrt{g}} \frac{\partial \psi}{\partial \rho} \right) \right] \quad (37)$$

and the expression of the contravariant equilibrium toroidal current given by:

$$\mathbf{J}_{eq} = \frac{1}{\mu_0 \sqrt{g}} \left(0, \frac{1}{R_0} \frac{dJ}{d\rho} + \frac{2\rho}{a^2} \frac{1}{R_0} \frac{\partial \beta_*}{\partial \zeta}, \frac{1}{\rho} \frac{dI}{d\rho} - \frac{2\rho}{a^2} \frac{1}{\rho} \frac{\partial \beta_*}{\partial \theta} \right) \quad (38)$$

with J the toroidal current and I the poloidal current functions. If the equation 34 is multiplied by $\mathbf{B} \cdot$ and the expression obtained in the equation 36, an equation for the evolution of the poloidal flux perturbation is obtained:

$$\frac{\partial \tilde{\psi}}{\partial t} = \frac{\partial \Phi}{\partial \zeta} - \frac{\partial \Phi}{\partial \theta} + \frac{1}{B_0} \left(-\frac{1}{\rho} \frac{\partial \tilde{\psi}}{\partial \theta} \frac{\partial \Phi}{\partial \rho} + \frac{\partial \tilde{\psi}}{\partial \rho} \frac{1}{\rho} \frac{\partial \Phi}{\partial \theta} \right) + \eta \sqrt{g} J J^\zeta \quad (39)$$

Next, in order to derive a vorticity evolution equation, the momentum conservation equation is multiplied by the operator $\nabla \times \sqrt{g}$, thus:

$$\frac{\partial}{\partial t} [\nabla \times (\rho_m \sqrt{g} \mathbf{v})] + \nabla \times [\rho_m \sqrt{g} (\mathbf{v} \cdot \nabla) \mathbf{v}] = -\nabla \sqrt{g} \times \nabla P + \nabla \times (\sqrt{g} \mathbf{J} \times \mathbf{B}) \quad (40)$$

Here, the $\mathbf{V} \cdot \nabla$ operator is:

$$\mathbf{v} \cdot \nabla = \mathbf{v}_\perp \cdot \nabla + \tilde{v}_{||th} \nabla_{||} = \frac{1}{B_0} \left(-\frac{1}{\rho} \frac{\partial \Phi}{\partial \theta} \frac{\partial}{\partial \rho} + \frac{\partial \Phi}{\partial \rho} \frac{1}{\rho} \frac{\partial}{\partial \theta} \right) + \tilde{v}_{||th} \nabla_{||} \quad (41)$$

The toroidal component of the vorticity is:

$$U = \sqrt{g} [\nabla \times (\sqrt{g} \mathbf{v})]^\zeta \quad (42)$$

$$= \frac{1}{B_0} \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left[\rho \sqrt{g} \left(-g_{\rho\theta} \frac{1}{\rho} \frac{\partial \Phi}{\partial \theta} + g_{\theta\theta} \frac{\partial \Phi}{\partial \rho} \right) \right] + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left[\sqrt{g} \left(g_{\rho\rho} \frac{1}{\rho} \frac{\partial \Phi}{\partial \theta} - g_{\rho\theta} \frac{\partial \Phi}{\partial \rho} \right) \right] \right) \quad (43)$$

thus, the evolution of the toroidal component of the vorticity perturbation is:

$$\frac{\rho_m}{\sqrt{g}} \frac{\partial \tilde{U}}{\partial t} = -[\nabla \times \rho_m \sqrt{g} (\mathbf{V} \cdot \nabla) \mathbf{V}]^\zeta - [\nabla \sqrt{g} \times \nabla \tilde{p}_{th}]^\zeta + [\nabla \times (\sqrt{g} \mathbf{J} \times \mathbf{B})]^\zeta \quad (44)$$

The pressure equation is:

$$\frac{\partial \tilde{p}_{th}}{\partial t} = -\mathbf{v} \cdot \nabla \mathbf{p}_{th} - \Gamma \mathbf{p}_{th} \left[\nabla \cdot \mathbf{v}_\perp + B \nabla_\parallel \left(\frac{\tilde{\mathbf{v}}_{\parallel, th}}{B} \right) \right] \quad (45)$$

The equation of the thermal plasma parallel velocity is:

$$\frac{\partial \tilde{v}_{\parallel, th}}{\partial t} = -\mathbf{v} \cdot \nabla \mathbf{v}_{\parallel, th} - \frac{1}{\sqrt{g}} \frac{1}{\rho_m \mathbf{R}_0} \left(\frac{\partial}{\partial \zeta} - \iota \frac{\partial}{\partial \theta} \right) \tilde{\mathbf{p}} - \frac{1}{\rho_m \sqrt{g}} \frac{d\mathbf{p}_{th,0}}{d\rho} \frac{1}{\rho} \frac{\partial \tilde{\psi}}{\partial \theta} \quad (46)$$

The normalized equations are obtained by normalizing the time to the Alfvén velocity $\tau_A = R_0 \sqrt{\mu_0 m_i n_i(0)} / B_0$ with R_0 the major radius, the radial variable to the minor radius a considering the relation $2|\psi'| = B_0 a^2$, the poloidal flux to $B_0 a^2$, β_* to $B_0 R_0 / 2$, J and I to $B_0 R_0$, Φ to $B_0 a^2 / \tau_{A0}$, U to τ_{A0}^{-1} and the pressure to the value at the magnetic axis. The inverse aspect ratio is $\epsilon = a / R_0$, the magnetic Reynolds number $S = \tau_R / \tau_A$ with $\tau_R = \mu_0 a^2 / \eta(0)$ the resistive time. Thus, the normalized equations of the poloidal flux and vorticity are (dropping Doppler and diffusion terms):

$$\frac{\partial \tilde{\psi}}{\partial t} = \sqrt{g} B \nabla_\parallel \Phi + \frac{\eta}{S} \sqrt{g} J \tilde{J}^\zeta \quad (47)$$

$$\frac{\partial \tilde{U}}{\partial t} = -\mathbf{v} \cdot \nabla \tilde{U} + -\frac{B \sqrt{g}}{\rho_m} \nabla_\parallel J^\zeta - \frac{\beta_0}{2\epsilon^2} \sqrt{g} (\nabla \sqrt{g} \times \nabla \tilde{p})^\zeta \quad (48)$$

Here, the operators are:

$$\sqrt{g} B \nabla_\parallel f = \frac{\partial \tilde{f}}{\partial \zeta} - \iota \frac{\partial \tilde{f}}{\partial \theta} - \frac{1}{B_0} \left(\frac{\partial f_0}{\partial \rho} \frac{1}{\rho} \frac{\partial \tilde{\psi}}{\partial \theta} + \frac{1}{\rho} \frac{\partial f_0}{\partial \theta} \frac{\partial \tilde{\psi}}{\partial \rho} \right) \quad (49)$$

$$(\nabla \sqrt{g} \times \nabla f)^\zeta = \frac{\partial \sqrt{g}}{\partial \rho} \frac{1}{\rho} \frac{\partial \tilde{f}}{\partial \theta} - \frac{1}{\rho} \frac{\partial \sqrt{g}}{\partial \theta} \frac{\partial \tilde{f}}{\partial \rho} \quad (50)$$

The normalized pressure equation is:

$$\frac{\partial \tilde{p}_{th}}{\partial t} = -\mathbf{v} \cdot \nabla p + \Gamma p \left[(\nabla \sqrt{g} \times \nabla \Phi)^\zeta - B \nabla_\parallel \left(\frac{\tilde{v}_{\parallel, th}}{B} \right) \right] \quad (51)$$

where Γ is the adiabatic index and the velocity divergence is defined as:

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{v}_\perp + B \nabla_\parallel \left(\frac{v_\parallel}{B} \right) \quad (52)$$

The normalized equation of the parallel velocity of the thermal plasma is:

$$\frac{\partial \tilde{v}_{\parallel, th}}{\partial t} = -\mathbf{v} \cdot \nabla \tilde{v}_{\parallel, th} - \frac{\beta_0}{2n_{0, th}} \nabla_\parallel p \quad (53)$$

The perpendicular gradient square operator is:

$$\nabla_{\perp}^2 = \frac{1}{\sqrt{g}} \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho g_{\theta\theta} \frac{\partial}{\partial \rho} - g_{\rho\theta} \frac{\partial}{\partial \theta} \right) + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left(g_{\rho\rho} \frac{1}{\rho} \frac{\partial}{\partial \theta} - g_{\rho\theta} \frac{\partial}{\partial \rho} \right) \right], \quad (54)$$

with $g_{\rho\rho}$, $g_{\rho\theta}$ and $g_{\theta\theta}$ the metric covariant elements of the coordinate system transformation.

0.5 Gyro-fluid energetic particle equations

This section is dedicated to the derivation of the gyro-fluid moment equations included in the FAR3d model (nonlinear terms not included here for simplicity). The approach used is to begin with a local gyrokinetic moments analysis [5], and then extend this to a set of global model equations that capture basic Landau resonance phenomena through use of closure relations. These closure relations involve a set of scalar coefficients that are typically derived by optimizing the resonant response function fit associated with this model to that obtained from an analytic kinetic calculation [1]. As will be indicated below, an important closure term in the FAR3d gyro-fluid model is associated with the $|\nabla_{||}|$ operator. The fact that FAR3d is based on a straight-field-line coordinate system using Fourier representations in the toroidal/poloidal angle coordinates greatly facilitates the evaluation of this operator, since its application reduces to multiplication by a one-dimensional function involving mode numbers and the q-profile; as a result the absolute value can trivially be applied. For other coordinate systems and representations $|\nabla_{||}|$ becomes a non-local operator and its evaluation becomes more complex. However, methods for this have been developed and implemented, for example in the BOUT++ model [6].

Following [5], the EP density moment is:

$$(\omega - \omega_d)n_f = k_{||}n_{f0}v_{f,||} - \frac{q_f n_{f0}}{T_f} [\omega_* - \omega_d + (\omega - \omega_d)(1 - \Gamma_0)] \Phi \quad (55)$$

and the EP parallel velocity moment is:

$$(\omega - \omega_d)v_{||,f} = k_{||} \frac{T_f}{m_f} \left(\frac{n_f}{n_{f0}} + \frac{q_f \Phi}{T_f} \right) - v_{th,f}^2 (\omega - \omega_*) \Gamma_0 \frac{q_f A_{||}}{T_f} + (2a_0 - 1)k_{||} \left(\frac{n_f}{n_{f0}} + \frac{q_f \Phi}{T_f} \right) + \sqrt{2}a_1 k_{||} \frac{v_{||,f}}{v_{th,f}} \quad (56)$$

The above equation would normally couple to the next order EP perturbed temperature equation. However, a closure relation has been incorporated here which relates the EP perturbed temperature to the lower order EP moments. The parameters $\gamma_I = \eta_f = 0$, $\Gamma_0 = 1$, $L_{1,2,3,4,5} = 1$ and $\nu_j = 0$ are assumed for simplicity. Next, the ansatz is made that the equations can be converted from frequency / local wavenumber space to the time / global flux coordinate framework by multiplying the equations by $-i$ and using the operator definitions $-i\omega \Rightarrow \frac{\partial}{\partial t}$, $ik_{||} \Rightarrow \nabla_{||}$, $i\omega_d \Rightarrow \Omega_d$ and $i\omega_* \Rightarrow \Omega_*$, the EP density moment equation is:

$$\frac{\partial n_f}{\partial t} = -\Omega_d(n_f) - n_{f0}\nabla_{||}v_{||,f} - n_{f0}\Omega_d \left(\frac{q_f \Phi}{T_f} \right) + n_{f0}\Omega_* \left(\frac{q_f \Phi}{T_f} \right) \quad (57)$$

EP parallel velocity moment:

$$\frac{\partial m_f n_f v_{||,f}}{\partial t} = - \left(\Omega_d + \sqrt{2}a_1 v_{th,f} |\nabla_{||}| \right) m_f n_f v_{||,f} - (2a_0 - 1)T_f \nabla_{||} n_f + q_f n_{f0} \Omega_* \left(\frac{\psi}{R_0} \right) \quad (58)$$

In vectorial form, the drift and diamagnetic frequencies are represented by the operators:

$$\Omega_d = \frac{v_{th,f}^2}{\Omega_c} \frac{\vec{B}_0 \times \nabla \vec{B}_0}{B_0^2} \cdot \vec{\nabla} \quad (59)$$

$$\Omega_* = -\frac{T_f}{q_f B_0 n_{f0}} \vec{\nabla} n_{f0} \cdot \frac{\vec{B}_0}{B_0} \times \vec{\nabla} \quad (60)$$

using the contravariant form of $\vec{B}_0$:

$$\Omega_d = \frac{T_f}{2\rho R_0 q_f \sqrt{g} B_0^4} \left[\left(I \frac{\partial B_0^2}{\partial \zeta} + J \frac{\partial B_0^2}{\partial \theta} \right) \frac{\partial}{\partial \rho} - \left(\frac{2\rho\beta_*}{a^2} \frac{\partial B_0^2}{\partial \zeta} + J \frac{\partial B_0^2}{\partial \rho} \right) \frac{\partial}{\partial \theta} + \left(\frac{2\rho\beta_*}{a^2} \frac{\partial B_0^2}{\partial \theta} - I \frac{\partial B_0^2}{\partial \rho} \right) \frac{\partial}{\partial \zeta} \right] \quad (61)$$

$$\Omega_* = -\frac{T_f}{q_f B_0^2 n_{f0}} \frac{1}{\rho R_0 \sqrt{g}} \frac{dn_{f0}}{d\rho} \left(I \frac{\partial}{\partial \zeta} + J \frac{\partial}{\partial \theta} \right) \quad (62)$$

The normalized EP moment equations are then:

$$\frac{\partial n_f}{\partial t} = -\frac{v_{th,f}^2}{\epsilon \Omega_{cf}} \Omega_d(n_f) - n_{f0} \nabla_{||} v_{||,f} - n_{f0} \Omega_d(\Phi) + n_{f0} \Omega_*(\Phi) \quad (63)$$

$$\frac{\partial v_{||,f}}{\partial t} = -\frac{v_{th,f}^2}{\epsilon \Omega_{cf}} \Omega_d(v_{||,f}) - \sqrt{2} a_1 v_{th,f} |\nabla_{||} v_{||,f}| - 2a_0 \frac{v_{th,f}^2}{n_{f0}} \nabla_{||} n_f + v_{th,f}^2 \Omega_*(\psi) \quad (64)$$

The EP and thermal plasma equations are coupled through the last two terms on the right hand side of the vorticity equation:

$$\frac{\partial \tilde{U}}{\partial t} = -\mathbf{v} \cdot \nabla \tilde{U} + -\frac{B\sqrt{g}}{\rho_m} \nabla_{||} J\zeta - \frac{\beta_0}{2\epsilon^2} \sqrt{g} (\nabla \sqrt{g} \times \nabla p)\zeta - \frac{\beta_f}{2\epsilon^2} \sqrt{g} (\nabla \sqrt{g} \times \nabla \tilde{n}_f)\zeta \quad (65)$$

The normalized drift and diamagnetic operators can then be written as, using the relation:

$$B^2 = \frac{J - \iota I}{\sqrt{g}} \quad (66)$$

$$\Omega_d = \frac{1}{\sqrt{g} B_0^4} \left[\left(\frac{I}{\rho} \frac{\partial B_0^2}{\partial \zeta} + J \frac{1}{\rho} \frac{\partial B_0^2}{\partial \theta} \right) \frac{\partial}{\partial \rho} - \left(\rho \beta_* \frac{\partial B_0^2}{\partial \zeta} + J \frac{\partial B_0^2}{\partial \rho} \right) \frac{1}{\rho} \frac{\partial}{\partial \theta} + \left(\rho \beta_* \frac{1}{\rho} \frac{\partial B_0^2}{\partial \theta} - \frac{I}{\rho} \frac{\partial B_0^2}{\partial \rho} \right) \frac{\partial}{\partial \zeta} \right] \quad (67)$$

$$\Omega_* = -\frac{1}{\sqrt{g} B_0^2 n_{f0}} \frac{dn_{f0}}{d\rho} \left(\frac{I}{\rho} \frac{\partial}{\partial \zeta} + J \frac{1}{\rho} \frac{\partial}{\partial \theta} \right) \quad (68)$$

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