APPENDIX A

To continue the discussion in Remark 2, consider, for example, the condition ii), v) and vi) are relaxed, that is, all of the sensors are active, the positive real condition $P_iB = C_i^{\mathrm{T}}J_i^{\mathrm{T}}$ is satisfied with full column rank B, and the inputs are constant, then we can drop some of the leakage terms like "$-\gamma P_i^{-1}\hat{x}_i(t)$" and "$-(\sigma_i K_i + \gamma \mathrm{I}_p)\hat{w}_i(t)$" in our algorithm (7) and (8) to make it become

$$\dot{\hat{x}}_i(t) = A\hat{x}_i(t) + B\hat{w}_i(t) + L_i\big(y_i(t) - C_i\hat{x}_i(t)\big) - \alpha P_i^{-1}\sum_{i\sim j}(\hat{x}_i(t) - \hat{x}_j(t)),\ \hat{x}_i(0) = \hat{x}_{i0}, \tag{A.1}$$

$$\dot{\hat{w}}_i(t) = J_i(y_i(t) - C_i\hat{x}_i(t)) - \alpha\sum_{i\sim j}(\hat{w}_i(t) - \hat{w}_j(t)),\ \ \hat{w}_i(0) = \hat{w}_{i0}. \tag{A.2}$$

Going through the same procedure in the subsection 3.2, we obtain the compact form of the error dynamics as

$$\dot{\tilde{x}}(t) = \bar{A}\tilde{x}(t) - (\mathrm{I}_N \otimes B)\tilde{w}(t) - \alpha P^{-1}(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)\tilde{x}(t), \tag{A.3}$$

$$\dot{\tilde{w}}(t) = M\tilde{x}(t) - \alpha(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_p)\tilde{w}(t), \tag{A.4}$$

where $\bar{A}$, M and P are the same matrices as defined in (19), (20) and (23), with $g_i = 1$ for all nodes $i = 1, \ldots, N$. Note that the positive real condition $P_iB = C_i^{\mathrm{T}}J_i^{\mathrm{T}}$ where $P_i > 0$ can be written in the compact form as $P(\mathrm{I}_N \otimes B) = M^{\mathrm{T}}$. In addition, since the leakage terms are dropped, the condition (9) is now replaced with the condition $\bar{A}_i^{\mathrm{T}}P_i + P_i\bar{A}_i < 0$ with $P_i > 0$, which can be satisfied easily with appropriate choice of L_i since we assume the matrix A is Hurwitz. In this case, the matrix A is not necessary to be Hurwitz if we assume (A, C_i) is observable, and hence, we can always find L_i such that $\bar{A}$ is Hurwitz.

For the stability analysis of (A.3) and (A.4), we can choose the same Lyapunov function candidate (24), then taking the time derivative of $V(\tilde{x}, \tilde{w})$ along the trajectories of (A.3) and (A.4) yields

$$\begin{aligned}\dot{V}(\cdot) &= \tilde{x}^{\mathrm{T}}(t)(\bar{A}^{\mathrm{T}}P + P\bar{A})\tilde{x}(t) - 2\tilde{x}^{\mathrm{T}}(t)P(\mathrm{I}_N \otimes B)\tilde{w}(t) - 2\alpha\tilde{x}^{\mathrm{T}}(t)(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)\tilde{x}(t) \\ &\quad + 2\tilde{w}^{\mathrm{T}}(t)M\tilde{x}(t) - 2\alpha\tilde{w}^{\mathrm{T}}(t)(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_p)\tilde{w}(t) \\ &= -\tilde{x}^{\mathrm{T}}(t)Q\tilde{x}(t) - 2\alpha\tilde{x}^{\mathrm{T}}(t)(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n)\tilde{x}(t) - 2\alpha\tilde{w}^{\mathrm{T}}(t)(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_p)\tilde{w}(t) \leq 0,\end{aligned} \tag{A.5}$$

where $\bar{A}^{\mathrm{T}}P + P\bar{A} \triangleq -Q < 0$ and the last equation is obtained directly from using the condition $\bar{A}_i^{\mathrm{T}}P_i + P_i\bar{A}_i < 0$, and the positive real condition $P(\mathrm{I}_N \otimes B) = M^{\mathrm{T}}$. Note that (A.5) shows that the error dynamics given by (A.3) and (A.4) are Lyapunov stable for all initial conditions. Let $z(t) \triangleq [\tilde{x}^{\mathrm{T}}(t), \tilde{w}^{\mathrm{T}}(t)]^{\mathrm{T}}$ and $\mathcal{S} = \{z(t) \in \mathbb{R}^{N(n+p)} \mid \dot{V}(z(t)) = 0\}$. When $\dot{V}(z(t)) = 0$, we have $\tilde{x}(t) = 0$ since the matrix $Q + (\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n) > 0$. Thus, $\mathcal{S} = \{z(t) \in \mathbb{R}^{N(n+p)} \mid \tilde{x}(t) = 0\}$. Let $\tilde{x}(t)$ be a solution that belongs identically to $\mathcal{S}$, then $\tilde{x}(t) \equiv 0$ means $\dot{\tilde{x}}(t) \equiv 0$, and hence $(\mathrm{I}_N \otimes B)\tilde{w}(t) \equiv 0$ from (A.3). Since B is full column rank, then $(\mathrm{I}_N \otimes B)\tilde{w}(t) \equiv 0$ implies that $\tilde{w}(t) \equiv 0$. Therefore, the only solution that can stay identically in $\mathcal{S}$ is $z(t) \equiv 0$. By Theorem 3.5 in Khalil (2014), the origin is asymptotically stable. Since the system given by (A.3) and (A.4) is linear time-invariant, the matrix $\begin{bmatrix} \bar{A} - \alpha P^{-1}(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_n) & -(\mathrm{I}_N \otimes B) \\ M & -\alpha(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_p) \end{bmatrix}$ is Hurwitz. While the above result is immediate based on the strict assumptions only considered in this appendix to show asymptotic stability, it is still not identical to the other results cited in Section 1 of this paper as well as the results presented in Viegas et al. (2015); Hu et al. (2013); Sadikhov et al. (2014).

APPENDIX B

In this appendix, we provide the parameters that we use for our illustrative examples in sections 3.3 and 4.3 with 15 decimal places in case the reader want to regenerate our simulation results.

Section 3.3, Example 1: For the observer gain L_i, the odd index nodes are subjected to

$$L_i = \begin{bmatrix} 18.969160655470404 & -1.907268388916050 \\ -0.487391998697633 & -0.075498787949882 \\ -1.939393950211166 & 19.129788461945985 \\ -0.284730910528483 & 2.491934726180443 \end{bmatrix}, \tag{A.6}$$

and the even index nodes are subject to

$$L_i = \begin{bmatrix} -2.387919504735957 & 0.357718861908060 \\ 5.830659255697644 & -0.803813918212959 \\ 0.428177912573512 & -2.397655697276556 \\ -1.037509640179160 & 6.765442143562447 \end{bmatrix}. \tag{A.7}$$

In addition,

$$\sigma_1 = \sigma_5 = 0.002061806076927, \tag{A.8}$$

$$\sigma_2 = \sigma_6 = 1.833655794790509 \times 10^{-6}, \tag{A.9}$$

$$\sigma_3 = \sigma_4 = \sigma_7 = \sigma_8 = \sigma_9 = \sigma_{10} = \sigma_{11} = \sigma_{12} = 0.002400000072044, \tag{A.10}$$

and $P_1 = P_5$, $P_2 = P_6$ and $P_3 = P_4 = P_7 = P_8 = P_9 = P_{10} = P_{11} = P_{12}$, where

$$P_1 = 10^3 \times \begin{bmatrix} 1.439835424802165 & -0.034377714548575 & 0.055117661315125 & -0.005443202673014 \\ -0.034377714548575 & 0.004148603989598 & 0.003907233598367 & 0.000148021815711 \\ 0.055117661315125 & 0.003907233598367 & 0.977251053041284 & -0.054353877874992 \\ -0.005443202673014 & 0.000148021815711 & -0.054353877874992 & 0.026092188711795 \end{bmatrix}, \tag{A.11}$$

$$P_2 = 10^2 \times \begin{bmatrix} 0.299361647347857 & 0.000002820408509 & 0.035830529875908 & 0.000000221819347 \\ 0.000002820408509 & 0.696303881680737 & 0.000001019351069 & -0.001153520717465 \\ 0.035830529875908 & 0.000001019351069 & 0.298140534700503 & 0.000013277207542 \\ 0.000000221819347 & -0.001153520717465 & 0.000013277207542 & 3.999240098499779 \end{bmatrix}, \tag{A.12}$$

$$P_{12} = \begin{bmatrix} 1.440000008960819 & 0.000000001949013 & 0 & 0 \\ 0.000000001949013 & 0.999999999913420 & 0 & 0 \\ 0 & 0 & 1.341667967078101 & 0.529441205638835 \\ 0 & 0 & 0.529441205638835 & 3.496288127071053 \end{bmatrix}. \tag{A.13}$$

Section 3.3, Example 2:

$$\sigma_1 = \sigma_3 = \sigma_5 = \sigma_7 = 0.002061806076927, \tag{A.14}$$

$$\sigma_2 = \sigma_4 = \sigma_6 = \sigma_8 = 1.833655794790509 \times 10^{-6}, \tag{A.15}$$

$$\sigma_9 = \sigma_{10} = \sigma_{11} = \sigma_{12} = 0.002400000072044. \tag{A.16}$$

Section 3.3, Example 3: The observer gain L_i is chosen such that

$$L_1 = \begin{bmatrix} 19.068611174404705 & -3.813722234880941 \\ -0.485700606350880 & 0.097140121270176 \\ -0.000000000000000 & 0.000000000000000 \\ 0.000000000000000 & -0.000000000000000 \end{bmatrix}, \tag{A.17}$$

$$L_2 = \begin{bmatrix} -2.387919504735957 & 0.357718861908060 \\ 5.830659255697644 & -0.803813918212959 \\ 0.428177912573512 & -2.397655697276556 \\ -1.037509640179160 & 6.76544214356244 \end{bmatrix}, \tag{A.18}$$

$$L_3 = \begin{bmatrix} .000000000000002 & -0.000000000000012 \\ 0.000000000000002 & -0.000000000000009 \\ -3.844972067593563 & 19.224860337967815 \\ -0.501192168103495 & 2.505960840517474 \end{bmatrix}, \tag{A.19}$$

with $L_1 = L_5 = L_9$, $L_2 = L_4 = L_6 = L_8 = L_{10} = L_{12}$ and $L_3 = L_7 = L_{11}$. In addition,

$$\sigma_1 = \sigma_5 = 0.001982811972340, \tag{A.20}$$

$$\sigma_2 = \sigma_4 = \sigma_6 = \sigma_8 = 1.833655794790509 \times 10^{-6}, \tag{A.21}$$

$$\sigma_3 = \sigma_7 = 0.002400002327086, \tag{A.22}$$

$$\sigma_9 = \sigma_{10} = \sigma_{11} = \sigma_{12} = 0.002400000072044, \tag{A.23}$$

and

$$P_1 = 10^3 \times \begin{bmatrix} 1.478035081597691 & -0.034321006113263 & 0 & 0 \\ -0.034321006113263 & 0.004059666480104 & 0 & 0 \\ 0 & 0 & 0.001491086838912 & 0.000317849850723 \\ 0 & 0 & 0.000317849850723 & 0.005841716956271 \end{bmatrix}, \tag{A.24}$$

$$P_3 = 10^2 \times \begin{bmatrix} 0.014399993247139 & 0.000000000383925 & 0 & 0 \\ 0.000000000383925 & 0.010000000455351 & 0 & 0 \\ 0 & 0 & 6.978268681664298 & -0.230181821168044 \\ 0 & 0 & -0.230181821168044 & 0.188324170653730 \end{bmatrix}, \tag{A.25}$$

with $P_1 = P_5$, $P_2 = P_4 = P_6 = P_8$, $P_3 = P_7$, and $P_9 = P_{10} = P_{11} = P_{12}$, where P_2 and P_{12} are the same as (A.12) and (A.13), respectively.

Section 4.3, Example 4: For the observer gain L_i, the odd index nodes are sudject to

$$L_i = \begin{bmatrix} 71.783953996258958 & -7.252963875491536 \\ -1.358611885022830 & 0.000475040815981 \\ -7.256525958548080 & 71.801764411541683 \\ -0.011569149768944 & -0.128688835617882 \end{bmatrix}, \tag{A.26}$$

while the even index nodes are subject to

$$L_i = \begin{bmatrix} -21.724842048641705 & 2.256236946732968 \\ 70.098170640403481 & -7.134526961104474 \\ 2.326854788384256 & -21.731989850585336 \\ -7.373583349180001 & 71.293452580781121 \end{bmatrix}. \tag{A.27}$$

In addition,

$$\sigma_1 = \sigma_3 = \sigma_5 = \sigma_7 = \sigma_9 = \sigma_{11} = 0.04342963222631, \tag{A.28}$$

$$\sigma_2 = \sigma_4 = \sigma_6 = \sigma_8 = \sigma_{10} = \sigma_{12} = 0.027557828266522, \tag{A.29}$$

and

$$P_1 = \begin{bmatrix} 15.365812598325633 & 0.288438698583006 & 2.891134645171142 & -0.519291422390345 \\ 0.288438698583006 & 8.058139150438166 & 0.489805786441967 & 1.796242594832787 \\ 2.891134645171142 & 0.489805786441967 & 20.900129728630198 & 0.981256246505427 \\ -0.519291422390345 & 1.796242594832787 & 0.981256246505427 & 76.693334412288678 \end{bmatrix}, \tag{A.30}$$

$$P_2 = \begin{bmatrix} 10.026109249319704 & 2.195377067059541 & 0.281414382843850 & -1.215486262312381 \\ 2.195377067059541 & 10.170435495903876 & 0.066429931103742 & 1.090421131679111 \\ 0.281414382843850 & 0.066429931103742 & 9.472902963103000 & 2.834028735894950 \\ -1.215486262312381 & 1.090421131679111 & 2.834028735894950 & 25.585410287198304 \end{bmatrix}, \tag{A.31}$$

with $P_1 = P_3 = P_5 = P_7 = P_9 = P_{11}$ and $P_2 = P_4 = P_6 = P_8 = P_{10} = P_{12}$.

Section 4.3, Example 5: The observer gain L_i is chosen such that

$$L_{1a} = \begin{bmatrix} 3.43 & -2.06 \\ 1.96 & -1.78 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \end{bmatrix}, \tag{A.32}$$

$$L_{1b} = \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 4.90 \\ 0.00 & 2.03 \end{bmatrix}, \tag{A.33}$$

$$L_{2b} = \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.31 & -6.14 \\ -0.03 & 1.25 \end{bmatrix}, \tag{A.34}$$

$$L_{3a} = \begin{bmatrix} -16.7 & 0.00 \\ 5.10 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \end{bmatrix}, \tag{A.35}$$

$$L_{5} = \begin{bmatrix} 71.78 & -7.25 \\ -1.36 & 0.00 \\ -7.26 & 71.80 \\ -0.01 & -0.13 \end{bmatrix}, \tag{A.36}$$

$$L_{6} = \begin{bmatrix} -21.72 & 2.26 \\ 70.10 & -7.13 \\ 2.33 & -21.73 \\ -7.37 & 71.29 \end{bmatrix}, \tag{A.37}$$

with $L_{1a} = L_{2a}$, $L_{1b} = L_{3b}$, $L_{2b} = L_{4b}$, $L_{3a} = L_{4a}$, $L_5 = L_7 = L_9$ and $L_6 = L_8$. In addition,

$$\sigma_{1a} = \sigma_{2a} = 0.176910352176191, \tag{A.38}$$

$$\sigma_{1b} = \sigma_{3b} = 1.899948288575375, \tag{A.39}$$

$$\sigma_{2b} = \sigma_{4b} = 0.088022361660829, \tag{A.40}$$

$$\sigma_{3a} = \sigma_{4a} = 0.622664424352870, \tag{A.41}$$

$$\sigma_{5} = \sigma_{7} = \sigma_{9} = 0.042867890125997, \tag{A.42}$$

$$\sigma_{6} = \sigma_{8} = 0.027071950457169, \tag{A.43}$$

and

$$P_{1a} = \begin{bmatrix} 15.694244208757699 & 2.033323268063762 & 0.000000000000000 & 0.000000000000000 \\ 2.033323268063762 & 4.361865209185537 & 0.000000000000000 & 0.000000000000000 \\ 0.000000000000000 & 0.000000000000000 & 11.186803471688481 & 8.425791501883902 \\ 0.000000000000000 & 0.000000000000000 & 8.425791501883902 & 28.234219806829103 \end{bmatrix}, \tag{A.44}$$

$$P_{1b} = \begin{bmatrix} 58.803434580028032 & 14.695626639657313 & 0.000000000000000 & 0.000000000000000 \\ 14.695626639657313 & 23.324653578136509 & 0.000000000000000 & 0.000000000000000 \\ 0.000000000000000 & 0.000000000000000 & 18.374966669627405 & 2.314132184510947 \\ 0.000000000000000 & 0.000000000000000 & 2.314132184510947 & 40.176291696525283 \end{bmatrix}, \tag{A.45}$$

$$P_{2b} = \begin{bmatrix} 5.835227092365172 & 1.443618793797672 & 0.000000000000000 & 0.000000000000000 \\ 1.443618793797672 & 3.236785013594619 & 0.000000000000000 & 0.000000000000000 \\ 0.000000000000000 & 0.000000000000000 & 4.712498556711108 & 6.096255186288076 \\ 0.000000000000000 & 0.000000000000000 & 6.096255186288076 & 41.684901733851333 \end{bmatrix}, \tag{A.46}$$

$$P_{3a} = \begin{bmatrix} 2.599136817168446 & 2.884026096694552 & 0.000000000000000 & 0.000000000000000 \\ 2.884026096694552 & 8.493188312361060 & 0.000000000000000 & 0.000000000000000 \\ 0.000000000000000 & 0.000000000000000 & 20.899094105514450 & 16.216490598122018 \\ 0.000000000000000 & 0.000000000000000 & 16.216490598122018 & 47.082566058448556 \end{bmatrix}, \tag{A.47}$$

$$P_{5} = \begin{bmatrix} 3.854149037906603 & 0.067878511647866 & 0.758175986616365 & -0.095402213615805 \\ 0.067878511647866 & 2.013031029413099 & 0.108516797222526 & 0.501148644494243 \\ 0.758175986616365 & 0.108516797222526 & 4.625387782296224 & 0.210098567192018 \\ -0.095402213615805 & 0.501148644494243 & 0.210098567192018 & 17.124367695246701 \end{bmatrix}, \tag{A.48}$$

$$P_6 = \begin{bmatrix} 2.468882758815266 & 0.534438597958104 & 0.047207215780697 & -0.289667612641583 \\ 0.534438597958104 & 2.527205119130702 & 0.006894277635073 & 0.244394471559582 \\ 0.047207215780697 & 0.006894277635073 & 2.258962041615192 & 0.668789059887803 \\ -0.289667612641583 & 0.244394471559582 & 0.668789059887803 & 5.943357864360471 \end{bmatrix}, \tag{A.49}$$

with $P_{1a} = P_{2a}$, $P_{1b} = P_{3b}$, $P_{2b} = P_{4b}$, $P_{3a} = P_{4a}$, $P_5 = P_7 = P_9$ and $P_6 = P_8$.

APPENDIX C

In this appendix, we summarize the effect of the design parameters $\alpha, \gamma, \sigma_i, J_i, K_i, L_i$ and P_i discussed in Remarks 4 and 5. The following main points recap the procedure of selecting design parameters:

- The observer gain matrix L_i should be judiciously chosen such that active agents can closely estimate a target of interest when its input is time-invariant. A good performance of active agents will improve the overall performance of the networked system.
- As discussed in Remark 4, parameters such as γ, σ_i and K_i should be chosen small to limit the effect of leakage terms. However, since σ_i and K_i contribute an important role in the feasibility of the linear matrix inequality condition, one should tune σ_i, K_i and J_i such that the linear matrix inequality condition is satisfied and the norm $\|\sigma_i K_i\|_2$ is small simultaneously.
- A large value can be chosen for α as discussed in Remark 5. Note that, a large value for α not only helps reduce the ultimate bound but also helps increase the convergence rate.
- Once σ_i, K_i, L_i and J_i are chosen, P_i is obtained from solving the linear matrix inequality given by (9).

Note that similar steps can be followed also for choosing the same design parameters used in Section 4.

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