Dynamic Statistical Models for Pyroclastic Density Current Generation at Soufrière Hills Volcano

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Appendix: Computations & Algorithms

The **data** we observe include three quantities for each of the N pyroclastic density currents during the period of time $[0, T]$:

$\{V_j\}$	Flow **Volumes** (m^3),		$1 \le j \le N$;
$\{\phi_j\}$	Initial **Directions** (deg),		$1 \le j \le N$;
$\{\tau_j\}$	Flow **Times** (yr),		$1 \le j \le N$.

Volumes for some PDCs are interval censored, so the data set includes only an interval $[V_j^{\mathsf{min}}, V_j^{\mathsf{max}}]$ containing V_j. Directions for all PDCs are known only up to their drainage $\Phi_j \subset S^1$, with $\phi_j \in \Phi_j$.

Hyperparameters

The hyperparameters we specify include:

ϵ	Minimum flow for model inclusion	$1.5 \cdot 10^5\,\mathrm{m}^3$
Ω	Minimum flow for new dome direction	$6.0 \cdot 10^6\,\mathrm{m}^3$
a_{lo}	Shape parameter for $\{\lambda_{\mathsf{lo}}\}$ dist'n	1.8
a_{hi}	Shape parameter for $\{\lambda_{\mathsf{hi}}\}$ dist'n	9.1
b	Rate parameter for $\{\lambda_{\mathsf{lo}}, \lambda_{\mathsf{hi}}\}$ dist'n	$(0.5/365)\,\mathrm{yr}$
r	Repulsion parameter for $\{\lambda_{\mathsf{lo}}, \lambda_{\mathsf{hi}}\}$ dist'n	2.0
α_{lo}	Shape parameter for $\{(s_m - t_{m-1})\}$ dist'n	1.7
α_{hi}	Shape parameter for $\{(t_m - s_m)\}$ dist'n	1.4
β	Rate parameter for $\{\vec{s}, \vec{t}\}$ dist'n	$0.57\,\mathrm{yr}^{-1}$
κ_μ	Concentration parameter for new $\{\mu_e\}$	0.67
κ_ϕ	Concentration parameter for new $\{\phi_j\}$	1.00
T	End of data time period $[0, T)$	$10.0\,\mathrm{yr}$
T'	End of forecast time period $[T, T')$	$12.6\,\mathrm{yr}$
M	Number of high/lo periods	6

Note M must be large enough to ensure $t_M > T'$ with high probability; select $M \gg \frac{\beta T'}{\alpha_{\mathsf{lo}} + \alpha_{\mathsf{hi}}}$ to ensure this.

Parameters

The parameters sampled within an MCMC loop include:

α	–	Pareto shape parameter for $\{V_j\}$ distribution
$\{\mu_e\}$	deg	Central directions during $(T_e, T_{e+1}]$
$\{\lambda_{\mathsf{lo}}, \lambda_{\mathsf{hi}}\}$	yr^{-1}	Event rates (or their logistics $\{\eta_1 := \log(\lambda_{\mathsf{lo}} + \lambda_{\mathsf{hi}}),\ \eta_2 := \frac{1}{2}\log(\lambda_{\mathsf{hi}}/\lambda_{\mathsf{lo}})\}$)
$\{s_m, t_m\}$	yr	Starts, ends of high-activity episodes.

Note that each new draw of $\{s_m, t_m\}$ will change the values of $\{N_{\mathsf{lo}}, N_{\mathsf{hi}}\}$ and $\{T_{\mathsf{lo}}, T_{\mathsf{hi}}\}$ (see Eqns (2, 3)), and hence the likelihood function.

Mathematical Spaces

Standard notation for some mathematical spaces used in this work include:

$\mathbb{R}$	$(-\infty, \infty)$	Real numbers
$\mathbb{R}_+$	$(0, \infty)$	Positive real numbers
$\mathbb{N}$	$\{1, 2, \dots\}$	Natural numbers
S^1	$(-180°, 180°]$	Unit circle (here in degrees counter-clockwise from East)

Data-dependent Derived Quantities

J_0	Indices of PDCs with uncensored volumes V_j
J_1	Indices of PDCs with interval censored volumes $V_j \in [V_j^{\mathsf{min}}, V_j^{\mathsf{max}}]$
J	Indices of all PDCs ($J_0 \cup J_1$)
T_{hi}	Total time in study period at high PDC rate λ_{hi}
T_{lo}	Total time in study period at low PDC rate λ_{lo} ($= T - T_{\mathsf{hi}}$)
N_{hi}	Total number of PDCs observed at high PDC rate λ_{hi}
N_{lo}	Total number of PDCs observed at low PDC rate λ_{lo} ($= N - N_{\mathsf{hi}}$)

Probability Distributions

Probability distributions used in this analysis include:

Poisson	$\mathsf{Po}(\lambda)$	$f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$ Mean $= \lambda$, Variance $= \lambda$	$x = 0, 1, 2, \ldots$
Gamma	$\mathsf{Ga}(\alpha, \beta)$	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\ e^{-\beta x}$ Mean $= \alpha/\beta$, Variance $= \alpha/\beta^2$	$0 \le x < \infty$
Normal	$\mathsf{No}(\mu, \sigma^2)$	$f(x) = (2\pi\sigma^2)^{-1/2}\ e^{-(x-\mu)^2/2\sigma^2}$ Mean $= \mu$, Variance $= \sigma^2$	$-\infty < x < \infty$
Pareto	$\mathsf{Pa}(\alpha, \epsilon)$	$f(x) = \alpha\epsilon^\alpha\ x^{-\alpha-1}$ $\text{Mean} = \begin{cases} \epsilon/(\alpha-1) & \alpha > 1 \\ \infty & \alpha \le 1 \end{cases}$	$\epsilon \le x < \infty$
Uniform	$\mathsf{Un}(a, b)$	$f(x) = \frac{1}{b-a} \mathbf{1}_{[a,b]}(x)$ Mean $= \frac{a+b}{2}$, Variance $= \frac{(b-a)^2}{12}$	$a \le x \le b$
von Mises	$\mathsf{vM}(\mu, \kappa)$	$f(x) = \big(360 I_0(\kappa)\big)^{-1}\ e^{\kappa \cos(x-\mu)}$	$-180° < x \le 180°$

Likelihood Function

This model can be described either as a marked inhomogeneous Poisson process with event times τ_j and marks (V_j, ϕ_j), or as an inhomogeneous Poisson random field with observed points $\{(V_j, \phi_j, \tau_j) :\ 1 \le j \le N\}$ on the three dimensional space $[\epsilon, \infty) \times S^1 \times [0, T]$. From either perspective the likelihood function is given by:

$$L = \Big\{ \prod_{j \in J_0 \cup J_1} (V_j/\epsilon) \Big\}^{-\alpha} (\alpha/\epsilon)^{|J_0|} \prod_{j \in J_1} \big[1 - (V_j^{\mathsf{min}}/V_j^{\mathsf{max}})^\alpha\big]$$
$$\times \prod_{j=1}^{N} \Big\{ \int_{\Phi_j} f_{\mathrm{vM}}(\phi_j \mid \mu_{e_j}, \kappa_\phi)\, d\phi_j\ \lambda(\tau_j) \Big\} \exp\Big(- \int_0^T \lambda(t)\, dt \Big) \tag{11}$$

This expression includes two specified hyperparameters ($\epsilon > 0$ and $\kappa_\phi > 0$), and two features that need more discussion: the the epoch-specific central directions μ_{e_j} and time-varying rate $\lambda(t)$, each a piecewise-constant latent dynamic process.

Central directions $\{\mu_e\}$

The probability distribution for PDC initial directions $\{\phi_j\} \sim \mathsf{vM}(\mu_e, \kappa_\phi)$ is constant (in this model) during "epochs" $\tau_j \in (T_e, T_{e+1}]$ between successive PDCs that are sufficiently large (say, that exceed a specified threshold volume $V > \Omega$) to collapse the volcano dome. We describe such PDCs as "major". Such a dome collapse will lead to new dome morphology and so to a new central direction μ_e for subsequent flows. In (11) "e_j" denotes the index e for the epoch $(T_e, T_{e+1}]$ that contains the time τ_j of the jth PDC, so μ_{e_j} is the central flow direction at the time of that PDC. Thus we set $T_0 = 0$ and, for $e \ge 1$,

$$T_e := \min\{\tau_j > T_{e-1} : \ V_j > \Omega\} \qquad e_j := \max\{e : \ T_e < \tau_j\}$$

The $\{T_e\}$ in $(0, T]$ are observed in the dataset, but those in our forecast simulation of events in the period $(T, T']$ subsequent to T (beyond our data) will depend on the random sample $\{(V_j, \phi_j, \tau_j) \mid \tau_j > T\}$ for which $V_j > \Omega$. These must be recomputed each time we resample the forecast future PDCs.

Rate function $\lambda(\tau)$

We model the rate of PDCs of volume $V \ge \epsilon$ as a function $\lambda(\tau)$ of time τ that takes just two values: a low one λ_{lo} and a high one λ_{hi}, with transitions from low to high at uncertain times $\{s_m\}$ and subsequently from high to low at times $\{t_m\}$. Thus with $0 = t_0 < s_1 < t_1 < s_2 < \cdots < t_M$ with $M \gg T\beta/(\alpha_{\mathsf{lo}} + \alpha_{\mathsf{hi}})$ chosen sufficiently large that $t_M \gg T$ with high probability, the rate (in PDC/ yr) at time τ is

$$\lambda(\tau) = \lambda_{\mathsf{lo}} \sum_{m=1}^{M} \mathbf{1}_{(t_{m-1}, s_m]}(\tau) + \lambda_{\mathsf{hi}} \sum_{m=1}^{M} \mathbf{1}_{(s_m, t_m]}(\tau) = \begin{cases} \lambda_{\mathsf{lo}} & \text{if } t_{m-1} < \tau \le s_m, \\ \lambda_{\mathsf{hi}} & \text{if } s_m \quad < \tau \le t_m, \end{cases}$$

illustrated in Figure (5).

Denote the total time and event counts in the high and low activity periods during $[0, T]$ by:

$$T_{\mathsf{hi}} := \sum_{m=1}^{M} \Big[(t_m \wedge T) - (s_m \wedge T) \Big] \qquad T_{\mathsf{lo}} := T - T_{\mathsf{hi}} \ = \sum_{m=1}^{M} \Big[(s_m \wedge T) - (t_{m-1} \wedge T) \Big]$$

$$N_{\mathsf{hi}} = \sum_{m=1, j=1}^{M,N} \mathbf{1}_{(s_m, t_m]}(\tau_j) \qquad N_{\mathsf{lo}} := N - N_{\mathsf{hi}} = \sum_{m=1, j=1}^{M,N} \mathbf{1}_{(t_{m-1}, s_m]}(\tau_j).$$

In the computations below we will treat N_{hi} (and hence $N_{\mathsf{lo}} \equiv N - N_{\mathsf{hi}}$) as known, and so must include its (binomial) conditional pmf, given α, $\{\mu_e\}$, and $\{\lambda(\cdot)\}$, in the likelihood.

Log likelihood

The logarithm $\ell := \log L$ of the likelihood for the augmented data can now be written as:

$$\ell = |J_0| \log \alpha + \sum_{j \in J_1} \log \left[1 - (V_j^{\mathsf{min}} / V_j^{\mathsf{max}})^{\alpha} \right] - \alpha \sum_{j \in J_0 \cup J_1} \log(V_j^{\mathsf{min}} / \epsilon) \qquad \text{(from } \{V_j\}) \qquad (12a)$$

$$+ \sum_{j=1}^{N} \log \left\{ \int_{\Phi_j} f_{\text{vM}}(\phi_j \mid \mu_{e_j}, \kappa_\phi) \, d\phi_j \right\} \qquad \text{(from } \{\phi_j\}) \qquad (12b)$$

$$+ N_{\mathsf{lo}} \log \lambda_{\mathsf{lo}} + N_{\mathsf{hi}} \log \lambda_{\mathsf{hi}} - (T_{\mathsf{lo}} \lambda_{\mathsf{lo}} + T_{\mathsf{hi}} \lambda_{\mathsf{hi}}) - \log N_{\mathsf{lo}}! - \log N_{\mathsf{hi}}! \qquad \text{(from } \{\tau_j\}) \qquad (12c)$$

Prior distributions

For the Pareto shape parameter α governing the PDC volumes we use the improper Jeffreys' Rule (or "Reference"— see Berger et al., 2009) prior $\alpha \sim \alpha^{-1}\mathbf{1}_{\{\alpha>0\}}$. With this choice the posterior distribution from uncensored observations would be the Gamma distribution

$$\alpha \mid \text{ Data } \sim \mathsf{Ga}\Big(N, \ \sum_{j\in J} \log(V_j/\epsilon)\Big), \tag{13}$$

which depends only on the count and volumes of the flows $\{V_j \ge \epsilon\}$ during $[0, T]$.

For the central initial flow parameters $\{\mu_e\}$ we begin with a uniform distribution $\mu_0 \sim \mathsf{Un}(S^1)$, and then at the start T_e of each new epoch we take a von Mises-distributed step

$$\mu_e \mid \text{ Past at time } T_e \ \sim \mathsf{vM}(\mu_{e-1}, \kappa_\mu). \tag{14}$$

This makes $\{\mu_e\}$ a von Mises random walk on the circle, *a priori*, whose step sizes depend on the concentration parameter κ_μ.

We model the levels $0 < \lambda_{\mathsf{lo}} < \lambda_{\mathsf{hi}} < \infty$ with a conjugate joint prior distribution (6), with log density

$$\log \pi(\lambda_{\mathsf{lo}}, \lambda_{\mathsf{hi}}) = c + (a_{\mathsf{lo}} - 1)\log \lambda_{\mathsf{lo}} + (a_{\mathsf{hi}} - 1)\log \lambda_{\mathsf{hi}} + r\log(\lambda_{\mathsf{hi}} - \lambda_{\mathsf{lo}}) - b(\lambda_{\mathsf{lo}} + \lambda_{\mathsf{hi}}) \tag{15a}$$

on $0 < \lambda_{\mathsf{lo}} < \lambda_{\mathsf{hi}}$ for constant c, unitless shape parameters $a_{\mathsf{lo}}, a_{\mathsf{hi}} > 0$ and repulsion parameter $r > -1$, and rate parameter $b > 0$ (in yr). For $r = 0$ this gives independent Gamma random variables conditioned to satisfy the order relation $\lambda_{\mathsf{lo}} < \lambda_{\mathsf{hi}}$, but taking $r > 0$ will encourage larger separation $|\lambda_{\mathsf{hi}} - \lambda_{\mathsf{lo}}|$ between the high and low rates.

The transition times $\{\vec{s}, \vec{t}\}$ are modeled as a Gamma renewal process, beginning with $t_0 := 0$ and proceeding sequentially for $m \in \mathbb{N}$ with increments

$$\{(s_m - t_{m-1})\} \overset{\text{iid}}{\sim} \mathsf{Ga}(\alpha_{\mathsf{lo}}, \beta) \qquad \{(t_m - s_m)\} \overset{\text{iid}}{\sim} \mathsf{Ga}(\alpha_{\mathsf{hi}}, \beta)$$

leading to log pdf from (5),

$$\begin{aligned}\log \pi(\vec{st}) = \text{const} &+ (\alpha_{\mathsf{lo}} - 1)\sum_{m=1}^{M} \log(s_m - t_{m-1}) + (\alpha_{\mathsf{hi}} - 1)\sum_{m=1}^{M} \log(t_m - s_m) \\ &+ M[\alpha_{\mathsf{lo}} \log \beta - \log \Gamma(\alpha_{\mathsf{lo}})] \qquad + M[\alpha_{\mathsf{hi}} \log \beta - \log \Gamma(\alpha_{\mathsf{hi}})] - \beta t_M\end{aligned} \tag{15b}$$

Posterior distributions

The **log posterior** based on likelihood (12) and prior from (13), (14), and (15) is:

$$\begin{aligned}
\ell(\theta) = &-\alpha \sum_{j\in J_0\cup J_1} \log(V_j^{\mathsf{min}}/\epsilon) + |J_0|\log\alpha \quad + \sum_{j\in J_1} \log\left[1-(V_j^{\mathsf{min}}/V_j^{\mathsf{max}})^\alpha\right] && \text{(16a)}\\
&+ \sum_{j=1}^{N} \log\left\{ \int_{\Phi_j} f_{\text{vM}}(\phi_j \mid \mu_{e_j}, \kappa_\phi)\, d\phi_j \right\} && \text{(16b)}\\
&+ N_{\mathsf{lo}}\log(\lambda_{\mathsf{lo}}) + N_{\mathsf{hi}}\log(\lambda_{\mathsf{hi}}) \quad - (T_{\mathsf{lo}}\lambda_{\mathsf{lo}} + T_{\mathsf{hi}}\lambda_{\mathsf{hi}}) - \log N_{\mathsf{lo}}! - \log N_{\mathsf{hi}}! && \text{(16c)}\\
&- \log\alpha && \text{(16d)}\\
&+ \sum_{e} \log f_{\text{vM}}(\mu_e \mid \mu_{e-1}, \kappa_\mu) && \text{(16e)}\\
&+ (a_{\mathsf{lo}}-1)\log\lambda_{\mathsf{lo}} \quad + (a_{\mathsf{hi}}-1)\log\lambda_{\mathsf{hi}} && \text{(16f)}\\
&+ r\log(\lambda_{\mathsf{hi}}-\lambda_{\mathsf{lo}}) \quad - b(\lambda_{\mathsf{lo}}+\lambda_{\mathsf{hi}}) &&\\
&+ (\alpha_{\mathsf{lo}}-1)\sum_{m=1}^{M}\log(s_m - t_{m-1}) \quad + (\alpha_{\mathsf{hi}}-1)\sum_{m=1}^{M}\log(t_m - s_m) && \text{(16g)}\\
&+ M[\alpha_{\mathsf{lo}}\log\beta - \log\Gamma(\alpha_{\mathsf{lo}})] \quad + M[\alpha_{\mathsf{hi}}\log\beta - \log\Gamma(\alpha_{\mathsf{hi}})] - \beta t_M && \text{(16h)}
\end{aligned}$$

where

$$\begin{aligned}
&T_0 := 0 && T_e := \min\{\tau_j > T_{e-1} :\ V_j > \Omega\} &&\\
&N_{\mathsf{hi}} := \sum_{m,j} \mathbf{1}_{(s_m,t_m]}(\tau_j) && N_{\mathsf{lo}} := \sum_{m,j} \mathbf{1}_{(t_{m-1},s_m]}(\tau_j) && = N - N_{\mathsf{hi}} \quad \text{(17a)}\\
&T_{\mathsf{hi}} := \sum \left[(t_m\wedge T) - (s_m \wedge T)\right] && T_{\mathsf{lo}} := \sum \left[(s_m\wedge T) - (t_{m-1}\wedge T)\right] && = T - T_{\mathsf{hi}} \quad \text{(17b)}
\end{aligned}$$

The terms in Eqns (16a,16b,16c) arise from the likelihood for α, $\{\mu_e\}$, and $\{\lambda(\tau)\}$, respectively; (16d) from the prior for α, (16e) from the prior for $\{\mu_e\}$, (16f) from the prior for $\{\alpha_{\mathsf{lo}}, \alpha_{\mathsf{hi}}\}$, and (16g, 16h) from the prior for $\{(s_m, t_m)\}$.

An MCMC algorithm

To draw parameter samples and forecasts from the posterior distribution we construct a Markov chain Monte Carlo (MCMC) computational scheme that employs a Metropolis-Hastings approach for the vectors $\{\mu_e\}$, $\{\eta_1, \eta_2\}$, and $\{s_m, t_m\}$, and Gibbs sampling for the scalar α whose posterior distribution is known in closed form. For each complete MCMC step we cycle through the four parameters in turn.

We implement these M-H steps by identifying for each parameter (let's call it "θ") the specific terms $\ell_\theta(\theta)$ of the log posterior pdf (16) that depend on that parameter. After generating the first t steps of the algorithm, arriving at value $\theta^{(t)}$ for the parameter, we make a proposal $\theta^* \sim q(\theta^* \mid \theta^{(t)})$ for a new value from a proposal distribution with symmetric pdf

$q(\theta_1 \mid \theta_2) = q(\theta_2 \mid \theta_1)$ described below. We "accept" the proposal and set $\theta^{(t+1)} := \theta^*$ if

$$\ell_\theta(\theta^*) + e^{(t)} > \ell_\theta(\theta^{(t)}) \tag{18}$$

for independent identically-distributed (iid) standard exponentially-distributed random variables $\{e^{(t)}\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$. Otherwise the proposal is rejected and $\theta^{(t+1)} := \theta^{(t)}$ remains unchanged. This is mathematically equivalent to, but numerically more stable than, accepting the proposal with probability $\min(H, 1)$ for the Hastings ratio $H := \exp\big(\ell_\theta(\theta^*)\big)/\exp\big(\ell_\theta(\theta^{(t)})\big)$, the ratio of posterior pdfs at the proposed θ^* and old $\theta^{(t)}$ parameter values. Typically the proposal distributions $q(\theta^* \mid \theta)$ are symmetric random walks with step sizes σ_θ chosen empirically to achieve acceptance rates in the range 5%–60%, near enough to the optimum 23.4% (Rosenthal, 2011). To accomplish this, acceptance rates must be monitored separately for each parameter θ.

For computational reasons it is helpful to re-parametrize the low and high rates $(\lambda_{\mathsf{lo}}, \lambda_{\mathsf{hi}})$ by logistics $(\eta_1, \eta_2) \in \mathbb{R} \times \mathbb{R}_+$, given by

$$\begin{aligned} \eta_1 &:= \log(\lambda_{\mathsf{lo}} + \lambda_{\mathsf{hi}}) & \eta_2 &:= \tfrac{1}{2}\log(\lambda_{\mathsf{hi}}/\lambda_{\mathsf{lo}}) \\ \lambda_{\mathsf{lo}} &= e^{\eta_1}/(1 + e^{2\eta_2}) & \lambda_{\mathsf{hi}} &= e^{\eta_1}/(1 + e^{-2\eta_2}) \\ &= \frac{\exp(\eta_1 - \eta_2)}{2\cosh(\eta_2)} & &= \frac{\exp(\eta_1 + \eta_2)}{2\cosh(\eta_2)} \end{aligned}$$

under which $(\lambda_{\mathsf{hi}} + \lambda_{\mathsf{lo}}) = \exp(\eta_1)$ and $(\lambda_{\mathsf{hi}} - \lambda_{\mathsf{lo}}) = \exp(\eta_1)\tanh(\eta_2)$. The Jacobian of this transformation is $\lambda_{\mathsf{lo}}^{-1}\lambda_{\mathsf{hi}}^{-1}\, d\lambda_{\mathsf{lo}}\, d\lambda_{\mathsf{hi}} = 2d\eta_1\, d\eta_2$, leading to the replacement of (16f) with $[a_{\mathsf{lo}} \log \lambda_{\mathsf{lo}} + a_{\mathsf{hi}} \log \lambda_{\mathsf{hi}}]$. Similarly, we employ a symmetric random walk for $\{(\vec{s}_m, \vec{t}_m)\}$ on the *log* scale, and so replace (16g) with $\big[+\alpha_{\mathsf{lo}} \sum_{m=1}^{M} \log(s_m - t_{m-1}) + \alpha_{\mathsf{hi}} \sum_{m=1}^{M} \log(t_m - s_m)\big]$. In both cases this amounts to simply removing each "-1" from $(\alpha_{\mathsf{lo}} - 1)$, $(\alpha_{\mathsf{hi}} - 1)$, $(a_{\mathsf{lo}} - 1)$, and $(a_{\mathsf{hi}} - 1)$ in (16).

The resulting algorithm begins with the specification of initial values $\{\theta^{(0)}\}$ at step $t = 0$ and step sizes $\{\sigma_\theta\}$ for the four parameters, and proceeds at each step $t \geq 0$ as follows:

1. α: Draw $\alpha^{(t+1)} \sim \mathsf{Ga}\big(N, \sum \log(V_j/\epsilon)\big)$, its posterior distribution.

2. $\{\mu_e\}$: Let $e_{\text{max}} := \#\{j : V_j > \Omega\}$ be the number of epochs in $(0, T]$. Choose one of the epochs e uniformly from $\{1, \cdots, e_{\text{max}}\}$. Add to μ_e a normally-distributed step $\delta \sim \mathsf{No}(0, \sigma_\mu^2)$ to get central angle proposal $\mu_e^* = \mu_e^{(t)} + \delta$ (mod 360), and (from Eqns (16a,16b,16e)) set

$$\begin{aligned} \ell_\mu(\mu_e) &:= \log f_{\text{vM}}(\mu_e \mid \mu_{e-1}, \kappa_\mu) + \log f_{\text{vM}}(\mu_{e+1} \mid \mu_e, \kappa_\mu) \\ &\quad + \sum_{j:\ e_j = e} \log f_{\text{vM}}(\phi_j \mid \mu_e, \kappa_\phi) \\ &= \kappa_\mu^2\big[\cos(\mu_e - \mu_{e-1}) + \cos(\mu_{e+1} - \mu_e)\big] + \kappa_\phi^2 \sum_{j:\ e_j = e} \cos(\phi_j - \mu_e) \end{aligned} \tag{19}$$

(note we neglect terms that do not include μ_e, since they will cancel in the M-H step). If the randomly-drawn epoch e is the first $e = 0$ or last $e = e_{\text{max}}$, omit the missing terms μ_{e-1} or μ_{e+1} in (19). Accept or reject the proposal as in (18).

An acceptable alternative is to add iid steps δ_e to *all* the $\{\mu_e\}$, and accept or reject the entire proposed vector using the sum

$$\ell_{\vec{\mu}}(\vec{\mu}) := \sum_{j=1}^{N} \log f_{\text{vM}}(\phi_j \mid \mu_{e_j}, \kappa_\phi) + \sum_{e} \log f_{\text{vM}}(\mu_e \mid \mu_{e-1}, \kappa_\mu). \tag{20}$$

3. $(\lambda_{\mathsf{lo}}, \lambda_{\mathsf{hi}})$: Keep track of the values of the logistic transforms $\eta_1 = \log(\lambda_{\mathsf{lo}} + \lambda_{\mathsf{hi}})$ and $\eta_2 = \log(\lambda_{\mathsf{hi}}/\lambda_{\mathsf{lo}})/2$. Add to $\{\eta_i^{(t)}\}$ increments $\delta_i \overset{\text{iid}}{\sim} \mathsf{No}(0, \sigma_\eta^2)$ and, if necessary, reflect to ensure $\eta_2^* > 0$ to get proposals:

$$\eta_1^* = \eta_1^{(t)} + \delta_1 \qquad \eta_2^* = |\eta_2^{(t)} + \delta_2|.$$

Compute the corresponding $\lambda_{\mathsf{lo}}^* = \exp(\eta_1^* - \eta_2^*)/2\cosh(\eta_2)$ and $\lambda_{\mathsf{hi}}^* = \exp(\eta_1^* + \eta_2^*)/2\cosh(\eta_2)$ and (from Eqns (16c, 16f), and using the Jacobian above) accept or reject the proposal (as in (18)) using

$$\begin{aligned}\ell_\eta(\eta_1, \eta_2) := &(N_{\mathsf{lo}} + a_{\mathsf{lo}}) \log \lambda_{\mathsf{lo}} + (N_{\mathsf{hi}} + a_{\mathsf{hi}}) \log \lambda_{\mathsf{hi}} \\ &- (T_{\mathsf{lo}}\lambda_{\mathsf{lo}} + T_{\mathsf{hi}}\lambda_{\mathsf{hi}}) + r \log(\lambda_{\mathsf{hi}} - \lambda_{\mathsf{lo}}) - b(\lambda_{\mathsf{lo}} + \lambda_{\mathsf{hi}}).\end{aligned}$$

4. $\{(s_m, t_m) : m \le M\}$: To generate proposal vectors $st^* = (\vec{s}^*, \vec{t}^*)$ at time step t, beginning with $st^{(t)} = (\vec{s}^{(t)}, \vec{t}^{(t)})$, fix $\sigma_{st} > 0$ and scale all the intervals $(s_m, t_m]$ and $(t_{m-1}, s_m]$ by independent log-normal factors as follows:

a) Set $\vec{x} := \big(s_1^{(t)}, (t_1^{(t)} - s_1^{(t)}), (s_2^{(t)} - t_1^{(t)}), ..., (t_M^{(t)} - s_M^{(t)})\big) \in \mathbb{R}_+^{2M}$;
b) Draw $\zeta_i \overset{\text{iid}}{\sim} \mathsf{No}(0,1)$ and set $x_i^* := x_i \exp(\sigma_{st}\zeta_i)$ for $1 \le i \le 2M$;
c) Set $s_1^* := x_1^*,\ t_1^* := (x_1^* + x_2^*),\ s_2^* := (x_1^* + x_2^* + x_3^*), \ldots, t_M^* := \sum x_i^*$;
d) Compute $N_{\mathsf{hi}}^* := \sum_{m,j} \mathbf{1}_{(s_m^*, t_m^*]}(\tau_j)$, $\quad N_{\mathsf{lo}}^* := N - N_{\mathsf{hi}}^*$;
and $T_{\mathsf{hi}}^* := \sum \big[(t_m^* \wedge T) - (s_m^* \wedge T)\big]$, $\quad T_{\mathsf{lo}}^* := T - T_{\mathsf{hi}}^*$.

Now accept or reject the proposal as in (18), using log Hastings numerator function

$$\begin{aligned}&= \alpha_{\mathsf{hi}} \sum_{m=1}^{M} \log(t_m^* - s_m^*) + \alpha_{\mathsf{lo}} \sum_{m=1}^{M} \log(s_m^* - t_{m-1}^*) - \beta t_M^* \\ &\quad + [N_{\mathsf{hi}}^* \log(\lambda_{\mathsf{hi}}) + N_{\mathsf{lo}}^* \log(\lambda_{\mathsf{lo}})] - [T_{\mathsf{hi}}^*\lambda_{\mathsf{hi}} + T_{\mathsf{lo}}^*\lambda_{\mathsf{lo}}] - [\log N_{\mathsf{hi}}^*! + \log N_{\mathsf{lo}}^*!]\end{aligned} \tag{21}$$

based on Eqns (16c,16g,16h).

5. Forecast $\{(V_j, \phi_j, \tau_j)\}$: For now, we ignore the initiation angles $\{\phi_i\}$ for future PDCs, and focus on their volumes $\{V_i \geq \epsilon\}$ and times $\{\tau_i > T\}$.
   Proposal: Select some $T' > T$ which also satisfies $T' \ll M(\alpha_{\mathsf{lo}} + \alpha_{\mathsf{hi}})/\beta$, to ensure that $t_M \gg T'$ with high probability. Now simulate those event times $\{\tau_i\}$ in $(T, T']$ and the associated volumes $\{V_i\}$, and make overlay plots of the cumulative volume during $(T, T']$ similar to Figure (12).

   One way to do that: Set

$$\begin{aligned}
s'_m &:= (s_m \vee T) \wedge T', \quad t'_m := (t_m \vee T) \wedge T' \\
T'_{\mathsf{hi}} &= \text{Time in } (T, T'] \text{ with high rate } \lambda(t) = \lambda_{\mathsf{hi}} \\
&= \sum_{m=1}^{M} (t'_m - s'_m) \\
T'_{\mathsf{lo}} &= (T' - T) - T'_{\mathsf{hi}} \\
N'_{\mathsf{hi}} &\sim \mathsf{Po}(T'_{\mathsf{hi}}\, \lambda_{\mathsf{hi}}), \quad N'_{\mathsf{lo}} \sim \mathsf{Po}(T'_{\mathsf{lo}}\, \lambda_{\mathsf{lo}}), \quad N' := N'_{\mathsf{hi}} + N'_{\mathsf{lo}} \\
\{V_i\} &\overset{\text{iid}}{\sim} \mathsf{Pa}(\alpha, \epsilon), \ 1 \leq i \leq N'
\end{aligned}$$

   and draw N'_{hi} random times uniformly $\{\tau_i\}$ from the union of the intervals $(s'_m, t'_m]$ and N'_{lo} times $\{\tau_i\}$ uniformly from $\cup\big(t'_{m-1}, s'_m\big]$; sort all the $\{\tau_i\}$, and plot the cumulative sum of the $\{V_i\}$ against $\{\tau_i\}$.

   A mathematically equivalent approach is to cycle through the intervals $\big(s'_m, t'_m\big]$ with positive length $(t'_m - s'_m)$ and, for each of these, draw $N'_m \sim \mathsf{Po}\big(\lambda_{\mathsf{hi}}(t'_m - s'_m)\big)$ pairs (τ_i, V_i) with $\tau_i \overset{\text{iid}}{\sim} \mathsf{Un}(s'_m, t'_m]$ and $V_i \sim \mathsf{Pa}(\alpha, \epsilon)$, and similarly for the intervals $\big(t'_{m-1}, s'_m\big]$.

   Now set $e'_{\max} := e_{\max} + \#\{j : \ j > N \text{ and } V_j > \Omega\}$, the number of epochs in the entire study and forecast period $[0, T']$ and, for $e_{\max} < e \leq e'_{\max}$, let $T_e := \min\{\tau_j > T_{e-1} : \ V_j > \Omega\}$ be the epoch ending times in the forecast period $(T, T']$. Draw forecast central angles successively as

$$\mu_e \sim \mathsf{vM}\big(\mu_{e-1}, \kappa_\mu\big), \qquad e_{\max} < e \leq e'_{\max}.$$

   Once again identify the epoch for each forecast PDC by $e_j := \max\{e : \ T_e < \tau_j\}$ for $j > N$ and, finally, draw initiation angles

$$\phi_j \overset{\text{ind}}{\sim} \mathsf{vM}\big(\mu_{e_j}, \kappa_\phi\big), \qquad j > N.$$

   This completes the simulation of PDCs $\{(V_j, \phi_j, \tau_j)\}$ in the forecast period $(T, T']$.

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