# Stability analysis of fluid flows using Lagrangian Perturbation Theory (LPT): application to the plane Couette flow.

**Sharvari Nadkarni-Ghosh** [1,*], **Jayanta K. Bhattacharjee** [2]

[1] *Department of Physics, I.I.T. Kanpur, Kanpur 208016, India*
[2] *Department of Physics, Harish-Chandra Research Institute, Jhusi, Allahabad 211019, India*

Correspondence*:
Sharvari Nadkarni-Ghosh
nsharvari@gmail.com, sharvari@iitk.ac.in

## ABSTRACT

We present a new application of Lagrangian Perturbation Theory (LPT): the stability analysis of fluid flows. As a test case that demonstrates the framework we focus on the plane Couette flow. The incompressible Navier-Stokes equation is recast such that the particle position is the fundamental variable, expressed as a function of Lagrangian coordinates. The displacement due to the steady state flow is taken to be the zeroth order solution and the position is formally expanded in terms of a small parameter (generally, the strength of the initial perturbation). The resulting hierarchy of equations is solved analytically at first order. We find that we recover the standard result in the Eulerian frame: the plane Couette flow is asymptotically stable for all Reynolds numbers. However, it is also well established that experiments contradict this prediction. In the Eulerian picture, one of the proposed explanations is the phenomenon of 'transient growth' which is related to the non-normal nature of the linear stability operator. The first order solution in the Lagrangian frame also shows this feature, albeit qualitatively. As a first step, and for the purposes of analytic manipulation, we consider only linear stability of 2D perturbations but the framework presented is general and can be extended to higher orders, other flows and/or 3D perturbations.

**Keywords: keyword, keyword, keyword, keyword, keyword, keyword, keyword, keyword**

### 0.1 Mathematical Transformations

We start with the expression for divergence of the velocity.

$$\nabla_r \cdot \dot{\mathbf{r}} = \frac{\partial \dot{r}_i}{\partial r_i} = \frac{\partial \dot{r}_i}{\partial R_l}\frac{\partial R_l}{\partial r_i}. \tag{1}$$

Einstein's repeated summation convention is followed. The inverse transformation from $R$-space to $r$-space is given as

$$\frac{\partial R_l}{\partial r_i} = \frac{1}{2J}\epsilon_{ilm}\epsilon_{jl'm'}\frac{\partial r_l}{\partial R_{l'}}\frac{\partial r_m}{\partial R_{m'}}, \tag{2}$$

where

$$J = \text{Det}\left(\frac{\partial r_i}{\partial R_j}\right) = \epsilon_{jlm}\frac{\partial r_1}{\partial R_j}\frac{\partial r_2}{\partial R_l}\frac{\partial r_3}{\partial R_m} = \frac{1}{6}\epsilon_{ipq}\epsilon_{jlm}\frac{\partial r_i}{\partial R_j}\frac{\partial r_p}{\partial R_l}\frac{\partial r_q}{\partial R_m} \tag{3}$$

and $\epsilon_{jlm}$ is the usual Levi-Civita symbol. The incompressibility condition reduces to

$$\epsilon_{ilm}\epsilon_{jl'm'}\dot{r}_{i,j}r_{l,l'}r_{m,m'} = 0, \tag{4}$$

where commas denote spatial derivatives with respect to the Lagrangian coordinate. For example, $r_{m,m'}$ denotes the derivative of the $m$-th component of the vector $\mathbf{r}$ with respect to the $m'$-th component of the Lagrangian coordinate $\mathbf{R} = \{X, Y, Z\}$. Note that this can also be written as

$$\epsilon_{ilm}\epsilon_{jl'm'}\dot{r}_{i,j}r_{l,l'}r_{m,m'} = \frac{1}{3}\epsilon_{ilm}\epsilon_{jl'm'}\frac{d}{dt}\left(r_{i,j}r_{l,l'}r_{m,m'}\right). \tag{5}$$

Consider the curl of the Navier-Stokes equation. The $i$-th component of the l.h.s. is

$$\begin{aligned} \nabla_r \times \ddot{\mathbf{r}} &= \epsilon_{ijk}\frac{\partial \ddot{r}_k}{\partial r_j} \\ &= \epsilon_{ijk}\frac{\partial \ddot{r}_k}{\partial R_l}\frac{\partial R_l}{\partial r_j} \\ &= \frac{1}{2J}\epsilon_{ijk}\epsilon_{jmn}\epsilon_{lm'n'}\ddot{r}_{k,l}r_{m,m'}r_{n,n'}, \end{aligned} \tag{6}$$

where the last equality follows from equation (2). The r.h.s. of the Navier-Stokes is proportional to $\nabla_r^2(\nabla_r \times \dot{\mathbf{r}})$. For any scalar $f_i$, $\nabla_r$ converted to Lagrangian coordinates is

$$\begin{aligned} \nabla_r^2 f_i &= \frac{\partial}{\partial r_l}\left(\frac{\partial f_i}{\partial r_l}\right) \\ &= \frac{\partial}{\partial R_q}\left(\frac{\partial f_i}{\partial R_p}\cdot\frac{\partial R_p}{\partial r_l}\right)\cdot\frac{\partial R_q}{\partial r_l}. \end{aligned} \tag{7}$$

Using equation (2) gives

$$\nabla_r^2 f_i = \frac{1}{2J}\epsilon_{lde}\epsilon_{qd'e'}r_{d,d'}r_{e,e'}\left(\frac{1}{2J}\epsilon_{lab}\epsilon_{pa'b'}r_{a,a'}r_{b,b'}f_{i,p}\right)_{,q}. \tag{8}$$

Substituting $f_i = \epsilon_{ijk}\frac{\partial \dot{r}_k}{\partial r_j}$ and again using equation (2) to transform derivatives gives

$$\nabla_r^2(\nabla_r \times \dot{\mathbf{r}})_i = \frac{1}{2J}\epsilon_{ijk}\epsilon_{jfg}\epsilon_{mf'g'}\epsilon_{lde}\epsilon_{qd'e'}\epsilon_{lab}\epsilon_{pa'b'}r_{d,d'}r_{e,e'}\left\{\frac{1}{2J}r_{a,a'}r_{b,b'}\left(\frac{1}{2J}\dot{r}_{k,m}r_{f,f'}r_{g,g'}\right)_{,p}\right\}_{,q}. \tag{9}$$

Thus the curl of the Navier-Stokes equation in Lagrangian coordinates reduces to

$$\epsilon_{ijk}\epsilon_{jmn}\epsilon_{lm'n'}\ddot{r}_{k,l}r_{m,m'}r_{n,n'} = \nu\epsilon_{ijk}\epsilon_{jfg}\epsilon_{mf'g'}\epsilon_{lde}\epsilon_{qd'e'}\epsilon_{lab}\epsilon_{pa'b'}r_{d,d'}r_{e,e'}\left\{\frac{1}{2J}r_{a,a'}r_{b,b'}\left(\frac{1}{2J}\dot{r}_{k,m}r_{f,f'}r_{g,g'}\right)_{,p}\right\}_{,q} \quad (10)$$

## 0.2 The background solution in Lagrangian coordinates

The solution for the physical position $\mathbf{r} = \{x, y, z\}$ in terms of the Lagrangian variable $\mathbf{R} = \{X, Y, Z\}$ at the zeroth order

$$\mathbf{r}(\mathbf{R}) = \mathbf{p}^{(0)}(\mathbf{R}) = \{X + cYt, Y, Z\}. \quad (11)$$

We check that this is an exact solution of the incompressible Navier-Stokes system given by equations (6) and (7) in the text. The transformation between the Lagrangian and Eulerian frame is

$$\frac{\partial r_i}{\partial R_j} = \begin{pmatrix} 1 & ct & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (12)$$

and the inverse $\frac{\partial R_i}{\partial r_j}$ is given by equation (2) (or can be easily computed for this simple case),

$$\frac{\partial R_i}{\partial r_j} = \begin{pmatrix} 1 & -ct & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (13)$$

Here $i$ is the row-wise index, $j$ is column-wise. To check if equations (6) and (7) are true, it suffices only to consider the $x$-component since the others are trivially zero. The divergence-less condition given by equation (6) in the main text is

$$\begin{aligned} \nabla_r \dot{r}_x &= \frac{\partial \dot{r}_x}{\partial x} \quad (14) \\ &= \frac{\partial \dot{r}_x}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial \dot{r}_x}{\partial Y} \cdot \frac{\partial Y}{\partial x} \\ &= 0 \cdot 1 + c \cdot 0 = 0 \end{aligned}$$

In equation (7), the l.h.s. is zero since there is no $t$ dependence in $\dot{\mathbf{r}}$ and Lagrangian derivative is just the total time derivative acting on $\dot{\mathbf{r}}$. So it remains to prove that r.h.s =0. The $x$-component is

$$\begin{aligned} \nabla_r^2 \dot{\mathbf{r}} &= \frac{\partial^2 \dot{r}_x}{\partial x^2} + \frac{\partial^2 \dot{r}_x}{\partial y^2} \quad (15) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial \dot{r}_x}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial \dot{r}_x}{\partial Y} \cdot \frac{\partial Y}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \dot{r}_x}{\partial X} \cdot \frac{\partial X}{\partial y} + \frac{\partial \dot{r}_x}{\partial Y} \cdot \frac{\partial Y}{\partial y}\right). \end{aligned}$$

Applying the change of derivatives again and using the fact that $\partial Y/\partial x = 0$, $\partial \dot{r}_x/\partial X = 0$ and $\partial^2 \dot{r}_x/\partial Y^2 = 0$, all terms become zero. Thus both Navier-Stokes and the incompressibility are satisfied when the base flow is expressed in Lagrangian coordinates. It is a natural candidate for the zeroth order particle position and one can examine the stability of the system by perturbing about this steady state solution.

## 0.3 First order perturbation Theory

The perturbation ansatz is $\mathbf{r} = \mathbf{p}^{(0)} + \mathbf{p}^{(1)}\Delta$, where $\Delta$ is just a book-keeping parameter. Substitute this ansatz in equations (4) and (10) and collect terms of first order. The zeroth order solution is determined by the background flow. We will assume that the first order perturbation is two dimensional.

### 0.3.1 Divergence Equation

At first order equation (5) reduces to

$$\epsilon_{ilm}\epsilon_{jl'm'}\frac{d}{dt}\left(p^{(0)}_{i,j}p^{(0)}_{l,l'}p^{(1)}_{m,m'}\right)\Delta = 0. \tag{16}$$

Using the symmetry properties of the Levi-Civita tensor and the fact that the background flow is laminar gives

$$\nabla_R \cdot \dot{\mathbf{p}}^{(1)} - \dot{p}^{(0)}_{X,Y}p^{(1)}_{Y,X} - p^{(0)}_{X,Y}\dot{p}^{(1)}_{Y,X} = 0. \tag{17}$$

Substituting for the plane Couette flow zeroth order solution from equation (**??**),

$$\nabla_R \cdot \dot{\mathbf{p}}^{(1)} = c\left(p^{(1)}_{Y,X} + \dot{p}^{(1)}_{Y,X}t\right).. \tag{18}$$

Note that a flow which is divergence-free in the Eulerian frame does not stay divergence-free in the Lagrangian frame.

### 0.3.2 Curl Equation

To simplify the curl equation we first note that from equation (3) the determinant $J$ to first order can be expanded as

$$J = \frac{1}{6}\epsilon_{ipq}\epsilon_{jlm}\left(\frac{\partial p^{(0)}_i}{\partial R_j}\frac{\partial p^{(0)}_p}{\partial R_l}\frac{\partial p^{(0)}_q}{\partial R_m} + 3\frac{\partial p^{(0)}_i}{\partial R_j}\frac{\partial p^{(0)}_p}{\partial R_l}\frac{\partial p^{(1)}_q}{\partial R_m}\cdot\Delta\right) + \mathcal{O}(\Delta^2). \tag{19}$$

When the background flow is laminar this gives

$$J = 1 + \left(\nabla_R \cdot \mathbf{p}^{(1)} - p^{(0)}_{X,Y}p^{(1)}_{Y,X}\right)\Delta + \mathcal{O}(\Delta^2). \tag{20}$$

But note that equation (17) can be re-written as

$$\frac{d}{dt}[\nabla_R \cdot \mathbf{p}^{(1)} - p^{(0)}_{X,Y}p^{(1)}_{Y,X}] = 0. \tag{21}$$

Comparing with the expression equation (20) for $J$, this gives $\dot{J} = 0$ to first order or in other words $J$ is conserved to first order. Since $J(t=0) = 1$, by definition of the Lagrangian coordinate, to first order we can set $J \approx 1$ in equation (10). The coefficient of the first order term in $\Delta$ in the l.h.s. of equation (10) is

$$2\left(\ddot{p}^{(1)}_{Y,X} - \ddot{p}^{(1)}_{X,Y}\right) + 2p^{(0)}_{X,Y}\ddot{p}^{(1)}_{X,X}. \tag{22}$$

We now consider the structure of the r.h.s. of equation (10). We remind the reader that the 'comma' subscript denotes differentiation w.r.t. the Lagrangian coordinate. First note that each $r$ will have an expansion in terms of $p^{(0)}$ and $p^{(1)}$ and, at first order, any term involving $p^{(1)}$ can only couple to other terms with $p^{(0)}$.

Depending on the location of $p^{(1)}$, we can classify these into four types of terms. The first type is when $p^{(1)}$ is located between the curly and regular bracket, the second is when $p^{(1)}$ is within the regular bracket and with the time derivative, the third is when it is within the regular bracket but the time derivative does not act on it and the fourth is when it is outside both brackets. Thus, the coefficient of the term which is first order in $\Delta$ in the r.h.s. of equation (10) is I + II + III + IV, with,

$$\text{I.}\quad 2\mathcal{E}\qquad \frac{\partial}{\partial R_q}\left\{p^{(1)}_{a,a'}p^{(0)}_{b,b'}\frac{\partial}{\partial R_p}\left(\dot{p}^{(0)}_{k,m},p^{(0)}_{f,f'},p^{(0)}_{g,g'}\right)\right\}p^{(0)}_{d,d'}p^{(0)}_{e,e'} \tag{23}$$

$$\text{II.}\quad \mathcal{E}\qquad \frac{\partial}{\partial R_q}\left\{p^{(0)}_{a,a'}p^{(0)}_{b,b'}\frac{\partial}{\partial R_p}\left(\dot{p}^{(1)}_{k,m},p^{(0)}_{f,f'},p^{(0)}_{g,g'}\right)\right\}p^{(0)}_{d,d'}p^{(0)}_{e,e'} \tag{24}$$

$$\text{III.}\quad 2\mathcal{E}\qquad \frac{\partial}{\partial R_q}\left\{p^{(0)}_{a,a'}p^{(0)}_{b,b'}\frac{\partial}{\partial R_p}\left(\dot{p}^{(0)}_{k,m},p^{(0)}_{f,f'},p^{(1)}_{g,g'}\right)\right\}p^{(0)}_{d,d'}p^{(0)}_{e,e'} \tag{25}$$

$$\text{IV.}\quad 2\mathcal{E}\qquad \frac{\partial}{\partial R_q}\left\{p^{(0)}_{a,a'}p^{(0)}_{b,b'}\frac{\partial}{\partial R_p}\left(\dot{p}^{(0)}_{k,m},p^{(0)}_{f,f'},p^{(0)}_{g,g'}\right)\right\}p^{(0)}_{d,d'}p^{(1)}_{e,e'} \tag{26}$$

where $\mathcal{E} = \frac{\nu}{4}\epsilon_{ijk}\epsilon_{jfg}\epsilon_{mf'g'}\epsilon_{lde}\epsilon_{qd'e'}\epsilon_{lab}\epsilon_{pa'b'}$. For the plane Couette flow terms of the type I and IV will be zero since there are two spatial derivatives acting on components of $\mathbf{p}^{(0)}$.

The terms of type II and III simplify to

$$\text{II} \rightarrow \qquad \nu\left(2\nabla^2_R - 4ct\frac{\partial^2}{\partial X\partial Y} + 2c^2t^2\frac{\partial}{\partial X^2}\right)\left(\dot{p}^{(1)}_{Y,X} - \dot{p}^{(1)}_{X,Y} + ct\dot{p}^{(1)}_{X,X}\right). \tag{27}$$

$$\text{III} \rightarrow \qquad \nu\left(2\nabla^2_R - 4ct\frac{\partial^2}{\partial X\partial Y} + 2c^2t^2\frac{\partial}{\partial X^2}\right)(-cp^{(1)}_{X,X}). \tag{28}$$

Putting together equations (22) and (27) and (28) gives

$$\left(\ddot{p}^{(1)}_{Y,X} - \ddot{p}^{(1)}_{X,Y}\right) + ct\ddot{p}^{(1)}_{X,X} = \nu\left(\nabla^2_R - 2ct\frac{\partial^2}{\partial X\partial Y} + c^2t^2\frac{\partial}{\partial X^2}\right)\left(\dot{p}^{(1)}_{Y,X} - \dot{p}^{(1)}_{X,Y} + ct\dot{p}^{(1)}_{X,X} - cp^{(1)}_{X,X}\right). \tag{29}$$

### 0.4 Conditions for the bounded Couette flow

For the bounded Couette flow, we can take $\phi$ to be of the general form

$$\phi(X,Y,t) = H(X,Y)f(t), \tag{30}$$

where $H(X,Y)$ will be an appropriately chosen 'basis function' which satisfies the boundary conditions at both plates. Substituting it in the system

$$\mathcal{A}\dot{\psi} = \phi, \tag{31}$$

$$\dot{\phi} = \nu\mathcal{A}\phi, \tag{32}$$

we get

$$\frac{df}{dt} = \nu\left((1+c^2t^2)\frac{H_{,XX}}{H} + \frac{H_{,YY}}{H} - 2ct\frac{H_{,XY}}{H}\right)f. \tag{33}$$

Here $H_{,XY} = \frac{\partial^2 H}{\partial X \partial Y}$ etc. Since $H$ is just a function of space, this can be integrated w.r.t. time to give

$$f(t) = f(0) \exp \left[ \nu t \left( \frac{H_{,XX} + H_{,YY}}{H} \right) + \frac{\nu c^2 t^3}{3} \frac{H_{,XX}}{H} - \nu c t^2 \frac{H_{,XY}}{H} \right] \tag{34}$$

where $f(0)$ is the integration constant to be set later. One can now solve equation (**??**) for $\dot{\psi}$ with the ansatz

$$\dot{\psi}(X, Y, t) = G(X, Y, t) f(t). \tag{35}$$

Here $\phi$ acts like a source term so the temporal dependence of $G(X, Y, t)$ can always be chosen to be of the above form. Substituting in equation (**??**) and using the form of $\phi$ gives the equation

$$(1 + c^2 t^2) G_{,XX} + G_{,YY} - 2ct G_{,XY} = H(X, Y). \tag{36}$$

Given a $H(X, Y)$, this equation is a PDE in two variables which needs to be solved numerically subject to the boundary conditions.

For infinite extent along $x$ direction, it is possible to use Fourier decomposition along the $x$-axis: $H(X, Y) = \sum_{k_x} \tilde{\phi}(k_x) e^{ik_x X} h(Y)$ and $G(X, Y, T) = \sum_{k_x} \tilde{\phi}(k_x) e^{ik_x X} g(Y, t)$. Equation (36) becomes,

$$-k_x^2 (1 + c^2 t^2) g(Y, t) - 2ik_x ct g'(Y, t) + g''(Y, t) = h(Y); \tag{37}$$

where the primes denote differentiation w.r.t $Y$. The solution for $g(Y, t)$ can be split into a homogeneous part and a particular solution: $g(Y, t) = g_{homo.}(Y, t) + g_{part.}(Y, t)$. The homogeneous solution is

$$g_{homo.}(Y, t) = C_1(t) g_1(Y) + C_2(t) g_2(Y) \tag{38}$$

where $g_1(Y) = e^{m_1 Y}$ and $g_2 = e^{m_2 Y}$ with $m_1 = -k_x + ik_x ct$ and $m_2 = k_x + ik_x ct$. The particular solution is given by

$$g_{part.}(Y, t) = a(Y) g_1(Y) + b(Y) g_2(Y) \tag{39}$$

where

$$a'(Y) = -\frac{h(y)}{W(g_1, g_2)} g_2(Y) \quad b'(Y) = \frac{h(y)}{W(g_1, g_2)} g_1(Y), \tag{40}$$

where the Wronskian $W(g_1, g_2) = g_1 g_2' - g_1' g_2$. In this case the two linearly independent homogeneous solutions are exponentials and $W(g_1, g_2) = (m_2 - m_1) g_1 g_2$. This gives the full solution as

$$g(Y, t) = c_1 g_1(Y) + c_2 g_2(Y) - \frac{g_1(Y)}{m_2 - m_1} \int_0^Y \frac{h(Y')}{g_1(Y')} dY' + \frac{g_2(Y)}{m_2 - m_1} \int_0^Y \frac{h(Y')}{g_2(Y')} dY \tag{41}$$

The boundary conditions given by

$$\left. \frac{\partial \dot{\psi}}{\partial Y} \right|_{Y=0} = c \left. \frac{\partial \psi}{\partial X} \right|_{Y=0} \quad \forall\, t, \tag{42}$$

$$\left. \frac{\partial \dot{\psi}}{\partial X} \right|_{Y=0} = 0 \qquad \forall\, t \tag{43}$$

and

$$\left.\frac{\partial \dot{\psi}}{\partial Y}\right|_{Y=0} = 0 \qquad \forall\, t, \tag{44}$$

get extended to

$$\left.\frac{\partial \dot{\psi}}{\partial X}\right|_{Y=0,h} = 0 \quad \text{and} \quad \left.\frac{\partial \dot{\psi}}{\partial Y}\right|_{Y=0,h} = 0. \qquad \forall\, t. \tag{45}$$

This imposes constraints on $c_1, c_2$ and the form of $h$. The important point to note here is that the temporal dependence for the bounded flow has also the same exponential factor as the semi-bounded case and it is plausible that it will exhibit the same late time behaviour: the flow will be stable for all Reynolds numbers (or all values of kinematic viscosity).

## 0.5 Effect of the $(\mathbf{v}\cdot\nabla)\mathbf{v}$ term in Eulerian theory

Let $\mathbf{v}_{s.s} = \{cy, 0\}$ denote the steady state solution for the plane Couette flow and let $\mathbf{v}$ denote the perturbation around this background solution. The full non-linear equation satisfied by $\mathbf{v}$ is

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v}_{s.s}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v}_{s.s} + \nu\nabla^2\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}, \tag{46}$$

where the spatial derivatives are w.r.t. the Eulerian coordinate $\mathbf{r} \equiv \{x, y\}$. This can be written as

$$\frac{\partial \mathbf{v}}{\partial t} = \mathcal{L}\cdot\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}, \tag{47}$$

where the operator $\mathcal{L}$ for the plane Couette flow is,

$$\mathcal{L} = \begin{pmatrix} -cy\frac{\partial}{\partial x} + \nu\nabla^2 & -c \\ 0 & -cy\frac{\partial}{\partial x} + \nu\nabla^2 \end{pmatrix}. \tag{48}$$

We construct the solution to the full non-linear solution iteratively as follows. Let $\mathbf{v}^{(1)}$ be the linear solution that satisfies

$$\frac{\partial \mathbf{v}^{(1)}}{\partial t} = \mathcal{L}\cdot\mathbf{v}^{(1)}. \tag{49}$$

Integrating over $t$ for fixed $x, y$, we get

$$\mathbf{v}^{(1)}(x, y, t) = e^{\mathcal{L}t}\cdot\mathbf{v}_0, \tag{50}$$

where $\mathbf{v}_0$ is the perturbation at the initial time $t = 0$. We assume it to have the form

$$\mathbf{v}_0(x, y) = \{v_{x,0}, v_{y,0}\}e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{51}$$

Use this solution to construct the non-linear acceleration as $(\mathbf{v}^{(1)}\cdot\nabla)\mathbf{v}^{(1)}$. For early times, the second order solution is

$$\mathbf{v}^{(2)}(x, y, t) = \mathbf{v}^{(1)}(x, y, t) + [(\mathbf{v}^{(1)}\cdot\nabla)\mathbf{v}^{(1)}]\cdot t. \tag{52}$$

It can be seen from the form of the operator $\mathcal{L}$ that all modes will be stable at linear order. The form of the linear operator is such that there is only one Eigen direction along $(1, 0)$ and the linear mechanism

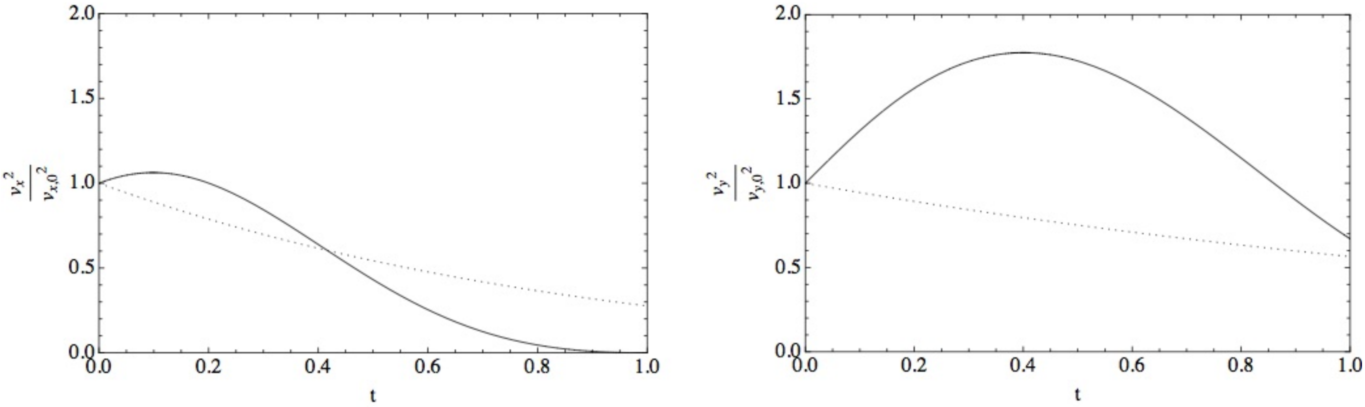


**Figure 1.** The ratio of the linear (dotted lines) and second order (solid lines) Eulerian perturbation solutions at early times. The left and right panels correspond to the $x$ and $y$-components respectively. Note that the second order solution which is obtained using by including the convective term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ (constructed from the first order solution) shows transient growth before eventually decaying away.

of non-normal growth which relies on the non-orthogonality of the eigenvectors is not applicable here. However, for some modes, the second order solution may show hints of transient growth. We evaluate numerically $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ with parameters $\nu = 0.1, c = 1, x = 1, y = 1, kx = 1, ky = 1$ (this choice allows us to demonstrate the transient growth). Figure 1 shows the plots of the ratio of final velocity to initial velocity for small time $t < c^{-1}$. The dotted (solid) line denotes the first (second) order solution and the left (right) panel is the $x$ ($y$) component. It is clear that the linear solution $\mathbf{v}^{(1)}$ decays exponentially, whereas the second order solution that is calculated using the term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ shows a growth at early times. Thus, in this case (plane Couette) the transient growth is not seen at linear order but is hinted at second order. The main point here is that the second order Eulerian solution obtained perturbatively is in qualitative agreement with the linear Lagrangian solution. As mentioned in the text, matching the two frames orderwise is ill-defined. A true comparison of the solutions can be made only in the same frame (Eulerian) and is beyond the scope of the current work.