Apendices to “Central Limit Theorem for Linear Eigenvalue Statistics for Submatrices of Wigner Random Matrices”

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Abstract

This text contains the Appendices to [1]. We discuss the Decoupling Formula in Appendix 1 and prove technical Proposition 3.7 and Proposition 3.9 from [1] in Appendices 2 and 3 correspondingly.

1 Appendix 1

Decoupling Formula is a valuable tool developed in the resolvent analysis of statistical properies of random matrices.

Theorem 1.1 (Decoupling Formula). *[3] Let ξ be a random variable such that $\mathbb{E}\{|\xi|^{p+2}\} < \infty$ for a certain nonnegative integer p. Then for any function $f : \mathbb{R} \to \mathbb{C}$ of the class C^{p+1} with bounded derivatives $f^{(l)}, l = 1, ..., p+1$, we have*

$$\mathbb{E}\{\xi f(\xi)\} = \sum_{l=0}^{p} \frac{\kappa_{l+1}}{l!} \mathbb{E}\{f^{(l)}(\xi)\} + \varepsilon_p. \tag{1.1}$$

where κ_l denotes the lth cumulant of ξ and the remainder term ε_p admits the bound

$$|\varepsilon_p| \le C_p \mathbb{E}\{|\xi|^{p+2}\} \sup_{t\in\mathbb{R}} f^{(p+1)}(t), \ C_p \le \frac{1 + (3+2p)^{p+2}}{(p+1)!}. \tag{1.2}$$

If ξ is a Gaussian random variable with zero mean,

$$\mathbb{E}\{\xi f(\xi)\} = \mathbb{E}\{\xi^2\}\mathbb{E}\{f'(\xi)\}. \tag{1.3}$$

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2 Appendix 2

Below we prove Proposition 3.7 formulated in Section 3 of [1].

Proof. Since $\langle x^k,\ x^q\rangle_{lr} = 0$ if $k+q$ is odd, it follows by linearity that

$$\langle U_k^{\gamma_l}, U_q^{\gamma_r}\rangle_{lr} = 0, \quad \text{if } k+q \text{ is odd.} \tag{2.1}$$

We begin by computing $\langle (\frac{x}{2\sqrt{\gamma_l}})^{2k}, U_{2q}^{\gamma_r}\rangle_{lr}$ and $\langle (\frac{x}{2\sqrt{\gamma_l}})^{2k+1}, U_{2q+1}^{\gamma_r}\rangle_{lr}$. We obtain

$$\begin{aligned}
&\langle (\frac{x}{2\sqrt{\gamma_l}})^{2k}, U_{2q}^{\gamma_r}\rangle_{lr} \\
&= (\frac{1}{\sqrt{\gamma_l}})^{2k}\langle x^{2k}, U_{2q}^{\gamma_r}(x)\rangle_{lr} \\
&= \gamma_l^{-k}\sum_{p=0}^{q}(-1)^p(\frac{1}{\sqrt{\gamma_l}})^{2q-2p}\binom{2q-p}{p}\langle x^{2k}, x^{2q-2p}\rangle_{lr} \\
&= \frac{\gamma_l^{-k}\gamma_r^{-q}}{2k+1}\sum_{j=0}^{k}\sum_{p=0}^{q-j}\frac{(-1)^p\gamma_l^p(2j+1)^2}{2q-2p+1}\binom{2k+1}{k+j+1}\binom{2q-p}{p}\binom{2q-2p+1}{q-p+j+1}\gamma_l^{k-j}\gamma_r^{q-p-j}\gamma_{lr}^{2j+1} \\
&= \frac{1}{2k+1}\sum_{j=0}^{k}(2j+1)^2\binom{2k+1}{k+j+1}\left[\sum_{p=0}^{q-j}\frac{(-1)^p(2q-p)!}{p!(q-p+j+1)!(q-p-j)!}\right]\gamma_l^{-j}\gamma_r^{-j}\gamma_{lr}^{2j+1}
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
&\langle (\frac{x}{2\sqrt{\gamma_l}})^{2k+1}, U_{2q+1}^{\gamma_r}\rangle_{lr} \\
&= (\frac{1}{\sqrt{\gamma_l}})^{2k+1}\langle x^{2k+1}, U_{2q+1}^{\gamma_r}(x)\rangle \\
&= (\frac{1}{\sqrt{\gamma_l}})^{2k+1}\sum_{p=0}^{q}(-1)^p(\frac{1}{\sqrt{\gamma_r}})^{2q-2p+1}\binom{2q-p+1}{p}\langle x^{2k+1}, x^{2q-2p+1}\rangle_{lr} \\
&= \frac{\gamma_l^{-\frac{1}{2}}\gamma_r^{-\frac{1}{2}}}{2k+2}\sum_{j=0}^{k}\sum_{p=0}^{q-j}\frac{(-1)^p\gamma_r^p(2j+2)^2}{2q-2p+2}\binom{2k+2}{k+j+2}\binom{2q-p+1}{p}\binom{2q-2p+2}{q-p+j+2}\gamma_l^{-j}\gamma_r^{-p-j}\gamma_{lr}^{2j+2} \\
&= \frac{\gamma_l^{-\frac{1}{2}}\gamma_r^{-\frac{1}{2}}}{2k+2}\sum_{j=0}^{k}(2j+2)^2\binom{2k+2}{k+j+2}\left[\sum_{p=0}^{q-j}\frac{(-1)^p(2q-p+1)!}{p!(q-p+j+2)!(q-p-j)!}\right]\gamma_l^{-j}\gamma_r^{-j}\gamma_{lr}^{2j+2}.
\end{aligned} \tag{2.3}$$

Denote by

$$H_1(q,j) = \sum_{p=0}^{q-j}\frac{(-1)^p(2q-p)!}{p!(q-p+j+1)!(q-p-j)!}, \tag{2.4}$$

$$H_2(q,j) = \sum_{p=0}^{q-j}\frac{(-1)^p(2q-p+1)!}{p!(q-p+j+2)!(q-p-j)!}. \tag{2.5}$$

Then

$$\langle(\frac{x}{2\sqrt{\gamma_l}})^{2k}, U_{2q}^{\gamma_r}\rangle_{lr} = \frac{1}{2k+1}\sum_{j=0}^{k}(2j+1)^2\binom{2k+1}{k+j+1}H_1(q,j)\gamma_l^{-j}\gamma_r^{-j}\gamma_{lr}^{2j+1}, \tag{2.6}$$

$$\langle(\frac{x}{2\sqrt{\gamma_l}})^{2k+1}, U_{2q+1}^{\gamma_r}\rangle_{lr} = \frac{\gamma_l^{-\frac{1}{2}}\gamma_r^{-\frac{1}{2}}}{2k+2}\sum_{j=0}^{k}(2j+2)^2\binom{2k+2}{k+j+2}H_2(q,j)\gamma_l^{-j}\gamma_r^{-j}\gamma_{lr}^{2j+2}. \tag{2.7}$$

It follows from (2.4-2.5) that

$$H_1(q,j) = \frac{(2q)!}{(q-j)!(q+j+1)!} = {}_2F_1\left(\begin{matrix} -(q-j), -(q+j+1) \\ -2q \end{matrix} ;1\right), \tag{2.8}$$

$$H_2(q,j) = \frac{(2q+1)!}{(q-j)!(q+j+1)!} = {}_2F_1\left(\begin{matrix} -(q-j), -(q+j+2) \\ -(2q+1) \end{matrix} ;1\right), \tag{2.9}$$

where ${}_2F_1$ is a hypergeometric function. See [2] for the definition of hypergeometric functions. Below let $(x)_n = x(x+1)\cdots(x+n-1)$ denote the rising factorial. By the Chu-Vandermonde identity (see e.g. [2]), it follows that

$$H_1(q,j) = \frac{(2q)!}{(q-j)!(q+j+1)!}\ \frac{(j-q+1)_{q-j}}{(-2q)_{q-j}} = \begin{cases} 0 & 0 \le j < q \\ \frac{1}{2q+1} & j = q \end{cases} \tag{2.10}$$

$$H_2(q,j) = \frac{(2q+1)!}{(q-j)!(q+j+2)!}\ \frac{(j-q+1)_{q-j}}{(-2q-1)_{q-j}} = \begin{cases} 0 & 0 \le j < q \\ \frac{1}{2q+2} & j = q \end{cases} \tag{2.11}$$

Therefore, for $k = 0, 1, \cdots, q-1$, we get that $\langle(\frac{x}{2\sqrt{\gamma_l}})^{2k}, U_{2q}^{\gamma_r}\rangle_{lr} = 0$ and also $\langle(\frac{x}{2\sqrt{\gamma_l}})^{2k+1}, U_{2q+1}^{\gamma_r}\rangle_{lr} = 0$. With $k = q$ we obtain

$$\begin{aligned} \langle(\frac{x}{2\sqrt{\gamma_l}})^{2k}, U_{2k}^{\gamma_r}\rangle_{lr} &= \frac{1}{2k+1}\sum_{j=0}^{k}(2j+1)^2\binom{2k+1}{k+j+1}H_1(k,j)\gamma_l^{-j}\gamma_r^{-j}\gamma_{lr}^{2j+1} \\ &= \frac{(2k+1)^2}{2k+1}\binom{2k+1}{2k+1}H_1(k,k)\gamma_l^{-k}\gamma_r^{-k}\gamma_{lr}^{2k+1} \\ &= \frac{\gamma_{lr}^{2k+1}}{\gamma_l^k\gamma_r^k} \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \langle(\frac{x}{2\sqrt{\gamma_l}})^{2k+1}, U_{2k+1}^{\gamma_r}\rangle_{lr} &= \frac{\gamma_l^{-\frac{1}{2}}\gamma_r^{-\frac{1}{2}}}{2k+2}\sum_{j=0}^{k}(2j+2)^2\binom{2k+2}{k+j+2}H_2(k,j)\gamma_l^{-j}\gamma_r^{-j}\gamma_{lr}^{2j+2} \\ &= \gamma_l^{-\frac{1}{2}}\gamma_r^{-\frac{1}{2}}\frac{(2k+1)^2}{2k+1}\binom{2k+1}{2k+1}H_2(k,k)\gamma_l^{-k}\gamma_r^{-k}\gamma_{lr}^{2k+1} \\ &= \frac{\gamma_{lr}^{2k+2}}{\sqrt{\gamma_l\gamma_r}^{2k+1}}. \end{aligned} \tag{2.13}$$

Thus, for $k < q$,

$$\langle U_{2k}^{\gamma_l}, U_{2q}^{\gamma_r}\rangle_{lr} = 0, \quad \langle U_{2k+1}^{\gamma_l}, U_{2q+1}^{\gamma_r}\rangle_{lr} = 0, \tag{2.14}$$

and for $k = q$

$$\langle U_{2k}^{\gamma_l}, U_{2k}^{\gamma_r}\rangle_{lr} = \langle (\frac{x}{2\sqrt{\gamma_l}})^{2k}, U_{2k}^{\gamma_r}\rangle_{lr} = \frac{\gamma_{lr}^{2k+1}}{\gamma_l^k \gamma_r^k}, \tag{2.15}$$

$$\langle U_{2k+1}^{\gamma_l}, U_{2k+1}^{\gamma_r}\rangle_{lr} = \langle (\frac{x}{2\sqrt{\gamma_l}})^{2k+1}, U_{2k+1}^{\gamma_r}\rangle_{lr} = \frac{\gamma_{lr}^{2k+2}}{\sqrt{\gamma_l \gamma_r}^{2k+1}}. \tag{2.16}$$

This completes the proof of Proposition 3.7, which is the diagonalization part of Lemma 2.5 of [1]. □

3 Appendix 3

Below we prove Proposition 3.9 formulated in Section 3 of [1].

Proof. First it will be argued by approximation that $\langle \cdot, \cdot \rangle_{lr}$ can be extended to the class of functions $\mathcal{H}_{\frac{3}{2}+\epsilon}$, and then the bilinear form will be explicitly computed. It will be sufficient to approximate f, g below by truncated polynomials with rational coefficients in $\mathcal{H}_{\frac{3}{2}+\epsilon}$, because of the estimate (3.3). Recall that functions of the Schwartz class are dense in $\mathcal{H}_s$, so after a triangle inequality argument it is in fact sufficient to suppose that $f, g \in \mathcal{S}(\mathbb{R})$. Let $h \in \mathcal{C}_c^\infty$ be a function so that $h(x) = 1$ for $x \in [-3, 3]$, $h(x) = 0$ for $x \notin [-4, 4]$ and is smoothly interpolated in between. Note that with overwhelming probability, the eigenvalues of the submatrices concentrate in the support of μ_{sc}. As a consequence we may suppose that f, g are supported in $[-3, 3]$. We give a density argument. It is sufficient to argue that $||hf - hp_j||_{\frac{3}{2}+\epsilon}$ and $||hg - hq_j||_{\frac{3}{2}+\epsilon}$ converge to 0 as $j \to \infty$, where $\{p_j\}, \{q_j\}$ are appropriately chosen sequences of polynomials with rational coefficients. Note that $hf = f$ and $hg = g$. We now focus on estimating $||f - hp_j||_{\frac{3}{2}+\epsilon}$. Since f is a Schwarz function, we have that $f \in \mathcal{H}_2$. We note that

$$\int_{-\infty}^{\infty} |\widehat{f}(t)|^2 \, (1 + |t|)^{3+\epsilon} \, dt \leq \int_{-\infty}^{\infty} |\widehat{f}(t)|^2 \, (1 + |t|)^4 \, dt, \tag{3.1}$$

so it will be sufficient to approximate f in the larger $|| \cdot ||_2$ norm. Also, since

$$||f||_2^2 = \int_{-\infty}^{\infty} |\widehat{f}(t)|^2 \, (1 + |t|)^4 \, dt \leq Const \left[\int_{-\infty}^{\infty} |\widehat{f}(t)|^2 dt + \int_{-\infty}^{\infty} t^4 |\widehat{f}(t)|^2 dt \right], \tag{3.2}$$

we only need to approximate the two terms on the right hand side. Consider polynomials $\{p_j\}$ with rational coefficients so that $\sup_{-4 \leq x \leq 4} |f''(x) - p_j(x)| \to 0$ as $j \to \infty$. Then denote by $\tilde{p}_j(x) = \int_{-4}^{x} p_j(t) dt$, and $\tilde{\tilde{p}}_j(x) = \int_{-4}^{x} \tilde{p}_j(t) dt$. As a consequence of Parseval's theorem, it will be sufficient to show that

$$||f - h\tilde{\tilde{p}}_j||_{L^2([-4,4])} \to 0 \text{ and } \; ||f'' - (h\tilde{\tilde{p}}_j)''||_{L^2([-4,4])} \to 0, \text{ as } j \to \infty. \tag{3.3}$$

But observe that

$$||f'' - (h\tilde{\tilde{p}}_j)''||_{L^2([-4,4])} \;\leq\; ||f'' - hp_j||_{L^2([-4,4])} + ||h''\tilde{\tilde{p}}_j + 2h'\tilde{p}_j||_{L^2([-4,4])}. \tag{3.4}$$

The first term on the right hand side converges to 0 because of the uniform approximation. Noting that $h'(x) = 0$ and $h''(x) = 0$ on $(-3, 3)$, and also that $\tilde{p}_j$ and $\tilde{\tilde{p}}_j$ converge to 0 uniformly on $[-4,-3)\cup(3,4]$, it follows that the second term on the right hand side converges to 0 as well. Finally we observe that

$$\begin{aligned} ||f-h\tilde{\tilde{p}}_j||^2_{L^2([-4,4])} &= \int_{-4}^{4}|f(x)-h(x)\tilde{\tilde{p}}_j(x)|^2dx \\ &\leq \int_{-4}^{4}h^2(x)\left|\int_{-4}^{x}\int_{-4}^{t}[f''(u)-p_j(u)]dudt\right|^2 dx \\ &\leq Const\cdot\left(\sup_{-4\leq u\leq 4}\left|f''(u)-p_j(u)\right|\right)^2 \end{aligned} \tag{3.5}$$

It follows that $||f-h\tilde{\tilde{p}}_j||^2_{L^2([-4,4])}\to 0$ because of the uniform approximation. This completes the approximation argument, so we now turn toward computing the bilinear form.

Setting

$$f_k=\frac{1}{2\pi\gamma_l}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}f(x)U_k^{\gamma_l}(x)\sqrt{4\gamma_l-x^2}dx,\quad g_k=\frac{1}{2\pi\gamma_r}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}g(y)U_k^{\gamma_r}(y)\sqrt{4\gamma_r-y^2}dy, \tag{3.6}$$

it follows that

$$\begin{aligned} &\langle f,g\rangle_{lr} \\ &= \langle \sum_{k=0}^{\infty}f_kU_k^{\gamma_l}(x),\ \sum_{p=0}^{\infty}g_pU_p^{\gamma_r}(x)\ \rangle_{lr} \\ &= \sum_{k=0}^{\infty}\sum_{p=0}^{\infty}f_kg_p\langle U_k^{\gamma_l},U_p^{\gamma_r}\rangle_{lr} \\ &= \sum_{k=0}^{\infty}f_kg_k\frac{\gamma_{lr}^{k+1}}{\gamma_l^{k/2}\gamma_r^{k/2}} \\ &= \frac{1}{4\pi^2\gamma_l\gamma_r}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}f(x)g(y)\left[\sum_{k=0}^{\infty}U_k^{\gamma_l}(x)U_k^{\gamma_r}(y)\frac{\gamma_{lr}^{k+1}}{\gamma_l^{k/2}\gamma_r^{k/2}}\right]\sqrt{4\gamma_l-x^2}\sqrt{4\gamma_r-y^2}dydx. \end{aligned}$$

It also follows, using (3.140), that a.s.

$$\begin{aligned} &\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}\left\{P^{(l)}f(M^{(l)})\cdot P^{(l,r)}\cdot g(M^{(r)})P^{(r)}\right\} \\ &= \frac{1}{4\pi^2\gamma_l\gamma_r}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}f(x)g(y)\left[\sum_{k=0}^{\infty}U_k^{\gamma_l}(x)U_k^{\gamma_r}(y)\frac{\gamma_{lr}^{k+1}}{\gamma_l^{k/2}\gamma_r^{k/2}}\right]\sqrt{4\gamma_l-x^2}\sqrt{4\gamma_r-y^2}dydx. \end{aligned} \tag{3.7}$$

Proposition 3.9 follows. □

References

[1] L.Li, M.Reed, A. Soshnikov. *Central Limit Theorem for Linear Eigenvalue Statistics for Submatrices of Wigner Random Matrices.* Front. Appl. Math. Stat. 6:17, 2020.

[2] G. Andrews, R. Askey, R. Roy. *Special Functions.* Cambridge University Press, New York, 1999.

[3] A. Lytova, L. Pastur. *Central limit theorem for linear eigenvalue statistics of random matrices with independent entries.* Annals of Probability 37(5): 1778–1840, 2009.