CARMA Approximations and Estimation
Supplementary Material

Silvia Lavagnini
Department of Mathematics, University of Oslo
0316 Blindern, Norway
Email: silval@math.uio.no

Appendix A. Proofs of the main results

We report in this section the proofs to the main results in the order they appear in the paper.

A.1 Proof of Lemma 2.1

PROOF. By integration via parts applied to the product $L(s)\phi_\varepsilon(t-s)$, for $0 \le s \le t$, we get

$$L(t)\phi_\varepsilon(0) = L(0)\phi_\varepsilon(t) + \int_0^t \phi_\varepsilon(t-s)dL(s) - \int_0^t L(s)\phi'_\varepsilon(t-s)ds.$$

The claim follows since $\phi_\varepsilon(0) = 0$ and $L(0) = 0$. $\square$

A.2 Proof of Proposition 2.2

PROOF. Let us look at the increments of the process for $0 < t_1 < t_2 < t_3 < t_4 < T$, namely,

$$\begin{aligned} L_\varepsilon(t_2) - L_\varepsilon(t_1) &= \int_0^{t_2} \phi_\varepsilon(t_2-s)dL(s) - \int_0^{t_1} \phi_\varepsilon(t_1-s)dL(s) \\ &= \int_0^{t_1} \left(\phi_\varepsilon(t_2-s) - \phi_\varepsilon(t_1-s)\right) dL(s) + \int_{t_1}^{t_2} \phi_\varepsilon(t_2-s)dL(s), \\ L_\varepsilon(t_4) - L_\varepsilon(t_3) &= \int_0^{t_3} \left(\phi_\varepsilon(t_4-s) - \phi_\varepsilon(t_3-s)\right) dL(s) + \int_{t_3}^{t_4} \phi_\varepsilon(t_4-s)dL(s). \end{aligned}$$

We see that these are not independent, and L_ε is thus not a Lévy process. $\square$

A.3 Proof of Theorem 2.4

PROOF. We need to show that

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\left[\int_0^T \Big(L_\varepsilon(t) - L(t)\Big)^2 dt\right] = 0. \tag{A.1}$$

For the process $Z(t) := L_\varepsilon(t) - L(t)$, from equation (2.3) and the identity $L(t) = \int_0^t dL(s)$, we write that $Z(t) = \int_0^t \left(\phi_\varepsilon(t-s) - 1\right) dL(s)$. By means of the Fubini theorem, switching in the integration equation (A.1) with respect to t with certain expectations, we get something of the form $\int_0^T \mathbb{E}\left[Z(t)^2\right] dt$, where

$\mathbb{E}\left[Z(t)^2\right]$ is the second moment of $Z(t)$. Denoting $M_{Z(t)}(u) := \mathbb{E}[e^{iuZ(t)}]$, the characteristic function of $Z(t)$, we know that

$$\mathbb{E}\left[Z(t)^2\right] = -\left.\frac{d^2 M_{Z(t)}(u)}{du^2}\right|_{u=0}. \tag{A.2}$$

From (Benth, Di Persio, and Lavagnini 2018, Appendix A) the function $M_{Z(t)}$ takes the form

$$M_{Z(t)}(u) = \exp\left\{\int_0^t \eta\big(u\cdot\big(\phi_\varepsilon(t-s)-1\big)\big)\,ds\right\}, \tag{A.3}$$

η being the characteristic function of the Lévy process L. In particular, since L is a martingale, as a consequence of equation (2.2), from (Tankov 2003, Theorem 3.1) the function η takes the form

$$\eta(w) = -\frac{1}{2}\Sigma w^2 + \int_{-\infty}^{+\infty}\left(e^{iwx}-1-iwx\right)\nu(dx),$$

so that $\eta(0) = 0$, but also $\eta'(0) = 0$ since L has zero mean, and $\eta''(0) = -\Sigma - \int_{-\infty}^{+\infty} x^2\nu(dx)$. From equation (A.2) and (A.3), we then get

$$\mathbb{E}\left[Z(t)^2\right] = \left(\Sigma + \int_{-\infty}^{+\infty} x^2\nu(dx)\right)\int_0^t \left(\phi_\varepsilon(t-s)-1\right)^2 ds = \sigma^2\int_0^t \left(\phi_\varepsilon(s)-1\right)^2 ds,$$

and the expectation in equation (A.1) is bounded by

$$\mathbb{E}\left[\int_0^T \Big(L_\varepsilon(t)-L(t)\Big)^2 dt\right] \le T\sigma^2\int_0^T \left(\phi_\varepsilon(s)-1\right)^2 ds.$$

From equation (2.6), we know that $|\phi_\varepsilon(s)| \le 1$ and $\lim_{\varepsilon\downarrow 0}\phi_\varepsilon(s) = 1$ for every $s\in[0,T]$. The claim then follows by applying the Dominated Convergence Theorem. $\square$

A.4 Proof of Proposition 2.7

PROOF. Starting from Lemma 2.5, we consider the asymptotic expansion (2.9) truncated at the second order in $x = \frac{T}{\varepsilon}$ and $x = \frac{\sqrt{2}T}{\varepsilon}$, respectively. We get the following:

$$\begin{aligned}\mathbb{E}\left[\left(L_\varepsilon(T)-L(T)\right)^2\right] \le{}& 4\sigma^2 T\frac{e^{-\frac{T^2}{\varepsilon^2}}}{2\pi}\left(\frac{\varepsilon^3}{T^3}-\frac{\varepsilon}{T}\right)^2 + \\ &+ 4\sigma^2\varepsilon\left(\frac{1}{\sqrt{2\pi}} + 2\frac{e^{-\frac{T^2}{\varepsilon^2}}}{2\pi}\left(\frac{\varepsilon^3}{T^3}-\frac{\varepsilon}{T}\right) - \frac{1}{\sqrt{\pi}}\left(\frac{1}{2}+\frac{e^{-\frac{T^2}{\varepsilon^2}}}{\sqrt{2\pi}}\left(\frac{\varepsilon^3}{2\sqrt{2}T^3}-\frac{\varepsilon}{\sqrt{2}T}\right)\right)\right).\end{aligned}$$

After simplification, we obtain

$$\mathbb{E}\left[\left(L_\varepsilon(T)-L(T)\right)^2\right] \le 2\sigma^2\frac{e^{-\frac{T^2}{\varepsilon^2}}}{\pi}\left(\frac{\varepsilon^6}{T^5}-\frac{\varepsilon^4}{2T^3}\right) + \frac{2(\sqrt{2}-1)}{\sqrt{\pi}}\sigma^2\varepsilon.$$

By considering the Taylor expansion for the exponential function truncated at the first order, namely $e^x \approx 1 + \frac{1}{x}$ where $x = -\frac{T^2}{\varepsilon^2}$ converges to $-\infty$ when ε approaches 0, for $K_1 := \frac{2(\sqrt{2}-1)}{\sqrt{\pi}}\sigma^2$, we obtain the result. $\square$

A.5 Proof of Lemma 3.3

PROOF. The result follows by applying the integration by use of parts from itô's theory to equation (3.5). From the differentiability of L_ε and equation (2.7), we can write that

$$X_\varepsilon(t) = \phi'_\varepsilon(0)\int_0^t L(s)g(t-s)ds + \int_0^t \left(\int_0^s \phi''_\varepsilon(s-v)L(v)dv\right) g(t-s)ds. \tag{A.4}$$

By use of the stochastic Fubini Theorem, for the second term on the right-hand side, it holds that

$$\int_0^t \left(\int_0^s \phi''_\varepsilon(s-v)L(v)dv\right) g(t-s)ds = \int_0^t \left(\int_v^t \phi''_\varepsilon(s-v)g(t-s)ds\right) L(v)dv. \tag{A.5}$$

By integration of the parts, the inner integral on the right-hand side of equation (A.5) becomes

$$\int_v^t \phi''_\varepsilon(s-v)g(t-s)ds = \phi'_\varepsilon(t-v)g(0) - \phi'_\varepsilon(0)g(t-v) - \int_v^t \phi'_\varepsilon(s-v)\frac{\partial}{\partial s}g(t-s)ds,$$

where $\frac{\partial}{\partial s}g(t-s) = -g'(t-s)$. Let us focus on $p > q+1$. Since $g(0) = 0$, switching the roles of s and v and putting together the results of the last two equations, equation (A.4) becomes

$$X_\varepsilon(t) = \int_0^t \left(\int_v^t \phi'_\varepsilon(s-v)g'(t-s)ds\right) L(v)dv = \int_0^t \left(\int_0^{t-s} \phi'_\varepsilon(v)g'(t-s-v)dv\right) L(s)ds,$$

which proves the statement. Similarly, for $p = q+1$, since $g(0) = 1$, we get

$$X_\varepsilon(t) = \int_0^t \left(\phi'_\varepsilon(t-s) + \int_0^{t-s} \phi'_\varepsilon(v)g'(t-s-v)dv\right) L(s)ds,$$

which concludes the proof. $\square$

A.6 Proof of Proposition 3.4

PROOF. We want to prove that $\lim_{\varepsilon\downarrow 0} h_\varepsilon(x) = h(x)$ for $x > 0$. Let us first consider $p > q+1$. From Lemma 3.2 and equation (3.11), a sufficient condition is that

$$\lim_{\varepsilon\downarrow 0}\left\{\int_0^x e^{\lambda_i(x-v)}\psi_\varepsilon(v)dv\right\} = e^{\lambda_i x} \quad \text{for } i = 1, ..., p. \tag{A.6}$$

For ψ_ε in equation (2.5), the integral on the left hand side of equation (A.6) becomes

$$\int_0^x e^{\lambda_i(x-v)}\psi_\varepsilon(v)dv = 2e^{\lambda_i x + \frac{1}{2}\varepsilon^2\lambda_i^2}\left[\Phi\left(\frac{x+\varepsilon^2\lambda_i}{\varepsilon}\right) - \Phi\left(\varepsilon\lambda_i\right)\right] \xrightarrow{\varepsilon\downarrow 0} e^{\lambda_i x}, \tag{A.7}$$

for Φ the cumulative distribution function of a standard Gaussian variable. For $p = q+1$, we need $\lim_{\varepsilon\downarrow 0}\psi_\varepsilon(x) = 0$ for every $x > 0$, which is trivially satisfied. This concludes the proof. $\square$

A.7 Proof of Theorem 3.5

PROOF. By means of Lemma 3.3, we need to show that

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\int_0^T\left(\int_0^t\Big(h_\varepsilon(t-s)-h(t-s)\Big)L(s)ds\right)^2dt\right]=0.$$

By the Cauchy-Schwarz inequality, the inner integral of the last equation is bounded by

$$\left(\int_0^t\left(h_\varepsilon(t-s)-h(t-s)\right)L(s)ds\right)^2\leq\int_0^t\left(h_\varepsilon(s)-h(s)\right)^2ds\cdot\int_0^t L^2(s)ds,$$

and $\mathbb{E}\left[\int_0^T L^2(s)ds\right]=\sigma^2\frac{T^2}{2}$, for σ^2 in equation (2.1). The whole expectation is bounded by

$$\mathbb{E}\left[\int_0^T\left(\int_0^t\Big(h_\varepsilon(t-s)-h(t-s)\Big)L(s)ds\right)^2dt\right]\leq\sigma^2\frac{T^3}{2}\int_0^T\Big(h_\varepsilon(s)-h(s)\Big)^2ds. \tag{A.8}$$

From equation (3.11) and (A.7), for $p>q+1$ we get

$$|h_\varepsilon(x)|\leq\sum_{i,j=1}^p\left|\tilde{b}_j\right|\left|\lambda_i^j\right||\gamma_i|\left|\int_0^x e^{\lambda_i(x-v)}\psi_\varepsilon(v)dv\right|\leq 4\sum_{i,j=1}^p\left|\tilde{b}_j\right|\left|\lambda_i^j\right||\gamma_i|\left|e^{\lambda_i x+\frac{1}{2}\varepsilon^2\lambda_i^2}\right|<+\infty,$$

so that, from the boundedness of h_ε and Proposition 3.4, by use of the Dominated Convergence Theorem we conclude that $X_\varepsilon\xrightarrow{\varepsilon\downarrow 0}X$ in $L^2\left([0,T]\times\Omega\right)$. The case $p=q+1$ is similarly performed from the boundedness of ψ_ε. □

A.8 Proof of Proposition 3.6

PROOF. Proceeding as in the proof of Theorem 3.5, similarly to equation (A.8), we write that

$$\mathbb{E}\left[\left(X_\varepsilon(T)-X(T)\right)^2\right]=\mathbb{E}\left[\left(\int_0^T\Big(h_\varepsilon(t-s)-h(t-s)\Big)L(s)ds\right)^2\right]\leq\sigma^2\frac{T^3}{2}\Big(h_\varepsilon(T)-h(T)\Big)^2.$$

The proof comes from the spectral representation of h and h_ε in Lemma 3.2 and equation (3.11), respectively, together with the Taylor series for the Gaussian distribution function in equation (2.8) for small values of x, and the asymptotic expansion in equation (2.9) for large values of x, which are both truncated at the first order. Introducing the constant

$$K_2:=\sigma^2\frac{T^3}{\pi}\sum_{i,j,\ell,k=1}^p\tilde{b}_j\tilde{b}_k\lambda_i^{j+1}\lambda_\ell^{k+1}\gamma_i\gamma_\ell e^{(\lambda_i+\lambda_\ell)T},$$

we conclude the proof. □

A.9 Proof of Proposition 3.7

We start recalling the following result.

THEOREM A.1. *If $Z_n : \Omega \to \mathbb{R}^k$ is a Gaussian random variable for every n—and $Z_n \to Z$ in $L^2(\Omega)$ as $n \to \infty$—then Z also has Gaussian distribution.*

PROOF. See (Øksendal 2013, Appendix A) .

Since, for fixed $t \geq 0$, $X_\varepsilon(t)$ is a random variable, the idea is to apply Theorem A.1 to prove that $X_\varepsilon(t)$ has Gaussian distribution for every $t \geq 0$. Considering for $n > 0$ the same partition $\Pi(n)$ introduced in Section 4, for $m = m(t)$, $1 \leq m \leq n$ such that $s_m^{(n)} = t$, and $\Delta s_j^{(n)} = \Delta$ for every $j = 1, \ldots, n$, we define the process $\{X_\varepsilon^n(t),\, t \geq 0\}$ as the approximating Riemann sum of the process $\{X_\varepsilon(t),\, t \geq 0\}$ in Lemma 3.3, namely,

$$X_\varepsilon^n(t) := \sum_{j=1}^{m} h_\varepsilon(t - s_j^{(n)}) B(s_j^{(n)}) \Delta s_j^{(n)}. \tag{A.9}$$

This is normally distributed as states the following lemma.

LEMMA A.2. *The process $\{X_\varepsilon^n(t),\, t \geq 0\}$ has Gaussian distribution.*

PROOF. From equation (A.9), $X_\varepsilon^n(t)$ is the linear combination of samples from a Brownian motion, and it thus has Gaussian distribution. □

To prove that X_ε has Gaussian distribution, by means of Theorem A.1 and Lemma A.2, it is then sufficient to show that X_ε^n converges to X_ε in $L^2([0,T] \times \Omega)$.

PROPOSITION A.3. *For every $\varepsilon > 0$, the following convergence holds:*

$$\lim_{n\uparrow\infty} \mathbb{E}\left[\int_0^T \left| X_\varepsilon(t) - X_\varepsilon^n(t) \right|^2 dt \right] = 0. \tag{A.10}$$

PROOF. For the partition $\Pi(n)$, we rewrite X_ε in Lemma 3.3 and X_ε^n in equation (A.9) by

$$X_\varepsilon(t) = \sum_{j=1}^{m} \int_{s_{j-1}^{(n)}}^{s_j^{(n)}} h_\varepsilon(t-s) B(s) ds \qquad \text{and} \qquad X_\varepsilon^n(t) = \sum_{j=1}^{m} \int_{s_{j-1}^{(n)}}^{s_j^{(n)}} h_\varepsilon(t - s_j^{(n)}) B(s_j^{(n)}) ds.$$

By introducing

$$a_j := \int_{s_{j-1}^{(n)}}^{s_j^{(n)}} \big(h_\varepsilon(t-s) B(s) - h_\varepsilon(t - s_j^{(n)}) B(s_j^{(n)}) \big) ds, \quad j = 1, \ldots, n,$$

we then estimate the expectation in equation (A.10) by

$$\mathbb{E}\left[\int_0^T \left| \sum_{j=1}^{m} a_j \right|^2 dt \right] \leq \int_0^T \sum_{j=1}^{n} \mathbb{E}\left[|a_j|^2\right] dt + 2 \int_0^T \sum_{\substack{j,k=1 \\ j<k}}^{n} \mathbb{E}\left[|a_j||a_k|\right] dt. \tag{A.11}$$

Let us focus on the product $|a_j||a_k|$. It holds that

$$|a_j||a_k| \leq \int_{s_{j-1}^{(n)}}^{s_j^{(n)}} \int_{s_{k-1}^{(n)}}^{s_k^{(n)}} \Big\{ \Big|h_\varepsilon(t-s)h_\varepsilon(t-v)B(s)B(v)\Big| + \Big|h_\varepsilon(t-s)h_\varepsilon(t-s_k^{(n)})B(s)B(s_k^{(n)})\Big| + \\ + \Big|h_\varepsilon(t-s_j^{(n)})h_\varepsilon(t-v)B(s_j^{(n)})B(v)\Big| + \Big|h_\varepsilon(t-s_j^{(n)})h_\varepsilon(t-s_k^{(n)})B(s_j^{(n)})B(s_k^{(n)})\Big|\Big\} dvds,$$

where, for $j < k$, it holds that $s_{j-1}^{(n)} \leq s \leq s_j^{(n)} \leq s_{k-1}^{(n)} \leq v \leq s_k^{(n)} \leq T$, so that, by Lemma A.4,

$$\mathbb{E}\left[|a_j||a_k|\right] \leq \int_{s_{j-1}^{(n)}}^{s_j^{(n)}} \int_{s_{k-1}^{(n)}}^{s_k^{(n)}} \Big\{ \Big|h_\varepsilon(t-s)h_\varepsilon(t-v)\Big| \left(s + \sqrt{s}\sqrt{v-s}\right) + \\ + \Big|h_\varepsilon(t-s)h_\varepsilon(t-s_k^{(n)})\Big| \left(s + \sqrt{s}\sqrt{s_k^{(n)}-s}\right) + \Big|h_\varepsilon(t-s_j^{(n)})h_\varepsilon(t-v)\Big| \left(s_j^{(n)} + \sqrt{s_j^{(n)}}\sqrt{v-s_j^{(n)}}\right) + \\ + \Big|h_\varepsilon(t-s_j^{(n)})h_\varepsilon(t-s_k^{(n)})\Big| \left(s_j^{(n)} + \sqrt{s_j^{(n)}}\sqrt{s_k^{(n)}-s_j^{(n)}}\right)\Big\} dvds \leq 8K^2T\Delta^2,$$

and $K > 0$ is a constant such that $|h_\varepsilon(x)| \leq K$. We focus now on $|a_j|^2$ in equation (A.11). By adding the terms $\pm h_\varepsilon(t-s_j^{(n)})B(s)$ and using the inequality $(a+b)^2 \leq 2(a^2+b^2)$, we get

$$|a_j|^2 \;\leq\; 2\left|\int_{s_{j-1}^{(n)}}^{s_j^{(n)}} B(s)\big(h_\varepsilon(t-s) - h_\varepsilon(t-s_j^{(n)})\big)ds\right|^2 \;+\; 2\left|\int_{s_{j-1}^{(n)}}^{s_j^{(n)}} h_\varepsilon(t-s_j^{(n)})\big(B(s) - B(s_j^{(n)})\big)ds\right|^2,$$

and by Hölder's inequality

$$\mathbb{E}\left[\left|\int_{s_{j-1}^{(n)}}^{s_j^{(n)}} B(s)\big(h_\varepsilon(t-s) - h_\varepsilon(t-s_j^{(n)})\big)ds\right|^2\right] \leq \int_{s_{j-1}^{(n)}}^{s_j^{(n)}} sds \int_{s_{j-1}^{(n)}}^{s_j^{(n)}} \big|h_\varepsilon(t-s) - h_\varepsilon(t-s_j^{(n)})\big|^2 ds,$$

$$\mathbb{E}\left[\left|\int_{s_{j-1}^{(n)}}^{s_j^{(n)}} h_\varepsilon(t-s_j^{(n)})\big(B(s) - B(s_j^{(n)})\big)ds\right|^2\right] \leq \int_{s_{j-1}^{(n)}}^{s_j^{(n)}} \Big|h_\varepsilon(t-s_j^{(n)})\Big|^2 ds \int_{s_{j-1}^{(n)}}^{s_j^{(n)}} (s_j^{(n)} - s)ds,$$

so that, with similar argumentations as before, we can write that

$$\int_0^T \sum_{j=1}^n \mathbb{E}\left[|a_j|^2\right] dt \leq 12K^2T^3\Delta.$$

Combining all the results, for $\Delta = T/n$, we get that

$$\mathbb{E}\left[\int_0^T \Big|X_\varepsilon(t) - X_n(t)\Big|^2 dt\right] \leq 28K^2T^3\Delta \xrightarrow{n\uparrow+\infty} 0,$$

which proves the claim. $\square$

LEMMA A.4. *For every $t \leq s$ the inequality holds*

$$\mathbb{E}\left[|B(t)B(s)|\right] \leq t + \sqrt{t}\,\sqrt{s-t}.$$

PROOF. For $B(s) = B(s) - B(t) + B(t)$, since $B(t) \perp B(s) - B(t)$, by the triangular inequality

$$\mathbb{E}\left[|B(t)B(s)|\right] \leq \mathbb{E}\left[|B(t)|^2 + |B(t)|\,|B(s) - B(t)|\right] = t + \mathbb{E}\left[|B(t)|\right]\mathbb{E}\left[|B(s) - B(t)|\right].$$

For $B(t) \sim \mathcal{N}(0,t)$ and $B(s) - B(t) \sim \mathcal{N}(0, s-t)$, we also get $\mathbb{E}\left[|B(t)|\right] = \sqrt{\frac{2}{\pi}}\sqrt{t}$ and $\mathbb{E}\left[|B(s) - B(t)|\right] = \sqrt{\frac{2}{\pi}}\sqrt{s-t}$. Combining these results, we obtain the claim. □

A.10 Proof of Proposition 4.1

PROOF. Starting from equation (4.1), we rewrite L_ε^Δ at time $t = T$ as

$$L_\varepsilon^\Delta(T) = \sum_{j=1}^{n} \int_{s_{j-1}^{(n)}}^{s_j^{(n)}} \phi_\varepsilon(T - s_{j-1}^{(n)})dL(s),$$

so that the difference $L_\varepsilon^\Delta(T) - L_\varepsilon(T)$ becomes

$$L_\varepsilon^\Delta(T) - L_\varepsilon(T) = \sum_{j=1}^{n} \int_{s_{j-1}^{(n)}}^{s_j^{(n)}} \left(\phi_\varepsilon(T - s_{j-1}^{(n)}) - \phi_\varepsilon(T-s)\right) dL(s). \tag{A.12}$$

By applying the integration by parts formula to the product $L(s)\left(\phi_\varepsilon(T - s_{j-1}^{(n)}) - \phi_\varepsilon(T-s)\right)$ for $s_{j-1}^{(n)} \leq s \leq s_j^{(n)}$, each of the integrals in the sum of equation (A.12) can be rewritten as

$$\begin{aligned}
&\int_{s_{j-1}^{(n)}}^{s_j^{(n)}} \left(\phi_\varepsilon(T - s_{j-1}^{(n)}) - \phi_\varepsilon(T-s)\right) dL(s) \\
&= L(s_j^{(n)})\left(\phi_\varepsilon(T - s_{j-1}^{(n)}) - \phi_\varepsilon(T - s_j^{(n)})\right) - \int_{s_{j-1}^{(n)}}^{s_j^{(n)}} L(s)\psi_\varepsilon(T-s)ds \\
&= \int_{s_{j-1}^{(n)}}^{s_j^{(n)}} \left(L(s_j^{(n)}) - L(s)\right)\psi_\varepsilon(T-s)ds,
\end{aligned} \tag{A.13}$$

for $\psi_\varepsilon = \phi'_\varepsilon$. Combining equation (A.12) and (A.13), and since the increments for L are independent, the error on the left-hand side in Proposition 4.1 becomes

$$\begin{aligned}
\mathbb{E}\left[\left(L^\Delta_\varepsilon(T) - L_\varepsilon(T)\right)^2\right] &= \mathbb{E}\left[\left(\sum_{j=1}^n \int_{s^{(n)}_{j-1}}^{s^{(n)}_j} \left(\phi_\varepsilon(T - s^{(n)}_{j-1}) - \phi_\varepsilon(T-s)\right) dL(s)\right)^2\right] \\
&= \sum_{j=1}^n \mathbb{E}\left[\left(\int_{s^{(n)}_{j-1}}^{s^{(n)}_j} \left(\phi_\varepsilon(T - s^{(n)}_{j-1}) - \phi_\varepsilon(T-s)\right) dL(s)\right)^2\right] \\
&= \sum_{j=1}^n \mathbb{E}\left[\left(\int_{s^{(n)}_{j-1}}^{s^{(n)}_j} \left(L(s^{(n)}_j) - L(s)\right) \psi_\varepsilon(T-s)ds\right)^2\right] \\
&\leq \sum_{j=1}^n \mathbb{E}\left[\int_{s^{(n)}_{j-1}}^{s^{(n)}_j} \left(L(s^{(n)}_j) - L(s)\right)^2 ds\right] \cdot \int_{s^{(n)}_{j-1}}^{s^{(n)}_j} \psi^2_\varepsilon(T-s)ds, \qquad \text{(A.14)}
\end{aligned}$$

where we used the Cauchy-Schwarz inequality. It can be calculated that

$$\int_{s^{(n)}_{j-1}}^{s^{(n)}_j} \psi^2_\varepsilon(T-s)ds = \frac{2}{\sqrt{\pi}\varepsilon}\left(\Phi\left(\frac{\sqrt{2}}{\varepsilon}(T - s^{(n)}_{j-1})\right) - \Phi\left(\frac{\sqrt{2}}{\varepsilon}(T - s^{(n)}_j)\right)\right),$$

while, by means of equation (2.1),

$$\mathbb{E}\left[\int_{s^{(n)}_{j-1}}^{s^{(n)}_j} \left(L(s^{(n)}_j) - L(s)\right)^2 ds\right] = \sigma^2 \int_{s^{(n)}_{j-1}}^{s^{(n)}_j} \left(s^{(n)}_j - s\right) ds = \sigma^2\frac{\Delta^2}{2}.$$

Equation (A.14) then becomes

$$\begin{aligned}
\mathbb{E}\left[\left(L^\Delta_\varepsilon(T) - L_\varepsilon(T)\right)^2\right] &\leq \sigma^2\frac{\Delta^2}{\sqrt{\pi}\varepsilon}\sum_{j=1}^n\left(\Phi\left(\frac{\sqrt{2}}{\varepsilon}(T - s^{(n)}_{j-1})\right) - \Phi\left(\frac{\sqrt{2}}{\varepsilon}(T - s^{(n)}_j)\right)\right) \\
&= \sigma^2\frac{\Delta^2}{\sqrt{\pi}\varepsilon}\left(\Phi\left(\frac{\sqrt{2}}{\varepsilon}(T - s^{(n)}_0)\right) - \Phi\left(\frac{\sqrt{2}}{\varepsilon}(T - s^{(n)}_n)\right)\right) \\
&= \sigma^2\frac{\Delta^2}{\sqrt{\pi}\varepsilon}\left(\Phi\left(\frac{\sqrt{2}}{\varepsilon}T\right) - \frac{1}{2}\right) \leq \frac{\sigma^2}{2\sqrt{\pi}}\frac{\Delta^2}{\varepsilon}
\end{aligned}$$

which concludes the proof of the theorem for $K_3 := \frac{\sigma^2}{2\sqrt{\pi}}$. □

A.11 Proof of Proposition 4.3

PROOF. We notice that for $\mathbf{b} = \mathbf{e}_1$, $X_\varepsilon(T)$ and $X^\Delta_\varepsilon(T)$ are given by the first coordinate of $\mathbf{Y}_\varepsilon(T)$, respectively $\mathbf{Y}^\Delta_\varepsilon(T)$. Without loss of generality, we assume $\mathbf{Y}_\varepsilon(0) = \mathbf{Y}^\Delta_\varepsilon(0) = \mathbf{0}$, since at time $t = 0$ the two processes coincide and their difference is 0. By recursive substitution we get that

$$\mathbf{Y}_\varepsilon^{(\Delta)}(T) = \mathbf{Y}_\varepsilon^{(\Delta)}(s_n^{(n)}) = \sum_{j=1}^{n} (I + \Delta A)^{n-j}\, \mathbf{e}_p \Delta L_\varepsilon^{(\Delta)}(s_j^{(n)}),$$

where we write Δ in parentheses to indicate that the formula holds both for $\mathbf{Y}_\varepsilon$ and $\mathbf{Y}_\varepsilon^\Delta$. For $\tilde{A} := I + \Delta A$, as $X_\varepsilon^{(\Delta)}(T) = \mathbf{e}_1^\top \mathbf{Y}_\varepsilon^{(\Delta)}(T)$, we write that

$$\mathbb{E}\left[\left(X_\varepsilon^\Delta(T) - X_\varepsilon(T)\right)^2\right] = \mathbf{e}_1^\top \mathbb{E}\left[\left(\sum_{j=1}^{n} \tilde{A}^{n-j}\mathbf{e}_p \left(\Delta L_\varepsilon^\Delta(s_j^{(n)}) - \Delta L_\varepsilon(s_j^{(n)})\right)\right)^2\right]\mathbf{e}_1,$$

which means we only need the first coordinate of the expectation. We focus on it:

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{j=1}^{n} \tilde{A}^{n-j}\mathbf{e}_p \left(\Delta L_\varepsilon^\Delta(s_j^{(n)}) - \Delta L_\varepsilon(s_j^{(n)})\right)\right)^2\right] \\
&= \sum_{j=1}^{n} \tilde{A}^{n-j}\mathbf{e}_p\mathbf{e}_p^\top \left(\tilde{A}^\top\right)^{n-j} \mathbb{E}\left[\left(\Delta L_\varepsilon^\Delta(s_j^{(n)}) - \Delta L_\varepsilon(s_j^{(n)})\right)^2\right] + \\
&+2 \sum_{\substack{j=1\\ i=j+1}}^{n} \tilde{A}^{n-j}\mathbf{e}_p\mathbf{e}_p^\top \left(\tilde{A}^\top\right)^{n-i} \mathbb{E}\left[\left(\Delta L_\varepsilon^\Delta(s_j^{(n)}) - \Delta L_\varepsilon(s_j^{(n)})\right)\left(\Delta L_\varepsilon^\Delta(s_i^{(n)}) - \Delta L_\varepsilon(s_i^{(n)})\right)\right].
\end{aligned} \tag{A.15}$$

By use of equation (2.3), we find that

$$\begin{aligned}
&\Delta L_\varepsilon(s_j^{(n)}) = L_\varepsilon(s_j^{(n)}) - L_\varepsilon(s_{j-1}^{(n)}) \\
&= 2\int_0^{s_{j-1}^{(n)}} \left(\Phi\left(\frac{s_j^{(n)} - s}{\varepsilon}\right) - \Phi\left(\frac{s_{j-1}^{(n)} - s}{\varepsilon}\right)\right) dL(s) + 2\int_{s_{j-1}^{(n)}}^{s_j^{(n)}} \left(\Phi\left(\frac{s_j^{(n)} - s}{\varepsilon}\right) - \frac{1}{2}\right) dL(s) \\
&= 2\sum_{k=1}^{j} \int_{s_{k-1}^{(n)}}^{s_k^{(n)}} \left(\Phi\left(\frac{s_j^{(n)} - s}{\varepsilon}\right) - \Phi\left(\frac{s_{j-1}^{(n)} - s}{\varepsilon}\right)\right) dL(s),
\end{aligned}$$

while, from equation (4.1), we get that

$$\begin{aligned}
\Delta L_\varepsilon^\Delta(s_j^{(n)}) &= L_\varepsilon^\Delta(s_j^{(n)}) - L_\varepsilon^\Delta(s_{j-1}^{(n)}) \\
&= 2\sum_{k=1}^{j} \left(\Phi\left(\frac{s_j^{(n)} - s_{k-1}^{(n)}}{\varepsilon}\right) - \Phi\left(\frac{s_{j-1}^{(n)} - s_{k-1}^{(n)}}{\varepsilon}\right)\right) \Delta L(s_k^{(n)}) \\
&= 2\sum_{k=1}^{j} \int_{s_{k-1}^{(n)}}^{s_k^{(n)}} \left(\Phi\left(\frac{s_j^{(n)} - s_{k-1}^{(n)}}{\varepsilon}\right) - \Phi\left(\frac{s_{j-1}^{(n)} - s_{k-1}^{(n)}}{\varepsilon}\right)\right) dL(s).
\end{aligned}$$

The difference $\Delta L_\varepsilon^\Delta(s_j^{(n)}) - \Delta L_\varepsilon(s_j^{(n)})$ then becomes

$$\Delta L_\varepsilon^\Delta(s_j^{(n)}) - \Delta L_\varepsilon(s_j^{(n)}) = 2\sum_{k=1}^{j}\int_{s_{k-1}^{(n)}}^{s_k^{(n)}} F(s)dL(s), \tag{A.16}$$

$$F(s) := \Phi\left(\frac{s_j^{(n)} - s_{k-1}^{(n)}}{\varepsilon}\right) - \Phi\left(\frac{s_{j-1}^{(n)} - s_{k-1}^{(n)}}{\varepsilon}\right) - \left(\Phi\left(\frac{s_j^{(n)} - s}{\varepsilon}\right) - \Phi\left(\frac{s_{j-1}^{(n)} - s}{\varepsilon}\right)\right).$$

By applying the integration by parts formula to the product $F(s)L(s)$ for $s_{k-1}^{(n)} \leq s \leq s_k^{(n)}$, each of the integrals on the right hand side of equation (A.16) can be rewritten as

$$\int_{s_{k-1}^{(n)}}^{s_k^{(n)}} F(s)dL(s) = F(s_k^{(n)})L(s_k^{(n)}) - \frac{1}{\varepsilon}\int_{s_{k-1}^{(n)}}^{s_k^{(n)}}\left(\varphi\left(\frac{s_j^{(n)} - s}{\varepsilon}\right) - \varphi\left(\frac{s_{j-1}^{(n)} - s}{\varepsilon}\right)\right) L(s)ds.$$

Moreover, by noticing that

$$F(s_k^{(n)}) = \frac{1}{\varepsilon}\int_{s_{k-1}^{(n)}}^{s_k^{(n)}}\left(\varphi\left(\frac{s_j^{(n)} - s}{\varepsilon}\right) - \varphi\left(\frac{s_{j-1}^{(n)} - s}{\varepsilon}\right)\right) ds,$$

equation (A.16) becomes

$$\begin{aligned}\Delta L_\varepsilon^\Delta(s_j^{(n)}) &- \Delta L_\varepsilon(s_j^{(n)}) \\ &= 2\sum_{k=1}^{j}\frac{1}{\varepsilon}\int_{s_{k-1}^{(n)}}^{s_k^{(n)}}\left(\varphi\left(\frac{s_j^{(n)} - s}{\varepsilon}\right) - \varphi\left(\frac{s_{j-1}^{(n)} - s}{\varepsilon}\right)\right)\Big(L(s_k^{(n)}) - L(s)\Big) ds. \end{aligned} \tag{A.17}$$

Since the aim is to estimate equation (A.15), for the first sum, we proceed as in the proof of Proposition 4.1 and get that

$$\begin{aligned}\mathbb{E}\left[\Big(\Delta L_\varepsilon^\Delta(s_j^{(n)}) - \Delta L_\varepsilon(s_j^{(n)})\Big)^2\right] \leq \frac{\sigma^2}{\sqrt{\pi}}\frac{\Delta^2}{\varepsilon}\Bigg\{&\Phi\left(\sqrt{2}\frac{\Delta}{\varepsilon}j\right) - \frac{1}{2} + \Phi\left(\sqrt{2}\frac{\Delta}{\varepsilon}(j-1)\right) + \\ &-\Phi\left(-\sqrt{2}\frac{\Delta}{\varepsilon}\right) + e^{-\frac{\Delta^2}{4\varepsilon^2}}\left(\Phi\left(\sqrt{2}\frac{\Delta}{\varepsilon}\left(j - \frac{1}{2}\right)\right) - \Phi\left(-\frac{\sqrt{2}}{2}\frac{\Delta}{\varepsilon}\right)\right)\Bigg\},\end{aligned}$$

and for ε small enough, we can write that for every $j = 1, \dots, n$,

$$\mathbb{E}\left[\Big(\Delta L_\varepsilon^\Delta(s_j^{(n)}) - \Delta L_\varepsilon(s_j^{(n)})\Big)^2\right] \leq \frac{3\sigma^2}{2\sqrt{\pi}}\frac{\Delta^2}{\varepsilon}. \tag{A.18}$$

For the second sum of equation (A.15), by use of the Lipschitz property of φ (where the Lipschitz constant is found by taking the second derivative equal to zero, and substituting the value found for s into the first derivative), we get

$$\frac{1}{\varepsilon}\varphi\left(\frac{s_j^{(n)}-s}{\varepsilon}\right)-\frac{1}{\varepsilon}\varphi\left(\frac{s_{j-1}^{(n)}-s}{\varepsilon}\right)\leq\frac{e^{-1/2}}{\sqrt{2\pi}}\frac{\Delta}{\varepsilon^2}, \tag{A.19}$$

Since the increments of a Lévy process are independent, from equation (A.17) we get that

$$\begin{aligned}
&\mathbb{E}\left[\left(\Delta L_\varepsilon^\Delta(s_j^{(n)})-\Delta L_\varepsilon(s_j^{(n)})\right)\left(\Delta L_\varepsilon^\Delta(s_i^{(n)})-\Delta L_\varepsilon(s_i^{(n)})\right)\right]\\
&=\mathbb{E}\Bigg[2\sum_{k=1}^{j}\frac{1}{\varepsilon}\int_{s_{k-1}^{(n)}}^{s_k^{(n)}}\left(\varphi\left(\frac{s_j^{(n)}-s}{\varepsilon}\right)-\varphi\left(\frac{s_{j-1}^{(n)}-s}{\varepsilon}\right)\right)\left(L(s_k^{(n)})-L(s)\right)ds\,\cdot\\
&\qquad\cdot 2\sum_{v=1}^{i}\frac{1}{\varepsilon}\int_{s_{v-1}^{(n)}}^{s_v^{(n)}}\left(\varphi\left(\frac{s_i^{(n)}-w}{\varepsilon}\right)-\varphi\left(\frac{s_{i-1}^{(n)}-w}{\varepsilon}\right)\right)\left(L(s_v^{(n)})-L(w)\right)dw\Bigg]\\
&=\frac{8\sigma^2}{\varepsilon^2}\sum_{k=1}^{j}\int_{s_{k-1}^{(n)}}^{s_k^{(n)}}\int_{s_{k-1}^{(n)}}^{w}\left(\varphi\left(\frac{s_j^{(n)}-s}{\varepsilon}\right)-\varphi\left(\frac{s_{j-1}^{(n)}-s}{\varepsilon}\right)\right)\cdot\\
&\qquad\cdot\left(\varphi\left(\frac{s_i^{(n)}-w}{\varepsilon}\right)-\varphi\left(\frac{s_{i-1}^{(n)}-w}{\varepsilon}\right)\right)\left(s_k^{(n)}-w\right)dsdw,
\end{aligned}$$

And, by use of equation (A.19), after calculation we get

$$\mathbb{E}\left[\left(\Delta L_\varepsilon^\Delta(s_j^{(n)})-\Delta L_\varepsilon(s_j^{(n)})\right)\left(\Delta L_\varepsilon^\Delta(s_i^{(n)})-\Delta L_\varepsilon(s_i^{(n)})\right)\right]\leq\frac{2e^{-1}\sigma^2T}{3\pi}\frac{\Delta^4}{\varepsilon^4}. \tag{A.20}$$

With equation (A.18) and (A.20), equation (A.15) is estimated by

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{j=1}^{n}\tilde{A}^{n-j}\mathbf{e}_p\left(\Delta L_\varepsilon^\Delta(s_j^{(n)})-\Delta L_\varepsilon(s_j^{(n)})\right)\right)^2\right]\\
&\leq\frac{3\sigma^2}{2\sqrt{\pi}}\frac{\Delta^2}{\varepsilon}\sum_{j=1}^{n}\tilde{A}^{n-j}\mathbf{e}_p\mathbf{e}_p^\top\left(\tilde{A}^\top\right)^{n-j}+\frac{2e^{-1}\sigma^2T}{3\pi}\frac{\Delta^4}{\varepsilon^4}\sum_{\substack{j=1\\ i=j+1}}^{n}\tilde{A}^{n-j}\mathbf{e}_p\mathbf{e}_p^\top\left(\tilde{A}^\top\right)^{n-i}.
\end{aligned}\tag{A.21}$$

Since we are interested only in the first coordinate of the expectation, we can look at the two summations in equation (A.21) as two geometric sums. Then, to have convergence, we need the eigenvalues of $\tilde{A}=I+\Delta A$ to have modulus bounded by 1. From the basic algebra, the eigenvalues of $I+\Delta A$ are of the form $1+\Delta\lambda_1,\ldots,1+\Delta\lambda_p$. Moreover, because of the causality condition (3.4), we know that $Re(\lambda_j)<0$ for every $j=1,\ldots,p$. To have

$$|1+\Delta\lambda_j|=\sqrt{\left(1+\Delta Re(\lambda_j)\right)^2+\Delta^2 Im^2(\lambda_j)}<1,\qquad j=1,\ldots,p,$$

we then need Δ to be bounded as in the statement of the theorem. This condition guarantees the existence of $\tilde{K}_4$ and $\tilde{K}_5$ as positive real constants, such that

$$\mathbf{e}_1^\top \sum_{j=1}^{n} \tilde{A}^{n-j}\mathbf{e}_p\mathbf{e}_p^\top \left(\tilde{A}^\top\right)^{n-j}\mathbf{e}_1 \le \tilde{K}_4 \qquad \text{and} \qquad \mathbf{e}_1^\top \sum_{\substack{j=1\\ i=j+1}}^{n} \tilde{A}^{n-j}\mathbf{e}_p\mathbf{e}_p^\top \left(\tilde{A}^\top\right)^{n-i}\mathbf{e}_1 \le \tilde{K}_5.$$

Combining these bounds with equation (A.21), we obtain that

$$\mathbb{E}\left[\left(X_\varepsilon^\Delta(T) - X_\varepsilon(T)\right)^2\right] \le \frac{3\sigma^2}{2\sqrt{\pi}}\tilde{K}_4\frac{\Delta^2}{\varepsilon} + \frac{2e^{-1}\sigma^2 T}{3\pi}\tilde{K}_5\frac{\Delta^4}{\varepsilon^4},$$

And, by introducing $K_4 := \frac{3\sigma^2}{2\sqrt{\pi}}\tilde{K}_4$ and $K_5 := \frac{2e^{-1}\sigma^2 T}{3\pi}\tilde{K}_5$, we conclude the proof. □

A.12 Proof of Proposition 4.5

PROOF (SKETCH). We notice that, by means of equation (2.6), the terms of the sum in equation (4.1), for $j = 1, \dots, m$, $1 \le m \le n$, can be expressed by

$$\phi_\varepsilon(s_{m+1-j}^{(n)}) = 2\Phi\left(\frac{s_{m+1-j}^{(n)}}{\varepsilon}\right) - 1 = 2\Phi\left(\{m+1-j\}\frac{\Delta}{\varepsilon}\right) - 1, \tag{A.22}$$

for $m+1-j$, an integer number, which shows that L_ε^Δ is the function of the ratio $\frac{\Delta}{\varepsilon}$. For the increment $\Delta L_\varepsilon^\Delta(t+\Delta) = L_\varepsilon^\Delta(t+\Delta) - L_\varepsilon^\Delta(t)$, by means of equation (4.1) and (A.22), we write

$$\Delta L_\varepsilon^\Delta(t+\Delta) = 2\sum_{j=1}^{m+1}\left\{\Phi\left(\{m+2-j\}\frac{\Delta}{\varepsilon}\right) - \Phi\left(\{m+1-j\}\frac{\Delta}{\varepsilon}\right)\right\}\Delta L(s_j^{(n)}). \tag{A.23}$$

In the proof of Proposition 2.2 we showed that the increments of L_ε are not independent. The same trivially holds also for L_ε^Δ. More precisely, we notice that, when $\frac{\Delta}{\varepsilon} > 1$, only the last few terms of the sum contribute in equation (A.23). But if $\frac{\Delta}{\varepsilon} \le 1$, then more terms contribute, and $\Delta L_\varepsilon^\Delta(t+\Delta)$ depends on increments way more in the past. On the other hand, the increment $\Delta L_\varepsilon^\Delta(t+\Delta)$ is supposed to approximate $\Delta L(t+\Delta)$ when ε and Δ approach 0. As increments of Lévy processes are independent, then it seems reasonable to think that $\frac{\Delta}{\varepsilon} \le 1$ leads to some particular situations and the dynamics of X is not captured by the Euler approach. □

References

Benth, Fred Espen, Luca Di Persio, and Silvia Lavagnini (2018) "Stochastic Modeling of Wind Derivatives in Energy Markets". In: *Risks* 6.2. ISSN: 2227-9091. DOI: `10.3390/risks6020056`. URL: `https://www.mdpi.com/2227-9091/6/2/56`.

Øksendal, Bernt (2013) *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media.

Tankov, Peter (2003) *Financial Modelling with Jump Processes*. Vol. 2. CRC Press.