

Supplementary Material

1 GAMMA DISTRIBUTION

The pdf of Gamma distribution is given by Equation 13. Our aim is to derive the expression for differential entropy of this distribution. From Equation 9 we know that,

$$\begin{aligned}
 h(f_T) &= - \int_0^\infty f_T(t) \log(f_T(t)) dt \\
 &= - \int_0^\infty f_T(t) \log \left(\frac{b^a t^{a-1} e^{-bt}}{\Gamma(a)} \right) dt \\
 &= - \log \left(\frac{b^a}{\Gamma(a)} \right) \int_0^\infty f_T(t) dt - (a-1) \int_0^\infty \log(t) f_T(t) dt + b \int_0^\infty t f_T(t) dt \\
 &= - \log \left(\frac{b^a}{\Gamma(a)} \right) - \frac{(a-1)b^a}{\Gamma(a)} \int_0^\infty \log(t) t^{a-1} e^{-bt} dt + \frac{b^{a+1}}{\Gamma(a)} \int_0^\infty t^a e^{-bt} dt,
 \end{aligned}$$

The integrals in the second and third term can be solved by linear substitution and we arrive at the simplified form of $h(f_T)$,

$$\begin{aligned}
 h(f_T) &= - \log \left(\frac{b^a}{\Gamma(a)} \right) - \frac{(a-1)}{\Gamma(a)} [\Gamma'(a) - \log(b)\Gamma(a)] + \frac{1}{\Gamma(a)} \Gamma(a+1) \\
 &= \log \Gamma(a) - \log(b) - (a-1)\psi(a) + a \\
 &= \log \left(\frac{\Gamma(a)}{b} e^{a+(1-a)\psi(a)} \right)
 \end{aligned}$$

where $\Gamma(x) = \int_0^\infty z^{x-1} e^{-z} dz$ is the gamma function and $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function.

The entropy of the instantaneous rate distribution, with pdf given in Equation 17, is calculated in a similar manner,

$$\begin{aligned}
 h(f_R) &= - \int_0^\infty f_R(r) \log f_R(r) dr \\
 &= - \int_0^\infty f_R(r) \log \left(\frac{b^{a+1}}{\Gamma(a+1)} r^{-a-2} e^{-b/r} \right) dr \\
 &= - \log \left(\frac{b^{a+1}}{\Gamma(a+1)} \right) \int_0^\infty f_R(r) dr + (a+2) \int_0^\infty \log r f_R(r) dr + b \int_0^\infty \frac{1}{r} f_R(r) dr \\
 &= - \log \left(\frac{b^{a+1}}{\Gamma(a+1)} \right) + \frac{(a+2)}{\Gamma(a+1)} \int_0^\infty b^{a+1} r^{-a-2} e^{-b/r} \log r dr + \frac{1}{\Gamma(a+1)} \int_0^\infty b^{a+2} r^{-a-3} e^{-b/r} dr \\
 &= - \log \left(\frac{b^{a+1}}{\Gamma(a+1)} \right) + \frac{(a+2)}{\Gamma(a+1)} \int_0^\infty \left(\frac{b}{r} \right)^{a+1} \frac{1}{r} e^{-b/r} \log r dr + \frac{1}{\Gamma(a+1)} \int_0^\infty \left(\frac{b}{r} \right)^{a+2} \frac{1}{r} e^{-b/r} dr,
 \end{aligned}$$

The integrals in the second and third term are easily solved by using a substitution for b/r ,

$$\begin{aligned}
 &= -\log\left(\frac{b^{a+1}}{\Gamma(a+1)}\right) + \frac{(a+2)}{\Gamma(a+1)}[\Gamma(a+1)\log b - \Gamma'(a+1)] + \frac{1}{\Gamma(a+1)}\Gamma(a+2) \\
 &= -\log\left(\frac{b^{a+1}}{\Gamma(a+1)}\right) + (a+2)\log b - (a+2)\psi(a+1) + (a+1) \\
 &= \log(\Gamma(a+1)be^{(a+1)-(a+2)\psi(a+1)}).
 \end{aligned}$$

To derive the coefficient of variation $C_V(R)$ from Equation 8, we first need to find an expression for $\mathbb{E}(1/T)$,

$$\begin{aligned}
 \mathbb{E}\left(\frac{1}{T}\right) &= \int_0^\infty \frac{1}{t} f_T(t) dt \\
 &= \int_0^\infty \frac{1}{t} \frac{b^a t^{a-1} e^{-bt}}{\Gamma(a)} dt \\
 &= \frac{b^a}{\Gamma(a)} \int_0^\infty t^{a-2} e^{-bt} dt \\
 &= \frac{b^a}{\Gamma(a)} b^{1-a} \Gamma(a-1) \\
 &= \frac{b}{a-1}.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 C_V(R) &= \sqrt{\frac{\mathbb{E}(1/T)}{\lambda} - 1} \\
 &= \sqrt{\frac{b}{a-1} \frac{a}{b} - 1} \\
 C_V(R) &= \sqrt{\frac{1}{a-1}}.
 \end{aligned}$$

2 LOGNORMAL DISTRIBUTION

Entropy of the lognormal distribution with pdf given by Equation 26, is

$$\begin{aligned}
 h(f_T) &= - \int_0^\infty f_T(t) \log(f_T(t)) dt \\
 &= - \int_0^\infty f_T(t) \log \left(\frac{1}{\sigma t \sqrt{2\pi}} \exp \left\{ - \frac{(\log t - \log m)^2}{2\sigma^2} \right\} \right) dt \\
 &= - \int_0^\infty f_T(t) \left(- \log(\sigma \sqrt{2\pi}) - \log t - \frac{(\log t - \log m)^2}{2\sigma^2} \right) dt \\
 &= \log(\sqrt{2\pi}\sigma^2) \int_0^\infty f_T(t) dt + \int_0^\infty \log t f_T(t) dt + \frac{1}{2\sigma^2} \int_0^\infty (\log t - \log m)^2 f_T(t) dt \\
 &= \log(\sqrt{2\pi}\sigma^2) + \frac{1}{\sqrt{2\pi}\sigma^2} \int_0^\infty \frac{\log t}{t} \exp \left\{ - \frac{(\log t - \log m)^2}{2\sigma^2} \right\} dt \\
 &\quad + \frac{1}{\sqrt{2\pi}\sigma^2} \int_0^\infty \frac{(\log t - \log m)^2}{2\sigma^2 t} \exp \left\{ - \frac{(\log t - \log m)^2}{2\sigma^2} \right\} dt
 \end{aligned}$$

To simplify it further, we substitute $(\log t - \log m) = y$ into the integrals in the second and third term,

$$= \log(\sqrt{2\pi}\sigma^2) + \frac{1}{\sqrt{2\pi}\sigma^2} \int_{-\infty}^\infty (y + \log m) e^{-y^2/2\sigma^2} dy + \frac{1}{\sqrt{2\pi}\sigma^2} \int_{-\infty}^\infty \frac{y^2}{2\sigma^2} e^{-y^2/2\sigma^2} dy$$

Second term can be solved easily and to solve the third term integral, we again use the substitution method. Substituting $y/\sqrt{2}\sigma = x$ in the integral, we arrive at

$$= \log(\sqrt{2\pi}\sigma^2) + \log m + \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty x^2 e^{-x^2} dx. \quad (\text{S1})$$

To find the value of the third term integral, let's consider

$$I_s = \int_{-\infty}^\infty e^{-sm^2} dm$$

then,

$$\begin{aligned}
 -\frac{d}{ds} I_s|_{s=1} &= - \int_{-\infty}^\infty -sm^2 e^{-sm^2} dm|_{s=1} \\
 &= \int_{-\infty}^\infty m^2 e^{-m^2} dm.
 \end{aligned}$$

The last term in the equation above corresponds to the integral in the third term of Equation S1. Now we find the value of the integral I_s ,

$$\begin{aligned} I_s^2 &= \int_{-\infty}^{\infty} e^{-sm^2} dm \int_{-\infty}^{\infty} e^{-sn^2} dn \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-s(m^2+n^2)} dm dn, \end{aligned}$$

let $m = r\sin\theta$ and $n = r\cos\theta$,

$$\begin{aligned} I_s^2 &= \int_0^{2\pi} d\theta \int_0^{\infty} e^{-sr^2} r dr \\ &= 2\pi \int_0^{\infty} r e^{-sr^2} dr, \end{aligned}$$

this integral is easily solved by using linear substitution for r^2 ,

$$\begin{aligned} I_s^2 &= 2\pi \frac{1}{2s} \\ &= \frac{\pi}{s} \end{aligned}$$

Hence,

$$I_s = \sqrt{\frac{\pi}{s}}.$$

The integral in the third term of the Equation S1 is,

$$\begin{aligned} \int_{-\infty}^{\infty} x^2 e^{-x^2} dx &= -\frac{d}{ds} I_s \big|_{s=1} \\ &= -\frac{d}{ds} \sqrt{\frac{\pi}{s}} \big|_{s=1} \\ &= \frac{\sqrt{\pi}}{2}. \end{aligned}$$

Substituting this value back in Equation S1 gives us the expression for the ISI entropy,

$$\begin{aligned} h(f_T) &= \log(\sqrt{2\pi\sigma^2}) + \log m + \frac{1}{2} \\ &= \frac{1}{2} \log(2\pi e \sigma^2 m^2). \end{aligned}$$

The pdf of the instantaneous rate distribution is given in Equation 30 and the expression for the entropy is,

$$\begin{aligned}
 h(f_R) &= - \int_0^\infty f_R(r) \log f_R(r) dr \\
 &= - \int_0^\infty f_R(r) \log \left(\frac{1}{m e^{\sigma^2/2}} \frac{1}{r^2 \sigma \sqrt{2\pi}} \exp \left\{ - \frac{(\ln r + \ln m)^2}{2\sigma^2} \right\} \right) dr \\
 &= \log(\sqrt{2\pi} \sigma m e^{\sigma^2/2}) + 2 \int_0^\infty \log r f_R(r) dr + \frac{1}{2\sigma^2} \int_0^\infty (\log r - \log m)^2 f_R(r) dr \\
 &= \log(\sqrt{2\pi} \sigma m e^{\sigma^2/2}) + \frac{2}{\sigma m e^{\sigma^2/2} \sqrt{2\pi}} \int_0^\infty \frac{\log r}{r^2} \exp \left\{ - \frac{(\log r + \log m)^2}{2\sigma^2} \right\} dr \\
 &\quad + \frac{1}{\sigma m e^{\sigma^2/2} \sqrt{2\pi}} \int_0^\infty \frac{(\log r + \log m)^2}{2\sigma^2 r^2} \exp \left\{ - \frac{(\log r + \log m)^2}{2\sigma^2} \right\} dr.
 \end{aligned}$$

Substitute $\log r + \log m = y$ to simplify the integrals,

$$\begin{aligned}
 &= \log(\sqrt{2\pi} \sigma m e^{\sigma^2/2}) + \frac{2}{\sqrt{2\pi} \sigma e^{\sigma^2/2}} \int_{-\infty}^\infty (y - \log m) e^{-y} e^{-y^2/2\sigma^2} dy \\
 &\quad + \frac{1}{\sqrt{2\pi} \sigma e^{\sigma^2/2}} \int_{-\infty}^\infty \frac{y^2}{2\sigma^2} e^{-y} e^{-y^2/2\sigma^2} dy \\
 &= \log(\sqrt{2\pi} \sigma m e^{\sigma^2/2}) + \frac{2}{\sigma m e^{\sigma^2/2} \sqrt{2\pi}} \int_{-\infty}^\infty y e^{-y} e^{-y^2/2\sigma^2} dy \\
 &\quad - \frac{2 \log m}{\sigma \sqrt{2\pi} e^{\sigma^2/2}} \int_{-\infty}^\infty e^{-y} e^{-y^2/2\sigma^2} dy + \frac{1}{\sigma \sqrt{2\pi} e^{\sigma^2/2}} \int_{-\infty}^\infty \frac{y^2}{2\sigma^2} e^{-y} e^{-y^2/2\sigma^2} dy \\
 &= \log(\sqrt{2\pi} \sigma m e^{\sigma^2/2}) + \frac{2}{\sigma m e^{\sigma^2/2} \sqrt{2\pi}} I_1 - \frac{2 \log m}{\sigma \sqrt{2\pi} e^{\sigma^2/2}} I_2 + \frac{1}{\sigma \sqrt{2\pi} e^{\sigma^2/2}} I_3. \tag{S2}
 \end{aligned}$$

We will individually solve for the integrals I_1 , I_2 and I_3 . Note that,

$$e^{-y} e^{-y^2/2\sigma^2} = e^{-(\frac{y}{\sqrt{2}\sigma} + \frac{\sigma}{\sqrt{2}})^2} e^{\sigma^2/2}. \tag{S3}$$

Using the relationship S3 in I_1 ,

$$\begin{aligned}
 I_1 &= \int_{-\infty}^\infty y e^{-y} e^{-y^2/2\sigma^2} dy \\
 &= e^{\sigma^2/2} \int_{-\infty}^\infty y e^{-(\frac{y}{\sqrt{2}\sigma} + \frac{\sigma}{\sqrt{2}})^2} dy.
 \end{aligned}$$

This can be simplified further by the substituting $\frac{y}{\sqrt{2}\sigma} + \frac{\sigma}{\sqrt{2}} = n$ in the integral,

$$\begin{aligned}
 &= \sqrt{2}\sigma \int_{-\infty}^\infty (n\sqrt{2}\sigma - \sigma^2) e^{-n^2} dn \\
 &= 2\sigma^2 \int_{-\infty}^\infty n e^{-n^2} dn - \sqrt{2}\sigma^3 \int_{-\infty}^\infty e^{-n^2} dn.
 \end{aligned}$$

The first term has an odd integrand, hence equates to zero. The second term is the Gaussian integral $\int_{-\infty}^{\infty} e^{-n^2} dn = \sqrt{\pi}$ (Weisstein, 2020). Hence,

$$I_1 = -\sqrt{2\pi}\sigma^3. \quad (\text{S4})$$

We use Equation S3 to simplify and solve the integral I_2 in the following way,

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty} e^{-y} e^{-y^2/2\sigma^2} dy \\ &= e^{\sigma^2/2} \int_{-\infty}^{\infty} e^{(\frac{y}{\sqrt{2}\sigma} + \frac{\sigma}{\sqrt{2}})^2} dy \\ &= e^{\sigma^2/2} \int_{-\infty}^{\infty} e^{\frac{1}{2\sigma^2}(y+\sigma^2)^2} dy \\ &= \sigma\sqrt{2\pi}e^{\sigma^2/2}. \end{aligned} \quad (\text{S5})$$

Integral I_3 in the last term of Equation S2 is solved through a slightly different approach,

$$\begin{aligned} I_3 &= \int_{-\infty}^{\infty} \frac{y^2}{2\sigma^2} e^{-(y+\frac{y^2}{2\sigma^2})} dy \\ &= \int_{-\infty}^{\infty} \frac{y^2}{2\sigma^2} e^{-\frac{1}{2\sigma^2}(t+\sigma^2)^2 + \frac{1}{2}\sigma^2} dy, \end{aligned}$$

let $z = y + \sigma^2$,

$$\begin{aligned} &= \frac{1}{2\sigma^2} e^{\sigma^2/2} \int_{-\infty}^{\infty} (z - \sigma^2)^2 e^{-\frac{z^2}{2\sigma^2}} dz \\ &= \frac{1}{2\sigma^2} e^{\sigma^2/2} \int_{-\infty}^{\infty} (z^2 + \sigma^4) e^{-\frac{z^2}{2\sigma^2}} dz. \end{aligned}$$

Let $a = 1/2\sigma^2$ and use $J(a) = \int_{-\infty}^{\infty} e^{-az^2} dz = \sqrt{\frac{\pi}{a}}$, $\int_{-\infty}^{\infty} z^2 e^{-az^2} dz = -J'(a) = \frac{\sqrt{\pi}}{2a^{3/2}}$, to obtain

$$\begin{aligned} I_3 &= \frac{1}{2\sigma^2} e^{\sigma^2/2} (\sqrt{2\pi}\sigma^3 + \sqrt{2\pi}\sigma^5) \\ &= \sqrt{\frac{\pi}{2}} e^{\sigma^2/2} \sigma (1 + \sigma^2). \end{aligned} \quad (\text{S6})$$

Plug the values from Equation S4, S5 and S6 into Equation S2 to get

$$\begin{aligned} h(f_R) &= \log(\sqrt{2\pi}\sigma m e^{\sigma^2/2}) - 2\sigma^2 - 2\log m + \frac{1}{2}(\sigma^2 + 1) \\ &= \frac{1}{2} \log \frac{1}{m} 2\pi\sigma^2 e^{1-2\sigma^2}. \end{aligned}$$

To derive the expression for $C_V(R)$, we first find the expression for $\mathbb{E}(1/T)$,

$$\begin{aligned}\mathbb{E}\left(\frac{1}{T}\right) &= \int_0^\infty \frac{1}{t} f_T(t) dt \\ &= \int_0^\infty \frac{1}{t} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\log t - \log m)^2}{2\sigma^2}\right\} dt \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty \frac{1}{t^2} \exp\left\{-\frac{(\log t - \log m)^2}{2\sigma^2}\right\} dt,\end{aligned}$$

let $(\log t - \log m) = y$,

$$\begin{aligned}&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty \frac{1}{me^y} e^{-y^2/2\sigma^2} dy \\ &= \frac{1}{m\sigma\sqrt{2\pi}} \int_{-\infty}^\infty e^{-y} e^{-y^2/2\sigma^2} dy,\end{aligned}$$

This is the previously mentioned integral I_2 , which is solved in Equation S5, substituting the value of I_2 , we arrive at

$$\begin{aligned}&= \frac{1}{m\sigma\sqrt{2\pi}} \sigma\sqrt{2\pi} e^{\sigma^2/2} \\ &= \frac{e^{\sigma^2/2}}{m}.\end{aligned}$$

Now, we can derive $C_V(R)$ from Equation 8,

$$\begin{aligned}C_V(R) &= \sqrt{\frac{\mathbb{E}(1/T)}{\lambda} - 1} \\ &= \sqrt{\frac{e^{\sigma^2/2}}{m} \frac{me^{\sigma^2/2}}{1} - 1} \\ &= \sqrt{e^{\sigma^2} - 1}.\end{aligned}$$

3 INVERSE GAUSSIAN DISTRIBUTION

Entropy of the ISI distribution with pdf given by Equation 37,

$$\begin{aligned}
 h(f_T) &= - \int_0^\infty f_T(t) \log(f_T(t)) dt \\
 &= \frac{1}{2} \log \frac{2\pi b}{a} \int_0^\infty f_T(t) dt + \frac{3}{2} \int_0^\infty \log t f_T(t) dt + \frac{1}{2b} \int_0^\infty \frac{(t-a)^2}{at} f_T(t) dt \\
 &= \frac{1}{2} \log \frac{2\pi b}{a} + \sqrt{\frac{a}{2\pi b}} e^{1/b} \left[\frac{3}{2} \int_0^\infty \log t t^{-\frac{3}{2}} e^{-\frac{1}{2b}(\frac{t}{a} + \frac{a}{t})} dt \right. \\
 &\quad \left. + \frac{1}{2b} \int_0^\infty \left(\frac{t}{a} - 2 + \frac{a}{t} \right) t^{-\frac{3}{2}} e^{-\frac{1}{2b}(\frac{t}{a} + \frac{a}{t})} dt \right].
 \end{aligned}$$

Note that we can write all integrals in $h(f_T)$ in terms of the master integral $J(a, b, n) := \frac{1}{2} \int_0^\infty t^{n-1} e^{-\frac{1}{2b}(\frac{t}{a} + \frac{a}{t})} dt$. Using this definition,

$$\begin{aligned}
 h(f_T) &= \frac{1}{2} \log \frac{2\pi b}{a} + \sqrt{\frac{a}{2\pi b}} e^{\frac{1}{b}} \left[3 \frac{\partial}{\partial n} J(a, b, n) \Big|_{n=-\frac{1}{2}} \right. \\
 &\quad \left. + \frac{1}{b} \left(\frac{1}{a} J\left(a, b, \frac{1}{2}\right) - 2J\left(a, b, -\frac{1}{2}\right) + aJ\left(a, b, -\frac{3}{2}\right) \right) \right]. \tag{S7}
 \end{aligned}$$

To work with the master integral, first change variables so that $t = ae^x$. Under this transformation, $t^{n-1} dt = a^n e^{nx} dx$, and $\frac{1}{2} \left(\frac{t}{a} + \frac{a}{t} \right) = \cosh x$, and the limits change to $\pm\infty$. We arrive at,

$$J(a, b, n) = \frac{a^n}{2} \int_{-\infty}^\infty e^{nx} e^{\frac{1}{b} \cosh t} dx = a^n \int_0^\infty \cosh(nx) e^{\frac{1}{b} \cosh t} dx = a^n K_n \left(\frac{1}{b} \right),$$

where in the second equality we split the integral into two terms, the first ranging over $[-\infty, 0]$ and the second over $[0, \infty]$, and change variables from x to $-x$ in the first term. The final equality uses a standard integral representation for the Bessel K function, as in (DLMF, 2020, Eq. 10.32.9). Substituting for this form of the master integral into Equation S7, we arrive at,

$$\begin{aligned}
 h(f_T) &= \frac{1}{2} \log \frac{2\pi b}{a} + \frac{1}{\sqrt{2\pi b}} e^{1/b} \left[3 \log a K_{\frac{1}{2}} \left(\frac{1}{b} \right) + 3 \frac{\partial}{\partial n} K_n \left(\frac{1}{b} \right) \Big|_{n=-\frac{1}{2}} \right. \\
 &\quad \left. + \frac{1}{b} \left(K_{\frac{1}{2}} \left(\frac{1}{b} \right) - 2K_{-\frac{1}{2}} \left(\frac{1}{b} \right) + K_{-\frac{3}{2}} \left(\frac{1}{b} \right) \right) \right]. \tag{S8}
 \end{aligned}$$

To simplify further, we require some identities for the Bessel K function. The following identities are found in (DLMF, 2020, Eq. 10.27.3, Eq. 10.39.2 and Eq. 10.38.7),

$$K_n(z) = K_{-n}(z), \quad (\text{S9})$$

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}, \quad (\text{S10})$$

$$K_{\frac{3}{2}}(z) = \sqrt{\frac{\pi}{2z}} \left(1 + \frac{1}{z}\right) e^{-z}. \quad (\text{S11})$$

Simplifying $h(f_T)$ using these results,

$$\begin{aligned} h(f_T) &= \frac{1}{2} \log \frac{2\pi b}{a} + \frac{3}{2} \log a + \frac{3e^{1/b}}{\sqrt{2\pi b}} K^{(1,0)}\left(-\frac{1}{2}, \frac{1}{b}\right), \\ &= \frac{1}{2} \log(2\pi a^2 b e) + \frac{3e^{1/b}}{\sqrt{2\pi b}} K^{(1,0)}\left(-\frac{1}{2}, \frac{1}{b}\right). \end{aligned}$$

The pdf of the instantaneous rate distribution is given by Equation 42, and we note that if ISI $T \sim IG(a, b)$ where $IG(a, b)$ is the inverse Gaussian distribution with mean $a > 0$ and scale parameter $b > 0$ then $R \sim IG(1/a, b)$. We can derive $h(f_R)$ through this relationship,

$$h(f_R) = \frac{1}{2} \log \left(\frac{2\pi b e}{a^2} \right) + \frac{3e^{1/b}}{\sqrt{2\pi b}} K^{(1,0)}\left(-\frac{1}{2}, \frac{1}{b}\right).$$

Next, we find out the expression for $\mathbb{E}(1/T)$ for the inverse Gaussian distribution,

$$\begin{aligned} \mathbb{E}\left(\frac{1}{T}\right) &= \int_0^\infty \frac{1}{t} f_T(t) dt \\ &= \int_0^\infty \frac{1}{t} \sqrt{\frac{a}{2\pi b t^3}} \exp\left\{-\frac{1}{2b} \frac{(t-a)^2}{at}\right\} dt, \end{aligned}$$

the integral part can be written in form of the master integral $J(a, b, n)$,

$$\begin{aligned} &= \sqrt{\frac{2a}{2\pi b}} e^{1/b} J(a, b, -3/2) \\ &= \sqrt{\frac{2a}{2\pi b}} e^{1/b} a^{-3/2} K_{-3/2}\left(\frac{1}{b}\right) \end{aligned}$$

Using Equation S9 and S11, we get

$$= \frac{1+b}{a}$$

The value of $\mathbb{E}(1/T)$ is substituted into the equation below to derive an expression for $C_V(R)$,

$$\begin{aligned} C_V(R) &= \sqrt{\frac{\mathbb{E}(1/T)}{\lambda} - 1} \\ &= \sqrt{\frac{(1+b)}{a}a - 1} \\ &= \sqrt{b}. \end{aligned}$$

4 SHIFTED EXPONENTIAL DISTRIBUTION

The pdf of the shifted exponential distribution is given in Equation 47, and the entropy of this distribution is,

$$\begin{aligned} h(f_T) &= - \int_{\tau}^{\infty} f_T(t) \log(f_T(t)) dt \\ &= - \int_{\tau}^{\infty} f_T(t) \log(ae^{-a(t-\tau)}) dt \\ &= - \log a \int_{\tau}^{\infty} f_T(t) dt + a \int_{\tau}^{\infty} (t - \tau) ae^{-a(t-\tau)} dt. \end{aligned}$$

Using the substitution $a(t - \tau) = x$ in the second term integral, the limit changes to $[0, \infty]$

$$\begin{aligned} &= - \log a + \int_0^{\infty} xe^{-x} dx \\ &= \log\left(\frac{1}{a}\right) + 1 \\ &= \log\left(\frac{e}{a}\right). \end{aligned}$$

Entropy of the instantaneous rate distribution with pdf given in Equation 51,

$$\begin{aligned} h(f_R) &= - \int_0^{1/\tau} f_R(r) \log f_R(r) dr \\ &= - \int_0^{1/\tau} f_R(r) \log\left(\frac{a^2}{1+a\tau} \frac{1}{r^3} e^{-a(\frac{1}{r}-\tau)}\right) dr \\ &= - \log\left(\frac{a^2}{1+a\tau}\right) \int_0^{1/\tau} f_R(r) dr + 3 \int_0^{1/\tau} \log r f_R(r) dr + a \int_0^{1/\tau} \left(\frac{1}{r} - \tau\right) f_R(r) dr \\ &= - \log\left(\frac{a^2}{1+a\tau}\right) + \frac{3a^2}{1+a\tau} \int_0^{1/\tau} \frac{\log r}{r^3} e^{-a(\frac{1}{r}-\tau)} dr + \frac{a^3}{(1+a\tau)} \int_0^{1/\tau} \frac{1}{r^3} \left(\frac{1}{r} - \tau\right) e^{-a(\frac{1}{r}-\tau)} dr \\ &= - \log\left(\frac{a^2}{1+a\tau}\right) + \frac{3a^2}{1+a\tau} I_4 + \frac{a^3}{(1+a\tau)} I_5. \end{aligned} \tag{S12}$$

Solving the integrals in the second and third term with the help of MATHEMATICA (Wolfram Research, Inc., 2020),

$$I_4 = -\frac{1 + e^{a\tau}\Gamma(0, a\tau) + (1 + a\tau) \log \tau}{a^2},$$

$$I_5 = \frac{2 + a\tau}{a^3}.$$

Substitute these values back in Equation S12,

$$h(f_R) = -\log\left(\frac{a^2}{1 + a\tau}\right) - \frac{3(1 + e^{a\tau}\Gamma(0, a\tau) + (1 + a\tau) \log \tau)}{1 + a\tau} + \frac{2 + a\tau}{1 + a\tau},$$

where $\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt$ is the upper incomplete gamma function. For the next step, we derive the expression for $\mathbb{E}(1/T)$,

$$\begin{aligned}\mathbb{E}\left(\frac{1}{T}\right) &= \int_\tau^\infty \frac{1}{t} f_T(t) dt \\ &= \int_\tau^\infty \frac{1}{t} a e^{-a(t-\tau)} dt \\ &= a e^{a\tau} \int_\tau^\infty t^{-1} e^{-at} dt \\ &= a^2 e^{a\tau} \int_{a\tau}^\infty (at)^{-1} e^{-at} dt \\ &= a e^{a\tau} \int_{a\tau}^\infty x^{-1} e^{-x} dx \\ &= a e^{a\tau} \Gamma(0, a\tau).\end{aligned}$$

Now, we can solve for the expression of $C_V(R)$ through Equation 8,

$$\begin{aligned}C_V(R) &= \sqrt{\frac{\mathbb{E}(1/T)}{\lambda} - 1} \\ &= \sqrt{\frac{a e^{a\tau} \Gamma(0, a\tau) (1 + a\tau)}{a} - 1} \\ &= \sqrt{(1 + a\tau) e^{a\tau} \Gamma(0, a\tau) - 1}.\end{aligned}$$

5 MIXTURE OF TWO EXPONENTIAL DISTRIBUTIONS WITH REFRACTORY PERIOD

The mean $(1/\lambda)$ of the mixture of two exponential distributions with refractory period τ , and pdf given by Equation 58 is,

$$\begin{aligned}
 \mathbb{E}(T) &= \int_{\tau}^{\infty} t f_T(t) dt \\
 &= \int_{\tau}^{\infty} t (p a e^{-a(t-\tau)} + (1-p) b e^{-b(t-\tau)}) dt \\
 &= a p \int_{\tau}^{\infty} t e^{-a(t-\tau)} dt + (1-p) b \int_{\tau}^{\infty} t e^{-b(t-\tau)} dt \\
 &= a p \frac{(1+a\tau)}{a^2} + (1-p) b \frac{(1+b\tau)}{b^2} \\
 &= \frac{p b (1+a\tau) + (1-p) b (1+b\tau)}{a b},
 \end{aligned}$$

Consequently, we get Equation 59,

$$\lambda = \frac{a b}{p b (1+a\tau) + (1-p) a (1+b\tau)}.$$

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