# Gaussian Local Phase Approximation in a Cylindrical Tissue Model: Appendices

### Appendix A: Phase approximation by cumulant expansion

The transverse local magnetization during magnetic resonance imaging is a function of the ensemble average, denoted by $\langle \dots \rangle$, of the accumulated spin phase at this position

$$m(\vec{r},t) = m_0 \left\langle \mathrm{e}^{-\mathrm{i}\Psi(\vec{r},t)} \right\rangle , \tag{A1}$$

where $\Psi(\vec{r},t)$ is a random variable dependent on the stochastic trajectory of a reflected Brownian motion process, that is

$$\Psi(\vec{r},t) = \int_0^t \mathrm{d}\xi \, \omega(\vec{R}(\vec{r},\xi)) \, . \tag{A2}$$

The spatial path variable $\vec{R}(\vec{r},\xi)$ represents a Wiener process in the three-dimensional dephasing domain with reflective boundary conditions [1]. The ensemble average can be approximated by a cumulant expansion [2, 3]

$$\ln \left\langle \mathrm{e}^{-\mathrm{i}\Psi(\vec{r},t)} \right\rangle = \sum_{k=1}^{\infty} \frac{[-\mathrm{i}]^k}{k!} \left\langle \Psi^k(\vec{r},t) \right\rangle_c , \tag{A3}$$

with $\langle \dots \rangle_c$ denoting the respective cumulant. Terminating the exponential expansion series after the second term leads to

$$\left\langle \mathrm{e}^{-\mathrm{i}\Psi(\vec{r},t)} \right\rangle \approx \exp\left( -\mathrm{i} \left\langle \Psi(\vec{r},t) \right\rangle_c - \frac{1}{2} \left\langle \Psi^2(\vec{r},t) \right\rangle_c \right) . \tag{A4}$$

The first two cumulants can be expressed in terms of the moments of the accumulated phase:

$$\left\langle \Psi(\vec{r},t) \right\rangle_c = \underbrace{\left\langle \Psi(\vec{r},t) \right\rangle}_{=\alpha(\vec{r},t)} , \tag{A5}$$

$$\left\langle \Psi^2(\vec{r},t) \right\rangle_c = \underbrace{\left\langle \Psi^2(\vec{r},t) \right\rangle}_{=2\beta(\vec{r},t)} - \left\langle \Psi(\vec{r},t) \right\rangle^2 . \tag{A6}$$

Therefore, the local magnetization in the GLP approximation with one cumulant is

$$m_I(\vec{r},t) = m_0 \exp\left( -\mathrm{i} \left\langle \Psi(\vec{r},t) \right\rangle \right) . \tag{A7}$$

If an additional second cumulant is taken into account, this becomes

$$m_{II}(\vec{r},t) = m_0 \exp\left( -\mathrm{i} \left\langle \Psi(\vec{r},t) \right\rangle - \frac{1}{2} \left\langle \Psi^2(\vec{r},t) \right\rangle + \frac{1}{2} \left\langle \Psi(\vec{r},t) \right\rangle^2 \right) . \tag{A8}$$

In the Gaussian Phase approximation [4], the total magnetization is described by

$$\frac{M_{GP}(t)}{M_0} = \exp\left( -\int_0^t \mathrm{d}\xi [t-\xi] K(\xi) \right) \tag{A9}$$

with the correlation function

$$K(t) = \frac{1}{V} \int_V \mathrm{d}^3\vec{r} \int_V \mathrm{d}^3\vec{r}_0 \, \omega(\vec{r}) p(\vec{r},\vec{r}_0,t) \omega(\vec{r}_0) \, . \tag{A10}$$

A connection between $\alpha(\vec{r},t)$ and the correlation function $K(t)$ can be found from their definitions:

$$K(t) = -\frac{1}{V}\int\limits_V \mathrm{d}^3\,\vec{r}\omega(\vec{r})\frac{\partial\alpha(\vec{r},t)}{\partial t}\,. \tag{A11}$$

Introducing the angular dependency from Eq. (28) results in

$$K(t) = \delta\omega\frac{\eta}{\eta-1}\underbrace{\frac{1}{R_\mathrm{C}^2}\int\limits_{R_\mathrm{C}}^{R_\mathrm{D}}\mathrm{d}r r}_{=\frac{1-\eta}{2\eta}\vec{1}^\intercal\mathbf{k}}\underbrace{\frac{R_\mathrm{C}^2}{r^2}}_{=\mathbf{r}^{-2}}\frac{\partial}{\partial t}\underbrace{\alpha_2(t)}_{=\vec{\alpha}_2(t)} \tag{A12}$$

$$= -\frac{\delta\omega}{2}\vec{1}^\intercal\mathbf{k}\mathbf{r}^{-2}\frac{\partial}{\partial t}\vec{\alpha}_2(t)\,. \tag{A13}$$

With the explicit expression for $\vec{\alpha}_2(t)$ from Eq. (73) and the general property $\mathbf{\Delta}_r\vec{1} = \vec{0}$ one finally gets

$$K(t) = \frac{\delta\omega^2}{2}\vec{1}^\intercal\mathbf{k}\mathbf{r}^{-2}\mathrm{e}^{\frac{t}{\tau}[\mathbf{\Delta}_r - 4\mathbf{r}^{-2}]}\mathbf{r}^{-2}\vec{1}\,. \tag{A14}$$

With this expression, the integral in the exponent of the right hand side of Eq. (A9) can be written as:

$$-\int\limits_0^t \mathrm{d}\xi[t-\xi]K(\xi) = \frac{\delta\omega^2}{2}\tau t\vec{1}^\intercal\mathbf{k}\mathbf{r}^{-2}[\mathbf{\Delta}_r - 4\mathbf{r}^{-2}]^{-1}\mathbf{r}^{-2}\vec{1} + \frac{[\tau\delta\omega]^2}{2}\vec{1}^\intercal\mathbf{k}\mathbf{r}^{-2}\left[\mathbf{1} - \mathrm{e}^{\frac{t}{\tau}[\mathbf{\Delta}_r - 4\mathbf{r}^{-2}]}\right][\mathbf{\Delta}_r - 4\mathbf{r}^{-2}]^{-2}\mathbf{r}^{-2}\vec{1}$$

The last term $[\mathbf{\Delta}_r - 4\mathbf{r}^{-2}]^{-2}\mathbf{r}^{-2}\vec{1}$ can easily be simplified using Eq. (72):

$$[\mathbf{\Delta}_r - 4\mathbf{r}^{-2}]^{-2}\mathbf{r}^{-2}\vec{1} = [\mathbf{\Delta}_r - 4\mathbf{r}^{-2}]^{-1}\underbrace{[\mathbf{\Delta}_r - 4\mathbf{r}^{-2}]^{-1}\mathbf{r}^{-2}\vec{1}}_{=-\frac{1}{4}\vec{1}}$$

$$= -\frac{1}{4}[\mathbf{\Delta}_r - 4\mathbf{r}^{-2}]^{-1}\vec{1} \tag{A15}$$

$$= -\frac{1}{4}\left[\mathbf{r}^2\frac{2\ln(\mathbf{r}) - \mathbf{1}}{8} + \frac{\mathbf{r}^2 + \mathbf{r}^{-2}}{8}\frac{\ln(\eta)}{1-\eta^2}\right]\vec{1}\,, \tag{A16}$$

resulting in

$$-\int\limits_0^t \mathrm{d}\xi[t-\xi]K(\xi) = -\frac{[\tau\delta\omega]^2}{8}\frac{t}{\tau}\vec{1}^\intercal\mathbf{k}\mathbf{r}^{-2}\vec{1} - \frac{[\tau\delta\omega]^2}{8}\vec{1}^\intercal\mathbf{k}\left[\frac{2\ln(\mathbf{r}) - \mathbf{1}}{8} + \frac{\mathbf{1} + \mathbf{r}^{-4}}{8}\frac{\ln(\eta)}{1-\eta^2}\right]\vec{1}$$
$$+ \frac{[\tau\delta\omega]^2}{8}\vec{1}^\intercal\mathbf{k}\mathbf{r}^{-2}\mathrm{e}^{\frac{t}{\tau}[\mathbf{\Delta}_r - 4\mathbf{r}^{-2}]}\left[\mathbf{r}^2\frac{2\ln(\mathbf{r}) - \mathbf{1}}{8} + \frac{\mathbf{r}^2 + \mathbf{r}^{-2}}{8}\frac{\ln(\eta)}{1-\eta^2}\right]\vec{1}\,.$$

Since the identity

$$\vec{1}^\intercal\mathbf{k}\mathbf{r}^{-2}\vec{1} = \frac{2\eta}{1-\eta}\frac{1}{R_\mathrm{C}^2}\int\limits_{R_\mathrm{C}}^{R_\mathrm{D}}\mathrm{d}r r\frac{R_\mathrm{C}^2}{r^2} = \frac{\eta\ln(\eta)}{\eta-1} \tag{A17}$$

holds, the matrix expression can be expressed as an integral:

$$\vec{1}^\intercal\mathbf{k}\left[\frac{2\ln(\mathbf{r}) - \mathbf{1}}{8} + \frac{\mathbf{1} + \mathbf{r}^{-4}}{8}\frac{\ln(\eta)}{1-\eta^2}\right]\vec{1} = \frac{2\eta}{1-\eta}\frac{1}{R_\mathrm{C}^2}\int\limits_{R_\mathrm{C}}^{R_\mathrm{D}}\mathrm{d}r r\left[\frac{1}{4}\ln\left(\frac{r}{R_\mathrm{C}}\right) - \frac{1}{8} + \frac{1}{8}\left[1 + \frac{R_\mathrm{C}^4}{r^4}\right]\frac{\ln(\eta)}{1-\eta^2}\right] \tag{A18}$$

$$= -\frac{1}{4}\,. \tag{A19}$$

With this, one gets:

$$-\int_0^t \mathrm{d}\xi[t-\xi]K(\xi) = \frac{[\tau\delta\omega]^2}{8}\left[\frac{t}{\tau}\frac{\eta\ln(\eta)}{1-\eta}+\frac{1}{4}\right]+\frac{[\tau\delta\omega]^2}{64}\vec{1}^\intercal\mathbf{k}\mathbf{r}^{-2}\mathrm{e}^{\frac{t}{\tau}[\boldsymbol{\Delta}_r-4\mathbf{r}^{-2}]}\left[\mathbf{r}^2\left[2\ln(\mathbf{r})-\mathbf{1}\right]+\left[\mathbf{r}^2+\mathbf{r}^{-2}\right]\frac{\ln(\eta)}{1-\eta^2}\right]\vec{1}\,.$$

Introducing this expression into Eq. (A9), one finally arrives at Eq. (90). The explicit analytical expression for the correlation function is (see Eq. (24) in [4])

$$K(t) = \delta\omega^2\sum_{n=1}^{\infty}F_n^2\mathrm{e}^{-\lambda_{n,2}^2\frac{t}{\tau}} \tag{A20}$$

with the coefficients $F_n$ given in Eq. (88) that can be summarized to the vector

$$\underbrace{F_n}_{=\vec{F}^\intercal} = \sqrt{\frac{\eta}{1-\eta}}\underbrace{\frac{1}{R_\mathrm{C}^2}\int_{R_\mathrm{C}}^{R_\mathrm{D}}\mathrm{d}r\,r}_{=\frac{1-\eta}{2\eta}\vec{1}^\intercal\mathbf{k}}\underbrace{\frac{R_\mathrm{C}^2}{r^2}}_{=\mathbf{r}^{-2}}\underbrace{\Phi_{n,2}(r)}_{=\boldsymbol{\Phi}_2}\underbrace{\frac{1}{\sqrt{N_{n,2}}}}_{=\boldsymbol{N}_2^{-\frac{1}{2}}}\,,$$

$$\vec{F}^\intercal = \frac{1}{2}\sqrt{\frac{1-\eta}{\eta}}\vec{1}^\intercal\mathbf{k}\mathbf{r}^{-2}\boldsymbol{\Phi}_2\boldsymbol{N}_2^{-\frac{1}{2}}\,, \tag{A21}$$

$$\vec{F} = \frac{1}{2}\sqrt{\frac{1-\eta}{\eta}}\boldsymbol{N}_2^{-\frac{1}{2}}\boldsymbol{\Phi}_2^\intercal\mathbf{r}^{-2}\mathbf{k}\vec{1}\,, \tag{A22}$$

with $\boldsymbol{\Phi}_2$ given in Eq. (C5). Thus, the correlation function in Eq. (A20) can be written in the form

$$\begin{aligned}K(t) &= \delta\omega^2\vec{F}^\intercal\mathrm{e}^{-\boldsymbol{\Lambda}_2^2\frac{t}{\tau}}\vec{F} \qquad (A23)\\ &= \frac{\delta\omega^2}{4}\frac{1-\eta}{\eta}\vec{1}^\intercal\mathbf{k}\mathbf{r}^{-2}\boldsymbol{\Phi}_2\underbrace{\boldsymbol{N}_2^{-\frac{1}{2}}\mathrm{e}^{-\boldsymbol{\Lambda}_2^2\frac{t}{\tau}}\boldsymbol{N}_2^{-\frac{1}{2}}}_{=\mathrm{e}^{-\boldsymbol{\Lambda}_2^2\frac{t}{\tau}}\boldsymbol{N}_2^{-1}}\boldsymbol{\Phi}_2^\intercal\mathbf{r}^{-2}\mathbf{k}\vec{1}\,.\end{aligned}$$

Using Eq. (C7) or Eq. (C8), the matrix $\boldsymbol{\Phi}_2^\intercal$ can be expressed as

$$\boldsymbol{\Phi}_2^\intercal = \frac{2\eta}{1-\eta}\boldsymbol{N}_2\boldsymbol{\Phi}_2^{-1}\mathbf{k}^{-1}\,, \tag{A24}$$

resulting in

$$K(t) = \frac{\delta\omega^2}{2}\vec{1}^\intercal\mathbf{k}\mathbf{r}^{-2}\underbrace{\boldsymbol{\Phi}_2\mathrm{e}^{-\boldsymbol{\Lambda}_2^2\frac{t}{\tau}}\boldsymbol{\Phi}_2^{-1}}_{=\mathrm{e}^{\frac{t}{\tau}[\boldsymbol{\Delta}_r-4\mathbf{r}^{-2}]}}\underbrace{\mathbf{k}^{-1}\mathbf{r}^{-2}\mathbf{k}}_{=\mathbf{r}^{-2}}\vec{1}\,, \tag{A25}$$

which coincides with Eq. (A14).

## Appendix B: Analytical moments

In order to obtain the form of the integrals shown in Eqs. (30) - (32), the two identities

$$\delta(\phi-\phi_0) = \frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\cos(n\phi)\cos(n\phi_0) \tag{B1}$$

and

$$\mathrm{e}^{D\xi[\Delta_r+\frac{1}{r^2}\Delta_\phi]}\cos(n\phi) = \cos(n\phi)\,\mathrm{e}^{D\xi[\Delta_r-\frac{n^2}{r^2}]} \tag{B2}$$

can be used to write the propagator from Eq. (8) as

$$p(r,\phi,r_0,\phi_0,\xi) = \sum_{n=0}^{\infty}\frac{2-\delta_{n,0}}{2\pi}\cos(n\phi)\cos(n\phi_0)\,\mathrm{e}^{D\xi[\Delta_r-\frac{n^2}{r^2}]}\frac{\delta(r-r_0)}{r}\,, \tag{B3}$$

directly leading to the respective integral forms. The integrals from Eqs. (30) - (32) can also be used to get the differential equations from Eqs. (33) - (35). These can be solved by introducing an appropriate ansatz for the respective moment function. The solution for the first moment is

$$\frac{\alpha_2(r,t)}{\tau\delta\omega} = \frac{1}{4}\left[\sum_{n=1}^{\infty}\frac{M^{+}_{n,2}}{N_{n,2}}\Phi_{n,2}(r)\mathrm{e}^{-\lambda^2_{n,2}\frac{t}{\tau}} - 1\right], \tag{B4}$$

with the normalization factors $N_{n,\nu}$ given in Eq. (14) and the integral coefficients $M^{\pm}_{n,\nu}$ defined below in Eq. (B8). The solutions for the second moments are

$$\frac{\beta_0(r,t)}{[\tau\delta\omega]^2} = \frac{1}{8}\frac{\eta\ln(\eta)}{\eta-1}\frac{t}{\tau} - \frac{1}{32} + \frac{1}{32}\sum_{n=1}^{\infty}\frac{M^{+}_{n,2}}{N_{n,2}}\Phi_{n,2}(r)\,\mathrm{e}^{-\lambda^2_{n,2}\frac{t}{\tau}} + \frac{1}{8}\sum_{n=2}^{\infty}\frac{M^{-}_{n,0}}{N_{n,0}}\Phi_{n,0}(r)\frac{1-\mathrm{e}^{-\lambda^2_{n,0}\frac{t}{\tau}}}{\lambda^2_{n,0}}, \tag{B5}$$

and

$$\frac{\beta_4(r,t)}{[\tau\delta\omega]^2} = \frac{1}{128} + \frac{1}{432}\sum_{n=1}^{\infty}\frac{M^{+}_{n,4}}{N_{n,4}}\Phi_{n,4}(r)\mathrm{e}^{-\lambda^2_{n,4}\frac{t}{\tau}} - \frac{1}{96}\sum_{n=1}^{\infty}\frac{M^{+}_{n,2}}{N_{n,2}}\Phi_{n,2}(r)\,\mathrm{e}^{-\lambda^2_{n,2}\frac{t}{\tau}}. \tag{B6}$$

The integral coefficients are

$$M^{\pm}_{n,\nu} = \frac{1}{R_{\mathrm{C}}^{1\pm1}}\int_{R_{\mathrm{C}}}^{R_{\mathrm{D}}}\mathrm{d}r\, r^{\pm1}\,\Phi_{n,\nu}(r) \tag{B7}$$

$$= \frac{2}{\pi\lambda_{n,\nu}^{1\pm1}}\left[\frac{J'_\nu(\lambda_{n,\nu})}{J'_\nu\left(\frac{\lambda_{n,\nu}}{\sqrt{\eta}}\right)}S'_{\pm1,\nu}\left(\frac{\lambda_{n,\nu}}{\sqrt{\eta}}\right) - S'_{\pm1,\nu}\left(\lambda_{n,\nu}\right)\right], \tag{B8}$$

with the Lommel functions $S_{n,\nu}(z)$. Obviously, the first sum appearing in Eq. (B5) contains the expression for $\alpha_2(r,t)$ given in Eq. (B4). Furthermore, it is advantageous to introduce the time independent function

$$f(r) = \frac{1}{8}\sum_{n=2}^{\infty}\frac{M^{-}_{n,0}}{N_{n,0}}\frac{\Phi_{n,0}(r)}{\lambda^2_{n,0}} \tag{B9}$$

$$= \frac{\pi}{8}\sum_{n=2}^{\infty}\Phi_{n,0}(r)J_1\left(\frac{\lambda_{n,0}}{\sqrt{\eta}}\right)\frac{J_1(\lambda_{n,0})S'_{-1,0}\left(\frac{\lambda_{n,0}}{\sqrt{\eta}}\right) - J_1\left(\frac{\lambda_{n,0}}{\sqrt{\eta}}\right)S'_{-1,0}(\lambda_{n,0})}{[J_1(\lambda_{n,0})]^2 - \left[J_1\left(\frac{\lambda_{n,0}}{\sqrt{\eta}}\right)\right]^2}. \tag{B10}$$

Thus, Eq. (B5) can be written as:

$$\frac{\beta_0(r,t)}{[\tau\delta\omega]^2} = \frac{1}{8}\frac{\eta\ln(\eta)}{\eta-1}\frac{t}{\tau} + \frac{1}{8}\frac{\alpha_2(r,t)}{\tau\delta\omega} + f(r) - \frac{1}{8}\sum_{n=2}^{\infty}\frac{M^{-}_{n,0}}{N_{n,0}}\Phi_{n,0}(r)\frac{\mathrm{e}^{-\lambda^2_{n,0}\frac{t}{\tau}}}{\lambda^2_{n,0}}. \tag{B11}$$

For long times, the sums containing the exponential functions in the expressions for $\alpha_2(r,t)$ in Eq. (B4) and $\beta_0(r,t)$ in Eq. (B11) vanish:

$$\frac{\alpha_2(r,t\gg\tau)}{\tau\delta\omega} = -\frac{1}{4} \tag{B12}$$

$$\frac{\beta_0(r,t\gg\tau)}{[\tau\delta\omega]^2} = \frac{1}{8}\frac{\eta\ln(\eta)}{\eta-1}\frac{t}{\tau} - \frac{1}{32} + f(r) \tag{B13}$$

and consequently the partial differential equation (34) reduces to the ordinary differential equation

$$R_{\mathrm{C}}^2\Delta_r f(r) = \frac{1}{8}\left[\frac{\eta\ln(\eta)}{\eta-1} - \frac{R_{\mathrm{C}}^2}{r^2}\right] \tag{B14}$$

which is solved by

$$f(r) = a(\eta) + \frac{1}{32}\frac{\eta\ln(\eta)}{\eta-1}\frac{r^2}{R_{\mathrm{C}}^2} - \frac{1}{64}\left[\frac{\eta\ln(\eta)}{\eta-1} + 2\ln\left(\frac{r}{R_{\mathrm{C}}}\right)\right]^2. \tag{B15}$$

To find the integration constant $a(\eta)$, on has to bear in mind that $f(r)$ is a linear combination of the functions $\Phi_{n,0}(r)$, as can been seen in Eq. (B10). Thus, from the orthogonality relation (see Eq. (70) in [5])

$$\int_{R_\mathrm{C}}^{R_\mathrm{D}} \mathrm{d}r\, r\Phi_{n,0}(r) = 0\,, \tag{B16}$$

it follows that

$$\int_{R_\mathrm{C}}^{R_\mathrm{D}} \mathrm{d}r\, r f(r) = 0\,. \tag{B17}$$

Introducing Eq. (B15) leads to

$$a(\eta) = \frac{1}{32} + \frac{\ln(\eta)}{64}\left[3\frac{1-\eta^2}{[1-\eta]^2} + \ln(\eta)\frac{1-\eta^3}{[1-\eta]^3}\right]\,, \tag{B18}$$

and finally results in the full form of $f(r)$:

$$f(r) = \frac{\ln(\eta)}{64}\left[3\frac{1-\eta^2}{[1-\eta]^2} + \ln(\eta)\frac{1-\eta^3}{[1-\eta]^3}\right] + \frac{1}{32}\left[\frac{\eta\ln(\eta)}{\eta-1}\frac{r^2}{R_\mathrm{C}^2}+1\right] - \frac{1}{64}\left[\frac{\eta\ln(\eta)}{\eta-1} + 2\ln\left(\frac{r}{R_\mathrm{C}}\right)\right]^2\,. \tag{B19}$$

Introducing this expression into Eq. (B11) one finally obtains Eq. (37). Alternatively, the moments can also be obtained by integrating over the diffusion propagator as defined in Eq. (8). The first moment is then

$$\frac{\alpha_2(r,t)}{\tau\delta\omega} = -\frac{1}{4} + 2\pi\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-\lambda_{n,2}^2\frac{t}{\tau}}}{\lambda_{n,2}}\Phi_{n,2}(r)\frac{[J_2'(\frac{\lambda_{n,2}}{\sqrt{\eta}})]^2 - \eta^{\frac{3}{2}}J_2'(\lambda_{n,2})J_2'(\frac{\lambda_{n,2}}{\sqrt{\eta}})}{\left[\lambda_{n,2}^2-4\eta\right][J_2'(\lambda_{n,2})]^2 - \left[\lambda_{n,2}^2-4\right][J_2'(\frac{\lambda_{n,2}}{\sqrt{\eta}})]^2} \tag{B20}$$

which coincides with Eq. (36), and the coefficients of the second moment are

$$\begin{aligned}\frac{\beta_0(r,t)}{[\tau\delta\omega]^2} =& \frac{2}{\pi}\sum_{n=1}^{\infty}\sum_{m=2}^{\infty}\frac{W_{m,n}^{0,2}}{N_{n,2}N_{m,0}}\frac{J_2'(\frac{\lambda_{n,2}}{\sqrt{\eta}}) - \eta^{\frac{3}{2}}J_2'(\lambda_{n,2})}{\lambda_{n,2}^3\,J_2'(\frac{\lambda_{n,2}}{\sqrt{\eta}})}\frac{\Phi_{m,0}(r)}{\lambda_{m,0}^2-\lambda_{n,2}^2}\left[\frac{\mathrm{e}^{-\lambda_{m,0}^2\frac{t}{\tau}}-1}{\lambda_{m,0}^2} + \frac{1-\mathrm{e}^{-\lambda_{n,2}^2\frac{t}{\tau}}}{\lambda_{n,2}^2}\right] \\ &+ \frac{16}{\pi^2}\frac{\eta}{1-\eta}\sum_{n=1}^{\infty}\left[1-\eta^{\frac{3}{2}}\frac{J_2'(\lambda_{n,2})}{J_2'\left(\frac{\lambda_{n,2}}{\sqrt{\eta}}\right)}\right]\frac{\mathrm{e}^{-\lambda_{n,2}^2\frac{t}{\tau}}-1+\lambda_{n,2}^2\frac{t}{\tau}}{N_{n,2}\lambda_{n,2}^{10}}\,,\end{aligned} \tag{B21}$$

and

$$\frac{\beta_4(r,t)}{[\tau\delta\omega]^2} = \frac{2}{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{W_{m,n}^{4,2}}{N_{n,2}N_{m,4}}\frac{J_2'(\frac{\lambda_{n,2}}{\sqrt{\eta}}) - \eta^{\frac{3}{2}}J_2'(\lambda_{n,2})}{\lambda_{n,2}^3\,J_2'(\frac{\lambda_{n,2}}{\sqrt{\eta}})}\frac{\Phi_{m,4}(r)}{\lambda_{m,4}^2-\lambda_{n,2}^2}\left[\frac{\mathrm{e}^{-\lambda_{m,4}^2\frac{t}{\tau}}-1}{\lambda_{m,4}^2} + \frac{1-\mathrm{e}^{-\lambda_{n,2}^2\frac{t}{\tau}}}{\lambda_{n,2}^2}\right]\,. \tag{B22}$$

The integral coefficients are the integrals over products of the eigenfunctions

$$W_{m,n}^{0,2} = \int_{R_\mathrm{C}}^{R_\mathrm{D}}\frac{\mathrm{d}r_0}{r_0}\Phi_{m,0}(r_0)\Phi_{n,2}(r_0)\,, \tag{B23}$$

$$W_{m,n}^{4,2} = \int_{R_\mathrm{C}}^{R_\mathrm{D}}\frac{\mathrm{d}r_0}{r_0}\Phi_{m,4}(r_0)\Phi_{n,2}(r_0)\,, \tag{B24}$$

for which no explicit summation formula is known [6].

### Appendix C: Matrix calculation of the spectral expansion

In analogy to Eq. (54), the radial eigenfunctions given in Eq. (12) can be written in a discretized version as a column vector of length $N$:

$$\Phi_{n,\nu}(r) \rightarrow \vec{\Phi}_{n,\nu} = \begin{pmatrix}\Phi_{n,\nu}(r_1)\\ \Phi_{n,\nu}(r_2)\\ \vdots \\ \Phi_{n,\nu}(r_N)\end{pmatrix} \tag{C1}$$

which is an eigenvector of the discretized version of the cylindrical Bessel differential equation (9)

$$\left[\mathbf{\Delta}_r - \nu^2 \mathbf{r}^{-2}\right] \vec{\Phi}_{n,\nu} = -\lambda_{n,\nu}^2 \vec{\Phi}_{n,\nu} \,. \tag{C2}$$

Combining the eigenvalues $\lambda_{n,\nu}$ to the diagonal matrix

$$\mathbf{\Lambda}_\nu = \operatorname{diag}\left(\lambda_{1,\nu}, \lambda_{2,\nu}, \ldots, \lambda_{N,\nu}\right) \,, \tag{C3}$$

and the eigenvectors $\vec{\Phi}_{n,\nu}$ to the matrix

$$\mathbf{\Phi}_\nu = \left(\vec{\Phi}_{1,\nu}, \vec{\Phi}_{2,\nu}, \ldots, \vec{\Phi}_{N,\nu}\right) \tag{C4}$$

$$= \begin{pmatrix} \Phi_{1,\nu}(r_1) & \Phi_{2,\nu}(r_1) & \ldots \Phi_{N,\nu}(r_1) \\ \Phi_{1,\nu}(r_2) & \Phi_{2,\nu}(r_2) & \ldots \Phi_{N,\nu}(r_2) \\ \vdots & \vdots & \ldots \vdots \\ \Phi_{1,\nu}(r_N) & \Phi_{2,\nu}(r_N) & \ldots \Phi_{N,\nu}(r_N) \end{pmatrix} , \tag{C5}$$

the discretized version of the cylindrical Bessel differential equation (C2) can be combined to the form

$$\left[\mathbf{\Delta}_r - \nu^2 \mathbf{r}^{-2}\right] \mathbf{\Phi}_\nu = -\mathbf{\Phi}_\nu \mathbf{\Lambda}_\nu^2 \,. \tag{C6}$$

Using the orthogonality relation of the eigenfunctions from Eq. (13), the discretized orthogonality relation can be written in the form

$$\frac{1-\eta}{2\eta} \mathbf{\Phi}_\nu^\intercal \mathbf{k} \mathbf{\Phi}_\nu = \boldsymbol{N}_\nu = \operatorname{diag}\left(N_{1,\nu}, N_{2,\nu}, \ldots, N_{N,\nu}\right) \,, \tag{C7}$$

with the diagonal matrix $\mathbf{k}$ given in Eq. (83) and the normalisation constants $N_{N,\nu}$ given in Eq. (14). This orthogonality relation can be used to express the inverse of the eigenfunction matrix in terms of its transpose:

$$\mathbf{\Phi}_\nu^{-1} = \frac{1-\eta}{2\eta} \boldsymbol{N}_\nu^{-1} \mathbf{\Phi}_\nu^\intercal \mathbf{k} \,. \tag{C8}$$

### Appendix D: Inversion of the discretized Laplacian

As the first eigenvalue of the discretized Laplacian is zero (see Eq. (15)), the matrix $\mathbf{\Delta}_r$ is singular and the integral appearing in Eq. (77) cannot be solved directly. Instead, the Drazin inverse has to be applied [7]. Due to $\mathbf{\Delta}_r$ being diagonalizable, its matrix index is one: $\operatorname{Ind}(\mathbf{\Delta}_r) = 1$. In this case, the Drazin inverse is equivalent to a group-inverse and defined by the three axioms (see Eqs. (1) - (3) in [8])

$$\mathbf{\Delta}_r \mathbf{\Delta}_r^D = \mathbf{\Delta}_r^D \mathbf{\Delta}_r , \tag{D1}$$

$$\mathbf{\Delta}_r^D \mathbf{\Delta}_r \mathbf{\Delta}_r^D = \mathbf{\Delta}_r^D , \tag{D2}$$

$$\mathbf{\Delta}_r^D \mathbf{\Delta}_r^2 = \mathbf{\Delta}_r \,. \tag{D3}$$

Applying Theorem 1 from [8], the integral over the exponential $\exp(\mathbf{\Delta}_r \xi/\tau)$ can be written as

$$\int \mathrm{d}\xi \, \mathrm{e}^{-\frac{\xi}{\tau}\mathbf{\Delta}_r} = -\tau \mathbf{\Delta}_r^D \mathrm{e}^{-\frac{\xi}{\tau}\mathbf{\Delta}_r} + \xi \left[\mathbf{1} - \mathbf{\Delta}_r \mathbf{\Delta}_r^D\right] \,, \tag{D4}$$

with $^D$ denoting the Drazin inverse. Therefore

$$\int_0^t \mathrm{d}\xi \, \mathrm{e}^{-\frac{\xi}{\tau}\mathbf{\Delta}_r} = \tau \mathbf{\Delta}_r^D \left[\mathbf{1} - \mathrm{e}^{-\frac{t}{\tau}\mathbf{\Delta}_r}\right] + t \left[\mathbf{1} - \mathbf{\Delta}_r \mathbf{\Delta}_r^D\right] \,, \tag{D5}$$

and finally

$$\int_0^t \mathrm{d}\xi \, \mathrm{e}^{-\frac{\xi}{\tau}\mathbf{\Delta}_r} \mathbf{r}^{-2} \vec{1} = \tau \mathbf{\Delta}_r^D \left[\mathbf{1} - \mathrm{e}^{-\frac{t}{\tau}\mathbf{\Delta}_r}\right] \mathbf{r}^{-2} \vec{1} + t \left[\mathbf{1} - \mathbf{\Delta}_r \mathbf{\Delta}_r^D\right] \mathbf{r}^{-2} \vec{1} \,. \tag{D6}$$

In general, the Drazin inverse of $\mathbf{\Delta}_r$ can be computed by converting it into its Jordan normal form

$$\mathbf{\Delta}_r = \mathbf{P}^{-1}\begin{pmatrix}\mathbf{L} & 0\\ 0 & 0\end{pmatrix}\mathbf{P}\,, \tag{D7}$$

where the single zero is due to its diagonalizability (rendering its Jordan form diagonal) and the eigenvalue zero. The Drazin inverse is then

$$\mathbf{\Delta}_r^D = \mathbf{P}^{-1}\begin{pmatrix}\mathbf{L}^{-1} & 0\\ 0 & 0\end{pmatrix}\mathbf{P}\,. \tag{D8}$$

For large matrix sizes, this is equivalent to the expression from Eq. (C6) for $\nu = 0$:

$$\begin{aligned}\mathbf{\Delta}_r &= -\mathbf{\Phi}_0\mathbf{\Lambda}_0^2\mathbf{\Phi}_0^{-1} &\text{(D9)}\\ &= -\mathbf{\Phi}_0\mathrm{diag}(0,\lambda_{2,0}^2,\ldots,\lambda_{N,0}^2)\mathbf{\Phi}_0^{-1}\,. &\text{(D10)}\end{aligned}$$

Thus, the Drazin inverse results in

$$\mathbf{\Delta}_r^D = -\mathbf{\Phi}_0\mathrm{diag}(0,\lambda_{2,0}^{-2},\ldots,\lambda_{N,0}^{-2})\mathbf{\Phi}_0^{-1}\,, \tag{D11}$$

and the product $\mathbf{\Delta}_r\mathbf{\Delta}_r^D$ appearing in the right hand side of Eq. (D6) is:

$$\mathbf{\Delta}_r\mathbf{\Delta}_r^D = \mathbf{\Phi}_0\mathrm{diag}(0,1,\ldots,1)\mathbf{\Phi}_0^{-1}\,, \tag{D12}$$

with consequently

$$\mathbf{1} - \mathbf{\Delta}_r\mathbf{\Delta}_r^D = \mathbf{\Phi}_0\mathrm{diag}(1,0,\ldots,0)\mathbf{\Phi}_0^{-1}\,. \tag{D13}$$

Introducing the matrix $\mathbf{\Phi}_0^{-1}$ given in Eq. (C8) for $\nu = 0$ one gets with the explicit expressions for the normalization constants $N_{n,0}$ given in Eq. (14) the result

$$\mathbf{1} - \mathbf{\Delta}_r\mathbf{\Delta}_r^D = \underbrace{\begin{pmatrix}1 & 1 & \ldots & 1\\ 1 & 1 & \ldots & 1\\ \vdots & \vdots & \ldots & \vdots\\ 1 & 1 & \ldots & 1\end{pmatrix}}_{=\vec{1}\vec{1}^\intercal}\mathbf{k}\,. \tag{D14}$$

Replacing the discretized differential area element $\mathbf{k}$ by its continuous variant (see. Eq. (64) in [9]), one gets

$$\begin{aligned}\left[\mathbf{1} - \mathbf{\Delta}_r\mathbf{\Delta}_r^D\right]\mathbf{r}^{-2}\vec{1} &= \vec{1}\quad\underbrace{\vec{1}^\intercal\mathbf{k}\mathbf{r}^{-2}\vec{1}}_{=\frac{2\eta}{1-\eta}\frac{1}{R_\mathrm{C}^2}\int_{R_\mathrm{C}}^{R_\mathrm{D}}\mathrm{d}r\,r\frac{R_\mathrm{C}^2}{r^2}} &\text{(D15)}\\ &= \frac{\eta\ln(\eta)}{\eta-1}\vec{1}\,. &\text{(D16)}\end{aligned}$$

Eq. (D6) can then be written as

$$\int_0^t \mathrm{d}\xi\,\mathrm{e}^{-\frac{\xi}{\tau}\mathbf{\Delta}_r}\mathbf{r}^{-2}\vec{1} = \tau\mathbf{\Delta}_r^D\left[\mathbf{1} - \mathrm{e}^{-\frac{t}{\tau}\mathbf{\Delta}_r}\right]\mathbf{r}^{-2}\vec{1} + t\frac{\eta\ln(\eta)}{\eta-1}\vec{1}\,. \tag{D17}$$

Inserting this result into Eq. (77) one obtains:

$$\begin{aligned}\frac{\vec{\beta}_0(t)}{[\tau\delta\omega]^2} &= \frac{1}{8}\frac{\eta\ln(\eta)}{\eta-1}\frac{t}{\tau}\vec{1} - \frac{1}{32}\vec{1} + \left[\frac{1}{8}\mathbf{\Delta}_r^D\left[\mathrm{e}^{\frac{t}{\tau}\mathbf{\Delta}_r} - \mathbf{1}\right]\mathbf{r}^{-2} + \frac{1}{32}\mathrm{e}^{\frac{t}{\tau}[\mathbf{\Delta}_r - 4\mathbf{r}^{-2}]}\right]\vec{1} &\text{(D18)}\\ &= \frac{1}{8}\frac{\vec{\alpha}_2(t)}{\tau\delta\omega} + \frac{1}{8}\frac{\eta\ln(\eta)}{\eta-1}\frac{t}{\tau}\vec{1} + \frac{1}{8}\mathbf{\Delta}_r^D\mathrm{e}^{\frac{t}{\tau}\mathbf{\Delta}_r}\mathbf{r}^{-2}\vec{1} - \frac{1}{8}\mathbf{\Delta}_r^D\mathbf{r}^{-2}\vec{1}\,. &\text{(D19)}\end{aligned}$$

Comparing the time-independent terms with the analytical expression from Eq. (37), the last term becomes

$$\mathbf{\Delta}_r^D \mathbf{r}^{-2}\vec{1} = \frac{1}{8}\left[\frac{\eta\ln(\eta)}{\eta-1}\mathbf{1} + 2\ln(\mathbf{r})\right]^2 \vec{1} - \frac{1}{4}\left[\frac{\eta\ln(\eta)}{\eta-1}\mathbf{r}^2 - \mathbf{1}\right]\vec{1} - \frac{\ln(\eta)}{8}\left[3\frac{1-\eta^2}{[1-\eta]^2} + \ln(\eta)\frac{1-\eta^3}{[1-\eta]^3}\right]\vec{1}\,. \tag{D20}$$

Since the discretized Laplacian commutes with its Drazin inverse (see Eq. (D1)), one has

$$\mathbf{\Delta}_r^D \mathrm{e}^{\frac{t}{\tau}\mathbf{\Delta}_r} = \sum_{l=0}^{\infty}\frac{\mathbf{\Delta}_r^D\mathbf{\Delta}_r^l}{l!}\left[\frac{t}{\tau}\right]^l = \sum_{l=0}^{\infty}\frac{\mathbf{\Delta}_r^l\mathbf{\Delta}_r^D}{l!}\left[\frac{t}{\tau}\right]^l = \mathrm{e}^{\frac{t}{\tau}\mathbf{\Delta}_r}\mathbf{\Delta}_r^D\,. \tag{D21}$$

Thus, the term $\mathbf{\Delta}_r^D \mathrm{e}^{\frac{t}{\tau}\mathbf{\Delta}_r}\mathbf{r}^{-2}\vec{1}$ in Eq. (D19) can be written as

$$\mathbf{\Delta}_r^D \mathrm{e}^{\frac{t}{\tau}\mathbf{\Delta}_r}\mathbf{r}^{-2}\vec{1} = \mathrm{e}^{\frac{t}{\tau}\mathbf{\Delta}_r}\mathbf{\Delta}_r^D\mathbf{r}^{-2}\vec{1} \tag{D22}$$

$$= \frac{1}{8}\mathrm{e}^{\frac{t}{\tau}\mathbf{\Delta}_r}\left[\frac{\eta\ln(\eta)}{\eta-1}\mathbf{1} + 2\ln(\mathbf{r})\right]^2\vec{1} - \frac{1}{4}\mathrm{e}^{\frac{t}{\tau}\mathbf{\Delta}_r}\left[\frac{\eta\ln(\eta)}{\eta-1}\mathbf{r}^2 - \mathbf{1}\right]\vec{1} - \mathrm{e}^{\frac{t}{\tau}\mathbf{\Delta}_r}\frac{\ln(\eta)}{8}\left[3\frac{1-\eta^2}{[1-\eta]^2} + \ln(\eta)\frac{1-\eta^3}{[1-\eta]^3}\right]\vec{1}\,. \tag{D23}$$

Introducing Eq. (D20) and Eq. (D23) into Eq. (D19) one finally obtains

$$\frac{\vec{\beta}_0(t)}{[\tau\delta\omega]^2} = \frac{1}{8}\frac{\vec{\alpha}_2(t)}{\tau\delta\omega} + \frac{1}{8}\frac{\eta\ln(\eta)}{\eta-1}\frac{t}{\tau}\vec{1} + \frac{1}{16}\left[\mathrm{e}^{\frac{t}{\tau}\mathbf{\Delta}_r} - \mathbf{1}\right]\left[\frac{\eta\ln(\eta)}{\eta-1}\left[\ln(\mathbf{r}) - \frac{\mathbf{r}^2}{2}\right] + [\ln(\mathbf{r})]^2\right]\vec{1}\,. \tag{D24}$$

or by replacing $\vec{\alpha}_2(t)$ from Eq. (73) one gets Eq. (78).


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