Supplementary Material: "A new criterion beyond divergence for judging the dissipation of a system: dissipative power"

1 SUPPLEMENTARY DATA

The linear system $\dot{x} = f(x)$ can be written as

$$\dot{x} = Fx, F \text{ is a constant matrix,} \tag{S1}$$

here, $x \in \mathbb{R}^n$.

Kwon, Ao and Thouless([Kwon et al., PNAS., 102(37): 13029-13033 (2005)]) have discussed the construction of Lyapunov function of linear system(S1). And some necessary formulas are given as follows

$$F = -[D + Q]U = -[S + T]^{-1}U, \tag{S2}$$

$$[D + Q]^{-1} = S + T, \tag{S3}$$

$$FQ + QF^{\tau} = FD - DF^{\tau}, \tag{S4}$$

$$\nabla\phi(x) = Ux, \tag{S5}$$

here, only the Q is unknown, U is a symmetric matrix.

Here, set $F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$, $Q = \begin{pmatrix} 0 & q_{12} \\ -q_{12} & 0 \end{pmatrix}$, $D = \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix}$, and d_{11}, d_{22}, d_{22} are chose to satisfy $d_{11}, d_{22} \geq 0, d_{11}d_{22} - d_{12}^2 \geq 0$. Then, by (S4), we have

$$\begin{cases} (f_{11} + f_{22})q_{12} = -f_{21}d_{11} + (f_{11} - f_{22})d_{12} + f_{12}d_{22}, \\ d_{11}, d_{22} \geq 0, d_{11}d_{22} - d_{12}^2 \geq 0. \end{cases} \tag{S6}$$

here, only q_{12} is unknown.

1.1 Two examples of the planar linear saddle system with zero divergence

Cansider the following linear system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 \end{cases} \tag{S7}$$

and

$$\begin{cases} \dot{y}_1 = y_1 \\ \dot{y}_2 = -y_2 \end{cases}, \tag{S8}$$

which have a saddle point and the divergence equals zero.

Here, system (S7) is written as

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x, \tag{S9}$$

here, $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, equation (S6) can be rewritten as

$$\begin{cases} -d_{11} + d_{22} = 0 \\ d_{11}, d_{22} \geq 0, d_{11}d_{22} - d_{12}^2 \geq 0 \end{cases}, \tag{S10}$$

then, we obtain

$$q_{12} \text{ and } d_{12} \text{ are arbitrary real numbers}, d_{11} = d_{22} \geq 0, d_{11}d_{22} - d_{12}^2 \geq 0. \tag{S11}$$

And then, we can separately rewrite system (S9) into a (generalized) Hamiltonian system or a (generalized) gradient system by choosing one group of values in (S11):

- A (generalized) gradient system:
  By (S11), we choose $q_{12} = 0$, $d_{11} = d_{22} = 1$ and obtain $Q = \mathbf{0}, D = D + Q = [D + Q]^{-1} = S = I, U = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.
  Then, by (S5), the Lyapunov function of system (S9) is obtained

$$\phi(x) = -x_1 x_2. \tag{S12}$$

And then, it can verify that Lyapunov function (S12) does not increase along the trajectory

$$\frac{d\phi}{dt} = -x_1^2 - x_2^2 \leq 0, \tag{S13}$$

which shows that $\frac{d\phi}{dt}$ is less than zero except for the equilibrium point.
  The corresponding divergence $divf(x)$ and dissipative power $H_P(x)$ are derived

$$H_P(x) = x_1^2 + x_2^2 \geq 0, \tag{S14}$$

$$divf(x) = trace(F) \equiv 0. \tag{S15}$$

By (S13) and (S14), we obtain

$$\frac{d\phi}{dt} = -H_p(x). \tag{S16}$$

  Finally, system (S9) can be rewritten as

$$\begin{aligned} \dot{x} =& -[D + Q]\nabla\phi(x) \\ =& -D\nabla\phi(x) \\ =& -\nabla\phi(x) \\ =& \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned} \tag{S17}$$

Obviously, system (S17) is a gradient system which is dissipative.

Here, (S14) and (S17) indicate that system (S7) is a dissipative system, which is consistent with the result obtained by combining the monograph([Borrelli and Coleman. Differential equations: a modeling perspective. New York: Wiley (1998). p.505.]) and monograph([Sachdev. Nonlinear ordinary differential equations and their applications. New York: CRC Press (1990).p.354.]). However, $divf(x) \equiv 0$ can not.

- A (generalized) Hamiltonian system:

  By (S11), we choose $q_{12} = 1$, $d_{11} = d_{22} = 0$ and obtain $D = S = \mathbf{0}$, $Q = D + Q = -[D+Q]^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, here $\mathbf{0}$ is the zero matrix.

  Then, the Lyapunov function of system (S9) is obtained by (S5)

$$\phi(x) = \frac{x_1^2 - x_2^2}{2}, \tag{S18}$$

And then, it can verify that Lyapunov function (S18) does not increase along the trajectory

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial x_1}\dot{x}_1 + \frac{\partial \phi}{\partial x_2}\dot{x}_2 \equiv 0. \tag{S19}$$

The corresponding divergence $divf(x)$ and dissipative power $H_P(x)$ are derived

$$H_P(x) = \dot{x}^{\tau} S \dot{x} \equiv 0, \tag{S20}$$

$$divf(x) \equiv 0. \tag{S21}$$

By (S19) and (S20), we obtain

$$\frac{d\phi}{dt} = -H_p(x). \tag{S22}$$

Finally, system (S9) can be rewritten as

$$\begin{aligned} \dot{x} &= -[D+Q]\nabla\phi(x) \\ &= -Q\nabla\phi(x) \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla\phi(x) \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned} \tag{S23}$$

Obviously, system (S23) is a Hamiltonian system which is conservative.

Here, (S20) and (S21) show that system (S7) is conservative at the same time. On the other hand, that system (S7) can be rewritten into a Hamiltonian system is consistent with the result obtained by Liouville's Theorem([Arnold. Ordinary differential equations. New York: Springer(1992). p.251.]).

By reversible linear transformation $x = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} y$, system (S7) can be transformed into system (S8), which can be rewritten as

$$\dot{y} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y, \tag{S24}$$

here, $F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and equation (S6) can be rewritten as

$$\begin{cases} d_{12} = 0 \\ d_{11}, d_{22} \geq 0 \end{cases}, \tag{S25}$$

then, we obtain

$$q_{12} \text{ is an arbitrary real number}, d_{12} = 0, d_{11}, d_{22} \geq 0. \tag{S26}$$

And then, we can separately rewrite system (S24) into a (generalized) Hamiltonian system or a (generalized) gradient system by choosing one group of values in (S26):

- A (generalized) gradient system:
  By (S26), we choose $q_{12} = 0$ and $d_{11} = d_{22} = 1$. And we obtain $D = D + Q = [D + Q]^{-1} = S = I$, $Q = 0, U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, here I is an identity matrix.
  Then, the Lyapunov function of system (S24) is obtained by (S5)

$$\phi(y) = -\frac{y_1^2 - y_2^2}{2}. \tag{S27}$$

  And then, it can verify that Lyapunov function (S27) does not increase along the trajectory

$$\frac{d\phi}{dt} = -(y_1^2 + y_2^2) \leq 0, \tag{S28}$$

  which shows that $\frac{d\phi}{dt}$ is less than zero except for the equilibrium point.
  The corresponding divergence $divf(y)$ and dissipative power $H_P(y)$ are derived

$$H_P(y) = y_1^2 + y_2^2 \geq 0, \tag{S29}$$

$$divf(y) = trace(F) \equiv 0. \tag{S30}$$

  By (S28) and (S29), we obtain

$$\frac{d\phi}{dt} = -H_p. \tag{S31}$$

Finally, system (S24) can be rewritten as

$$\begin{aligned}\dot{y} =& -[D+Q]\nabla\phi(y)\\ =& -D\nabla\phi(y)\\ =& -\nabla\phi(y)\\ =& \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix} y_1\\ y_2\end{pmatrix}.\end{aligned} \tag{S32}$$

Obviously, system (S32) is a gradient system, which is dissipative.

Here, (S29) and (S32) indicate that system (S8) is a dissipative system, which is consistent with the result obtained by combining the monograph([Borrelli and Coleman. Differential equations: a modeling perspective. New York: Wiley (1998). p.505.]) and monograph([Sachdev. Nonlinear ordinary differential equations and their applications. New York: CRC Press (1990).p.354.]). However, $divf(y) \equiv 0$ can not.

- A (generalized) Hamiltonian system:

  By (S26), we choose $q_{12} = 1$, $d_{11} = d_{22} = 0$ and obtain $D = S = 0, Q = D + Q = -[D+Q]^{-1} = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}, U = \begin{pmatrix} 0 & -1\\ -1 & 0\end{pmatrix}$.

  Then, the Lyapunov function of system (S24) is obtained by (S5)

$$\phi(y) = -y_1 y_2 \tag{S33}$$

And then, it can verify that Lyapunov function (S33) does not increase along the trajectory

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial y_1}\dot{y}_1 + \frac{\partial\phi}{\partial y_2}\dot{y}_2 \equiv 0. \tag{S34}$$

The corresponding divergence $divf(y)$ and dissipative power $H_P(y)$ are derived

$$H_P(y) = \dot{y}^{\tau} S\dot{y} \equiv 0, \tag{S35}$$

$$divf(y) = trace(F) \equiv 0. \tag{S36}$$

By (S34) and (S35), we obtain

$$\frac{d\phi}{dt} = -H_p. \tag{S37}$$

Finally, system (S24) can be rewritten as

$$\begin{aligned}\dot{y} =& -[D+Q]\nabla\phi(y)\\ =& -Q\nabla\phi(y)\\ =& \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\nabla\phi(y)\\ =& \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix} y_1\\ y_2\end{pmatrix}.\end{aligned} \tag{S38}$$

Obviously, system (S38) is a Hamiltonian system which is conservative.

Here, (S35) and (S38) show that system (S8) is conservative at the same time. On the other hand, that system (S8) can be rewritten into a Hamiltonian system is consistent with the result obtained by Liouville's Theorem([Arnold. Ordinary differential equations. New York: Springer(1992). p.251.]).

Here, we summarize the results obtained from systems (S7) and (S8) into the Table S1.

Table S1. Two criteria on the planar linear saddle system with zero divergence

H_P and $div\ f$ / Systems / Types	$\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x$	$\dot{y} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y$
(Generalized) gradient system	$H_P \geq 0$ (only $x_1 = x_2 = 0, H_P = 0$)	$H_P \geq 0$ (only $y_1 = y_2 = 0, H_P = 0$)
	$div\ f(x) \equiv 0$	$div\ f(y) \equiv 0$
(Generalized) Hamiltonian system	$H_P \equiv 0$	$H_P \equiv 0$
	$div\ f(x) \equiv 0$	$div\ f(y) \equiv 0$

1.2 The general planar linear system

By invertible linear transformation, the matrix $F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ has the following four types of Jordan's normal form([Ma and Zhou. Qualitative and stability methods for ordinary differential equations(in Chinese). Beijing: Science press (2013). p.100.])

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \tag{S39}$$

in which $\lambda_1,\ \lambda_2,\ \alpha \pm \beta i$ are the eigenvalues of F, $\lambda_1 \neq \lambda_2$, $\sqrt{-1} = i$ and $\beta \neq 0$.

Therefore, we will calculate the corresponding divergence $div f(x)$ and dissipative power $H_P(x)$ in four cases of (S39).

(1) When $F = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, (S6) can be rewritten as

$$\begin{cases} (\lambda_1 + \lambda_2)q_{12} = (\lambda_1 - \lambda_2)d_{12}, \\ d_{11}, d_{22} \geq 0, d_{11}d_{22} - d_{12}^2 \geq 0, \end{cases} \tag{S40}$$

(i) If $\lambda_1 + \lambda_2 = 0$, from (S40), we have

$$q_{12} \text{ is an arbitrary real number}, d_{12} = 0, d_{11}, d_{22} \geq 0. \tag{S41}$$

Then, it has

$$\begin{aligned}
D &= \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}, d_{11}, d_{22} \geq 0, \\
Q &= \begin{pmatrix} 0 & q_{12} \\ -q_{12} & 0 \end{pmatrix}, \\
D+Q &= \begin{pmatrix} d_{11} & q_{12} \\ -q_{12} & d_{22} \end{pmatrix}, \\
[D+Q]^{-1} &= \frac{1}{d_{11}d_{22}+q_{12}^2} \begin{pmatrix} d_{22} & -q_{12} \\ q_{12} & d_{11} \end{pmatrix}, \\
S &= \frac{1}{d_{11}d_{22}+q_{12}^2} \begin{pmatrix} d_{22} & 0 \\ 0 & d_{11} \end{pmatrix}, \\
U &= -[D+Q]^{-1}A \\
&= -\frac{\lambda_1}{d_{11}d_{22}+q_{12}^2} \begin{pmatrix} d_{22} & q_{12} \\ q_{12} & -d_{11} \end{pmatrix},
\end{aligned} \tag{S42}$$

here, $d_{11},\ d_{22},\ q_{12}$ satisfy $d_{11}d_{22}+q_{12}^2 \neq 0$.

By (S42) and (S5), it can derive the Lyapunov function

$$\phi(x) = -\frac{\lambda_1(d_{22}x_1^2 + 2q_{12}x_1x_2 - d_{11}x_2^2)}{2(d_{11}d_{22}+q_{12}^2)}. \tag{S43}$$

Then, it can verify that the Lyapunov function does not increase along the trajectory

$$\begin{aligned}
\frac{d\phi}{dt} &= \frac{\partial\phi}{\partial x_1}\dot{x}_1 + \frac{\partial\phi}{\partial x_2}\dot{x}_2 \\
&= -\frac{\lambda_1^2(d_{22}x_1^2 + d_{11}x_2^2)}{d_{11}d_{22}+q_{12}^2} \\
&\leq 0,
\end{aligned} \tag{S44}$$

which shows that $\frac{d\phi}{dt}$ is less than zero except for the equilibrium point.

The corresponding divergence $divf(x)$ and dissipative power $H_P(x)$ are derived

$$\begin{aligned}
H_P(x) &= \dot{x}^\tau S\dot{x} \\
&= \frac{\lambda_1^2(d_{22}x_1^2 + d_{11}x_2^2)}{d_{11}d_{22}+q_{12}^2} \qquad \text{(S45)}\\
&\geq 0, \qquad \text{(S46)}\\
divf(x) &= trace(F) \\
&= \lambda_1 + \lambda_2 \\
&\equiv 0. \qquad \text{(S47)}
\end{aligned}$$

By (S44) and (S45), we obtain

$$\frac{d\phi}{dt} = -H_p. \tag{S48}$$

In this case, $H_P(x) \geq 0$ in (S46) implies

- If $H_P(x) \geq 0$(only $x_1 = x_2 = 0, H_P = 0$), system (S1) is dissipative by $H_P \geq 0$. However, $divf(x) \equiv 0$ means that system (S1) is conservative. The conclusions by these two criteria are completely opposite.
- If $H_P(x) \equiv 0$, these two criteria consistently indicate that planar linear system (S1) is conservative by $H_P(x) \equiv 0$ and $divf(x) \equiv 0$.

In short, these two criteria are not always completely consistent, so further detailed discussion and analysis are needed. Systems (S7) and (S8) have launched a detailed discussion and analysis of this situation, and the obtained results are summarize in the Table S1.

(ii) If $\lambda_1 + \lambda_2 \neq 0$, from (S40), we have

$$q_{12} = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} d_{12}, \tag{S49}$$

and

$$\begin{aligned}
Q &= \begin{pmatrix} d_{11} & \frac{2\lambda_1}{\lambda_1+\lambda_2} d_{12} \\ \frac{2\lambda_2}{\lambda_1+\lambda_2} d_{12} & d_{22} \end{pmatrix}, \\
D + Q &= \begin{pmatrix} d_{11} & \frac{2\lambda_1}{\lambda_1+\lambda_2} d_{12} \\ \frac{2\lambda_2}{\lambda_1+\lambda_2} d_{12} & d_{22} \end{pmatrix}, \\
[D+Q]^{-1} &= \frac{(\lambda_1+\lambda_2)^2}{d_{11}d_{22}(\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2 d_{12}^2} \begin{pmatrix} d_{22} & -\frac{2\lambda_1}{\lambda_1+\lambda_2} d_{12} \\ -\frac{2\lambda_2}{\lambda_1+\lambda_2} d_{12} & d_{11} \end{pmatrix}, \\
S &= \frac{[D+Q]^{-1} + \left\{[D+Q]^{-1}\right\}^\tau}{2} \\
&= \frac{(\lambda_1+\lambda_2)^2}{d_{11}d_{22}(\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2 d_{12}^2} \begin{pmatrix} d_{22} & -d_{12} \\ -d_{12} & d_{11} \end{pmatrix}, \\
U &= -\frac{(\lambda_1+\lambda_2)^2}{(\lambda_1+\lambda_2)^2 d_{11}d_{22} - 4\lambda_1\lambda_2 d_{12}^2} \begin{pmatrix} \lambda_1 d_{22} & -\frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2} d_{12} \\ -\frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2} d_{12} & \lambda_2 d_{11} \end{pmatrix},
\end{aligned} \tag{S50}$$

here, $d_{11},\ d_{12},\ d_{22}$ satisfy $d_{11}d_{22} - d_{12}^2 > 0$.

By (S5) and(S50), it can derive the Lyapunov function

$$\phi(x) = -\frac{(\lambda_1+\lambda_2)^2 (d_{22}\lambda_1 x_1^2 - \frac{4\lambda_1\lambda_2}{\lambda_1+\lambda_2} d_{12} x_1 x_2 + d_{11}\lambda_2 x_2^2)}{2[d_{11}d_{22}(\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2 d_{12}^2]}. \tag{S51}$$

Then, it can verify that the Lyapunov function does not increase along the trajectory

$$\begin{aligned}\frac{d\phi}{dt} &= \frac{\partial\phi}{\partial x_1}\dot{x}_1 + \frac{\partial\phi}{\partial x_2}\dot{x}_2 \\ &= -\frac{(\lambda_1+\lambda_2)^2(d_{22}\lambda_1^2x_1^2 - 2\lambda_1\lambda_2 d_{12}x_1x_2 + d_{11}\lambda_2^2x_2^2)}{(\lambda_1+\lambda_2)^2 d_{11}d_{22} - 4\lambda_1\lambda_2 d_{12}^2} \\ &\leq 0,\end{aligned} \tag{S52}$$

which shows that $\frac{d\phi}{dt}$ is less than zero except for the equilibrium point.

And the corresponding divergence $divf(x)$ and dissipative power $H_P(x)$ are derived

$$\begin{aligned}H_P(x) &= \dot{x}^\tau S\dot{x} \\ &= \frac{(\lambda_1+\lambda_2)^2(d_{22}\lambda_1^2x_1^2 - 2d_{12}\lambda_1\lambda_2x_1x_2 + d_{11}\lambda_2^2x_2^2)}{d_{11}d_{22}(\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2d_{12}^2} && \text{(S53)} \\ &\geq 0, && \text{(S54)} \\ divf(x) &= trace(F) \\ &= \lambda_1+\lambda_2 \\ &\neq 0. && \text{(S55)}\end{aligned}$$

By (S52) and (S53), we obtain

$$\frac{d\phi}{dt} = -H_p(x). \tag{S56}$$

In this case, $H_P(x) \geq 0$ in (S54) indicates that system (S1) is dissipative and the dissipation is equal to zero at the equilibrium point. The $divf(x) \neq 0$ in (S55) implies $divf(x) > 0$ or $divf(x) < 0$. When $divf(x) < 0$, the results are always consistent by using the divergence and dissipative power to determine the dissipation of a planar linear system; When $divf(x) > 0$, the divergence can't judge the dissipation of the system, while dissipative power can judge the system being dissipative. Therefore, it is more advantageous to use the dissipative power than divergence to determine the dissipation of the system.

(2) When $F = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$, the (S6) can be rewritten as

$$\begin{cases} 2\lambda_1 q_{12} = 0, \\ d_{11}, d_{22} \geq 0,\ d_{11}d_{22} - d_{12}^2 \geq 0. \end{cases} \tag{S57}$$

(i) If $\lambda_1 \neq 0$, then $q_{12} = 0$, and

$$\begin{aligned}
Q &= \mathbf{0},\ \mathbf{0}\ \textit{is a zreo matrix},\\
D+Q &= \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix},\\
[D+Q]^{-1} &= \frac{1}{d_{11}d_{22}-d_{12}^2}\begin{pmatrix} d_{22} & -d_{12} \\ -d_{12} & d_{11} \end{pmatrix},\\
S &= \frac{1}{d_{11}d_{22}-d_{12}^2}\begin{pmatrix} d_{22} & -d_{12} \\ -d_{12} & d_{11} \end{pmatrix},\\
U &= -[D+Q]^{-1}F\\
&= -\frac{\lambda_1}{d_{11}d_{22}-d_{12}^2}\begin{pmatrix} d_{22} & -d_{12} \\ -d_{12} & d_{11} \end{pmatrix},
\end{aligned} \tag{S58}$$

here, $d_{11},\ d_{12},\ d_{22}$ satisfy $d_{11}d_{22} - d_{12}^2 > 0$.

By (S58) and (S5), the Lyapunov function can be obtained

$$\phi(x) = -\frac{\lambda_1(d_{22}x_1^2 - 2d_{12}x_1x_2 + d_{11}x_2^2)}{2(d_{11}d_{22}-d_{12}^2)}. \tag{S59}$$

Then, it can verify that the Lyapunov function does not increase along the trajectory

$$\begin{aligned}
\frac{d\phi}{dt} &= \frac{\partial\phi}{\partial x_1}\dot{x}_1 + \frac{\partial\phi}{\partial x_2}\dot{x}_2\\
&= -\frac{\lambda_1^2(d_{22}x_1^2 - 2d_{12}x_1x_2 + d_{11}x_2^2)}{d_{11}d_{22}-d_{12}^2}\\
&\leq 0,
\end{aligned} \tag{S60}$$

which shows that $\frac{d\phi}{dt}$ is less than zero except for the equilibrium point.

The corresponding divergence $divf(x)$ and dissipative power $H_P(x)$ are derived

$$\begin{aligned}
H_P(x) &= \dot{x}^\tau S\dot{x}\\
&= \frac{\lambda_1^2(d_{22}x_1^2 - 2d_{12}x_1x_2 + d_{11}x_2^2)}{d_{11}d_{22}-d_{12}^2} && \text{(S61)}\\
&\geq 0. && \text{(S62)}\\
divf(x) &= trace(F)\\
&= 2\lambda_1\\
&\neq 0. && \text{(S63)}
\end{aligned}$$

By (S60) and (S61), we obtain

$$\frac{d\phi}{dt} = -H_p(x). \tag{S64}$$

In this case, $H_P(x) \geq 0$ in (S62) indicates that system (S1) is dissipative and the dissipation is equal to zero at the equilibrium point. The $divf(x) \neq 0$ in (S63) implies $divf(x) > 0$ or $divf(x) < 0$. When $divf(x) < 0$, the results are always consistent by using the divergence and dissipative power to determine the dissipation of a planar linear system; When $divf(x) > 0$, the divergence can't judge the dissipation of the system, while dissipative power can judge the system being dissipative. Therefore, it is more advantageous to use the dissipative power than divergence to determine the dissipation of the system.

(ii) If $\lambda_1 = 0$, the system is conservative. It's easy to know

$$\begin{aligned} H_P(x) &= \dot{x}^\tau S \dot{x} \\ &\equiv 0, \end{aligned} \tag{S65}$$

$$\begin{aligned} divf(x) &= trace(F) \\ &= 2\lambda_1 \\ &\equiv 0. \end{aligned} \tag{S66}$$

And, we obtain

$$\frac{d\phi}{dt} = -H_p(x). \tag{S67}$$

So they are consistent in representing the system being conservative by $divf(x) \equiv 0$ and $H_P(x) \equiv 0$.

(3) When $F = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}$, the (S6) can be rewritten as

$$\begin{cases} 2\lambda_1 q_{12} = -d_{11}, \\ d_{11},\ d_{22} \geq 0,\ d_{11}d_{22} - d_{12}^2 \geq 0. \end{cases} \tag{S68}$$

(i) If $\lambda_1 \neq 0$, then

$$q_{12} = -\frac{d_{11}}{2\lambda_1}, \tag{S69}$$

and

$$\begin{aligned} Q &= \frac{-d_{11}}{2\lambda_1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ D + Q &= \begin{pmatrix} d_{11} & d_{12} - \frac{d_{11}}{2\lambda_1} \\ d_{12} + \frac{d_{11}}{2\lambda_1} & d_{22} \end{pmatrix}, \\ [D+Q]^{-1} &= m_1 \begin{pmatrix} d_{22} & -d_{12} + \frac{d_{11}}{2\lambda_1} \\ -d_{12} - \frac{d_{11}}{2\lambda_1} & d_{11} \end{pmatrix}, \\ S &= m_1 \begin{pmatrix} d_{22} & -d_{12} \\ -d_{12} & d_{11} \end{pmatrix}, \\ U &= -m_1 \begin{pmatrix} d_{22}\lambda_1 - d_{12} + \frac{d_{11}}{2\lambda_1} & -d_{12}\lambda_1 + \frac{d_{11}}{2} \\ -d_{12}\lambda_1 + \frac{d_{11}}{2} & d_{11}\lambda_1 \end{pmatrix}, \end{aligned} \tag{S70}$$

in which $m_1 = \frac{4\lambda_1^2}{4\lambda_1^2 d_{11}d_{22} - 4\lambda_1^2 d_{12}^2 + d_{11}^2}$, and $d_{11},\ d_{12},\ d_{22}$ satisfy $d_{11}d_{22} - d_{12}^2 > 0$.
By (S70) and (S5), the Lyapunov function can be obtained

$$\phi(x) = -\frac{2\lambda_1^2 \left[(d_{22}\lambda_1 - d_{12} + \frac{d_{11}}{2\lambda_1})x_1^2 + 2(-d_{12}\lambda_1 + \frac{d_{11}}{2})x_1x_2 + d_{11}\lambda_1 x_2^2\right]}{4\lambda_1^2 d_{11}d_{22} - 4\lambda_1^2 d_{12}^2 + d_{11}^2}, \tag{S71}$$

Then, it gets

$$\frac{d\phi}{dt} = -\frac{4\lambda_1^2 \left[(d_{22}\lambda_1^2 - 2d_{12}\lambda_1 + d_{11})x_1^2 + 2(d_{11} - d_{12}\lambda_1)\lambda_1 x_1x_2 + d_{11}\lambda_1^2 x_2^2\right]}{4\lambda_1^2 d_{11}d_{22} - 4\lambda_1^2 d_{12}^2 + d_{11}^2}, \tag{S72}$$

the discriminant of $g_1(x_1, x_2) \overset{\Delta}{=} (d_{22}\lambda_1^2 - 2d_{12}\lambda_1 + d_{11})x_1^2 + 2(-d_{12}\lambda_1 + d_{11})\lambda_1 x_1 x_2 + d_{11}\lambda_1^2 x_2^2 = 0$ about x_1 is

$$\begin{aligned}\Delta &= 4(-d_{12}\lambda_1 + d_{11})^2\lambda_1^2 - 4(d_{22}\lambda_1^2 - 2d_{12}\lambda_1 + d_{11})d_{11}\lambda_1^2 \\ &= 4\lambda_1^4\left[d_{12}^2 - d_{11}d_{22}\right] \\ &\leq 0,\end{aligned} \tag{S73}$$

it has $g_1(x_1, x_2) \geq 0$, then

$$\frac{d\phi}{dt} \leq 0, \tag{S74}$$

which shows that $\frac{d\phi}{dt}$ is less than zero except for the equilibrium point.
The corresponding divergence $divf(x)$ and dissipative power H_P are derived

$$H_P(x) = \frac{4\lambda_1^2 g(x_1, x_2)}{4\lambda_1^2 d_{11}d_{22} - 4\lambda_1^2 d_{12}^2 + d_{11}^2} \tag{S75}$$

$$\geq 0, \tag{S76}$$

$$\begin{aligned}divf(x) &= trace(F) \\ &= 2\lambda_1 \\ &\neq 0.\end{aligned} \tag{S77}$$

By (S72) and (S75), we obtain

$$\frac{d\phi}{dt} = -H_p(x). \tag{S78}$$

In this case, $H_P(x) \geq 0$ in (S76) indicates that system (S1) is dissipative and the dissipation is equal to zero at the equilibrium point. The $divf(x) \neq 0$ in (S77) implies $divf(x) > 0$ or $divf(x) < 0$. When $divf(x) < 0$, the results are always consistent by using the divergence and dissipative power to determine the dissipation of a planar linear system; When $divf(x) > 0$, the divergence can't judge the dissipation of the system, while dissipative power can judge the system being dissipative. Therefore, it is more advantageous to use the dissipative power than divergence to determine the dissipation of the system.

(ii) If $\lambda_1 = 0$, from (S68), we have $d_{11} = d_{12} = 0$, q_{12} is an arbitrary nonzero real number. And

$$\begin{aligned}
D &= \begin{pmatrix} 0 & 0 \\ 0 & d_{22} \end{pmatrix}, \\
Q &= \begin{pmatrix} 0 & q_{12} \\ -q_{12} & 0 \end{pmatrix}, \\
D+Q &= \begin{pmatrix} 0 & q_{12} \\ -q_{12} & d_{22} \end{pmatrix}, \\
[D+Q]^{-1} &= \frac{1}{q_{12}^2}\begin{pmatrix} d_{22} & -q_{12} \\ q_{12} & 0 \end{pmatrix}, \\
S &= \frac{1}{q_{12}^2}\begin{pmatrix} d_{22} & 0 \\ 0 & 0 \end{pmatrix}, \\
U &= -[D+Q]^{-1}F \\
&= \frac{1}{q_{12}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned} \tag{S79}$$

in which $q_{12} \neq 0$ and $d_{22} \geq 0$.

By (S79) and (S5), the Lyapunov function can be obtained

$$\phi(x) = \frac{1}{2q_{12}} x_1^2, \tag{S80}$$

then, it can verify that the Lyapunov function does not increase along the trajectory

$$\begin{aligned}
\frac{d\phi}{dt} &= \frac{\partial \phi}{\partial x_1}\dot{x}_1 + \frac{\partial \phi}{\partial x_2}\dot{x}_2 \\
&= \frac{\lambda_1 x_1^2}{q_{12}} \\
&\equiv 0.
\end{aligned} \tag{S81}$$

Then, the corresponding divergence $divf(x)$ and dissipative power $H_P(x)$ are derived

$$\begin{aligned}
H_P(x) &= \dot{x}^\tau S \dot{x} \\
&= x^\tau F^\tau S F x \\
&= \frac{1}{q_{12}^2}\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} d_{22} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&\equiv 0,
\end{aligned} \tag{S82}$$

$$\begin{aligned}
divf(x) &= trace(F) \\
&= 2\lambda_1 \\
&\equiv 0.
\end{aligned} \tag{S83}$$

By (S81) and (S82), we obtain

$$\frac{d\phi}{dt} = -H_p(x). \tag{S84}$$

In this case, they are consistent in representing the system being conservative by $divf(x) \equiv 0$ and $H_P(x) \equiv 0$.

(4) When $F = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, the (S6) can be rewritten as

$$\begin{cases} 2\alpha q_{12} = \beta(d_{11} + d_{22}), \\ d_{11},\ d_{22} \geq 0,\ d_{11}d_{22} - d_{12}^2 \geq 0,\ \beta \neq 0. \end{cases} \tag{S85}$$

(i) If $\alpha \neq 0$, we have

$$Q_{12} = \frac{\beta(d_{11} + d_{22})}{2\alpha}, \tag{S86}$$

and

$$\begin{aligned}
Q &= \frac{\beta(d_{11} + d_{22})}{2\alpha} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
D + Q &= \begin{pmatrix} d_{11} & d_{12} + \frac{\beta(d_{11}+d_{22})}{2\alpha} \\ d_{12} - \frac{\beta(d_{11}+d_{22})}{2\alpha} & d_{22} \end{pmatrix}, \\
[D + Q]^{-1} &= m_2 \begin{pmatrix} d_{22} & -d_{12} - \frac{\beta(d_{11}+d_{22})}{2\alpha} \\ -d_{12} + \frac{\beta(d_{11}+d_{22})}{2\alpha} & d_{11} \end{pmatrix}, \\
S &= m_2 \begin{pmatrix} d_{22} & -d_{12} \\ -d_{12} & d_{11} \end{pmatrix}, \\
U &= -m_2 \begin{pmatrix} d_{22}\alpha + d_{12}\beta + \frac{\beta^2(d_{11}+d_{22})}{2\alpha} & -d_{12}\alpha + \frac{\beta(d_{22}-d_{11})}{2} \\ -d_{12}\alpha + \frac{\beta(d_{22}-d_{11})}{2} & d_{11}\alpha - d_{12}\beta + \frac{\beta^2(d_{11}+d_{22})}{2\alpha} \end{pmatrix},
\end{aligned} \tag{S87}$$

in which $m_2 = \frac{4\alpha^2}{4\alpha^2(d_{11}d_{22} - d_{12}^2) + \beta^2(d_{11}+d_{22})^2}$, and $d_{11},\ d_{12},\ d_{22}$ satisfy $d_{11}d_{22} - d_{12}^2 \geq 0$.

By (S87) and (S5), the Lyapunov function can be obtained

$$\phi(x) = \frac{-2\alpha^2 g_2(x_1, x_2)}{4\alpha^2(d_{11}d_{22} - d_{12}^2) + \beta^2(d_{11} + d_{22})^2}, \tag{S88}$$

in which $g_2(x_1, x_2) = [d_{22}\alpha + d_{12}\beta + \frac{\beta^2(d_{11}+d_{22})}{2\alpha}]x_1^2 + 2[-d_{12}\alpha + \frac{\beta(d_{22}-d_{11})}{2}]x_1x_2 + [d_{11}\alpha - d_{12}\beta + \frac{\beta^2(d_{11}+d_{22})}{2\alpha}]x_2^2$. Then, it gets

$$\frac{d\phi}{dt} = \frac{-4\alpha^2 g_3(x_1, x_2)}{4\alpha^2(d_{11}d_{22} - d_{12}^2) + \beta^2(d_{11} + d_{22})^2}, \tag{S89}$$

in which $g_3(x_1, x_2) = (d_{22}\alpha^2 + 2d_{12}\alpha\beta + d_{11}\beta^2)x_1^2 + 2[d_{12}(\beta^2 - \alpha^2) + \alpha\beta(d_{22} - d_{11})]x_1x_2 + (d_{11}\alpha^2 + d_{22}\beta^2 - 2d_{12}\alpha\beta)x_2^2$. The discriminant of equation $g_3(x_1, x_2) = 0$ corresponding to the formula in (S89) is

$$\begin{aligned}\Delta &= 4[d_{12}(\beta^2-\alpha^2)+\alpha\beta(d_{22}-d_{11})]^2 \\ &\quad -(d_{22}\alpha^2+2d_{12}\alpha\beta+d_{11}\beta^2)(d_{11}\alpha^2+d_{22}\beta^2-2d_{12}\alpha\beta) \\ &= 4(\alpha^2+\beta^2)^2(d_{12}^2-d_{11}d_{22}) \\ &\leq 0,\end{aligned} \tag{S90}$$

then, it can verify that the Lyapunov function does not increase along the trajectory

$$\begin{aligned}\frac{d\phi}{dt} &= \frac{\partial\phi}{\partial x_1}\dot{x}_1+\frac{\partial\phi}{\partial x_2}\dot{x}_2 \\ &\leq 0,\end{aligned} \tag{S91}$$

which shows that $\frac{d\phi}{dt}$ is less than zero except for the equilibrium point.

Then, the corresponding divergence $divf(x)$ and dissipative powerH_P are derived

$$H_P(x) = \frac{4\alpha^2 g_3(x_1,x_2)}{4\alpha^2(d_{11}d_{22}-d_{12}^2)+\beta^2(d_{11}+d_{22})^2} \tag{S92}$$

$$\geq 0, \tag{S93}$$

$$\begin{aligned}divf(x) &= trace(F) \\ &= 2\alpha \\ &\neq 0.\end{aligned} \tag{S94}$$

By (S89) and (S92), we obtain

$$\frac{d\phi}{dt} = -H_p(x). \tag{S95}$$

In this case, $H_P(x) \geq 0$ in (S93) indicates that system (S1) is dissipative and the dissipation is equal to zero at the equilibrium point. The $divf(x) \neq 0$ in (S94) implies $divf(x) > 0$ or $divf(x) < 0$. When $divf(x) < 0$, the results are always consistent by using the divergence and dissipative power to determine the dissipation of a planar linear system; When $divf(x) > 0$, the divergence can't judge the dissipation of the system, while dissipative power can judge the system being dissipative. Therefore, it is more advantageous to use the dissipative power than divergence to determine the dissipation of the system.

(ii) If $\alpha = 0$, we have $d_{11} = d_{22} = d_{12} = 0$, q_{12} is an arbitrary nonzero real number. Then, we can obtain

$$\begin{aligned} Q &= \begin{pmatrix} 0 & q_{12} \\ -q_{12} & 0 \end{pmatrix}, \\ D &= \mathbf{0} \\ D+Q &= \begin{pmatrix} 0 & q_{12} \\ -q_{12} & 0 \end{pmatrix}, \\ [D+Q]^{-1} &= \frac{1}{q_{12}^2}\begin{pmatrix} 0 & -q_{12} \\ q_{12} & 0 \end{pmatrix}, \\ S &= \frac{[D+Q]^{-1} + \left\{[D+Q]^{-1}\right\}^{\tau}}{2} \\ &\equiv \mathbf{0} \\ U &= -[D+Q]^{-1}F \\ &= -\frac{\beta}{q_{12}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{S96}$$

By (S96) and (S5), the Lyapunov function can be obtained

$$\phi(x) = -\frac{\beta}{2q_{12}}(x_1^2 + x_2^2), \tag{S97}$$

it can verify that the Lyapunov function does not increase along trajectory

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{\partial\phi}{\partial x_1}\dot{x}_1 + \frac{\partial\phi}{\partial x_2}\dot{x}_2 \\ &= -\frac{\beta}{q_{12}}(x_1\dot{x}_1 + x_2\dot{x}_2) \\ &= -\frac{\beta}{q_{12}}[x_1\beta x_2 + x_2(-\beta x_1)] \\ &\equiv 0. \end{aligned} \tag{S98}$$

Then, the corresponding divergence $divf(x)$ and dissipative power H_P are derived

$$\begin{aligned} H_P(x) &= \dot{x}^{\tau}S\dot{x} \\ &\equiv 0. \end{aligned} \tag{S99}$$

$$\begin{aligned} divf(x) &= trace(F) \\ &\equiv 0. \end{aligned} \tag{S100}$$

By (S98) and (S99), we obtain

$$\frac{d\phi}{dt} = -H_p(x). \tag{S101}$$

In this case, they are consistent in representing the system being conservative by $divf(x) \equiv 0$ and $H_P(x) \equiv 0$.

For the planar linear system with four Jordan's normal forms, all the results are summarized into the Table S2.

Table S2. The H_p and $divf$ of planar linear system with four Jordan's normal forms as coefficient matrix

H_P / $div\ f$ / Jordan's normal norm	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$	$\begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}$	$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$
$div\ f(x) = 0$	$H_P \geq 0$ contains two case: $H_P \geq 0$(only $x_1 = x_2 = 0, H_P = 0$) $H_P \equiv 0$	$H_P \equiv 0$	$H_P \equiv 0$	$H_P \equiv 0$
$div\ f(x) \neq 0 (divf(x) > 0$ and $divf(x) < 0)$	$H_P \geq 0$ (only $x_1 = x_2 = 0, H_P = 0$)	$H_P \geq 0$ (only $x_1 = x_2 = 0, H_P = 0$)	$H_P \geq 0$ (only $x_1 = x_2 = 0, H_P = 0$)	$H_P \geq 0$ (only $x_1 = x_2 = 0, H_P = 0$)