APPENDIX B

Below we provide the derivation of the Landau equation, following [7].

Aiming to investigate evolution of the system near the instability threshold (7), we apply two-time-scale approach and present h and Θ as a series in powers of small parameter of supercriticality δ, see eqs. (8). Substituting (7) and (8) into governing equations (1)-(2) and collecting the terms of equal powers in δ, we obtain at the first order the linear stability problem, Eqs. (10)-(11). Its solution can be presented as a traveling wave

$$h_1 = A(t_2)e^{i(k_c x_0 + \omega_c t_0)} + c.c.,\ \theta_1 = (\alpha + 1)A(t_2)e^{i(k_c x_0 + \omega_c t_0)} + c.c. \tag{B1}$$

At higher orders in small δ one arrives at the inhomogeneous partial differential equations with the same linear operator as in linear stability problem. Inhomogeneities in these PDEs contains some secular terms, that should vanish according to the solvability condition (13).

At the second order in δ we obtain

$$\frac{\partial h_2}{\partial t_0} - \Delta\left(\frac{1}{3}P_2 + \frac{Ma_0}{2}f_2\right) = \nabla \cdot\ h_1 \nabla P_1 + Ma_0 h_1 \nabla f_1\ , \tag{B2}$$

$$\begin{aligned}\frac{\partial \theta_2}{\partial t_0} - \Delta\left(\theta_2 + \frac{1}{8}P_2 + \frac{Ma_0}{6}f_2\right) + (\beta - \varkappa_l)f_2 = -h_1\frac{\partial \theta_1}{\partial t_0} + \nabla\cdot(h_1\nabla\theta_1) - \frac{1}{2}(\nabla h_1)^2 + \varkappa_q f_1^2 \\ + \left(\frac{1}{3}\nabla P_1 + \frac{Ma_0}{2}\nabla f_1\right)\cdot\nabla f_1 + \nabla\cdot\left(\frac{h_1}{2}\nabla P_1 + \frac{Ma_0}{2}h_1\nabla f_1\right),\end{aligned} \tag{B3}$$

where $P_{(1,2)} = Gah_{(1,2)} - C\nabla^2 h_{(1,2)}$, $f_{(1,2)} = \theta_{(1,2)} - h_{(1,2)}$. One can see that this system is solvable for any right-hand sides, therefore it does not provide any restrictions on the amplitude A. The solution can be chosen in the form

$$h_2 = a_{22}A^2 e^{2i(kx_0 + \omega_0 t_0)} + c.c.,\ f_2 = b_{22}A^2 e^{2i(kx_0 + \omega_0 t_0)} + b_{20}\mid A\mid^2 + c.c. \tag{B4}$$

Constants a_{22}, b_{20} and b_{22} are given in Appendix C.

At the third order, inhomogeneities in equations contain terms proportional to $\exp(ikx_0 + i\omega_0 t_0)$ that produce secular terms. The elimination of the secular terms at the third order leads to the Landau equation (6) that governs the evolution of the wave amplitude.