APPENDICES

1 PROOF OF (39)

Let $B(0,\sqrt{\eta})$ be the closed ball of $\mathbb{R}^{N_0}$ with center 0 and radius $\sqrt{\eta}$ and let S_{N_0-1} be the surface of the unit-radius sphere in $\mathbb{R}^{N_0-1}$. We have

$$\begin{aligned}
\mathsf{P}(\mathsf{z}\in C_\eta) &= \frac{1}{(2\pi)^{N_0/2}(\det(\Omega))^{1/2}}\int_{C_\eta}\exp\Big(-\frac{1}{2}z^\top\Omega^{-1}z\Big)\,dz\\
&= \frac{1}{(2\pi)^{N_0/2}}\int_{B(0,\sqrt{\eta})}\exp\Big(-\frac{1}{2}\|u\|^2\Big)\,du\\
&= \frac{1}{(2\pi)^{N_0/2}}S_{N_0-1}\int_0^{\sqrt{\eta}}\rho^{N_0-1}\exp\Big(-\frac{\rho^2}{2}\Big)\,d\rho\\
&= \frac{1}{(2\pi)^{N_0/2}}\frac{2(\pi)^{N_0/2}}{\Gamma(N_0/2)}\int_0^{\sqrt{\eta}}\rho^{N_0-1}\exp\Big(-\frac{\rho^2}{2}\Big)\,d\rho\\
&= \frac{1}{\Gamma(N_0/2)}\int_0^{\frac{\eta}{2}}\zeta^{N_0/2-1}\exp(-\zeta)\,d\zeta\\
&= \frac{\gamma(N_0/2,\eta/2)}{\Gamma(N_0/2)}.
\end{aligned}\tag{66}$$

2 PROOFS OF PROPOSITION 3

(i): As $\epsilon\to 0$, the diagonal elements of $\Omega_{\epsilon,\mathbb{K}_0}$ with index $\ell\notin\mathbb{K}$ tend to zero, which in (47) amounts to assuming that the corresponding components of the input perturbation are zero. The existence of the limit $\Omega_{0,\mathbb{K}_0}$ is secured based on the remark at the end of Section3.2.
(ii): If $\epsilon=1$, then $\Omega_{\epsilon,\mathbb{K}_0}=\mathrm{Id}_{N_0}$ and (47) reduces to (20).
(iii): For every $i\in\{1,\ldots,m-1\}$, let $\Lambda_i\in\mathcal{D}_{N_i}(\{2\alpha_i-1,1\})$. Then, by using the triangle inequality,

$$\|W_m\Lambda_{m-1}\cdots\Lambda_1W_1(\Omega_{\epsilon,\mathbb{K}_0})^{1/2}\|_{p,q}\le\|W_m\Lambda_{m-1}\cdots\Lambda_1W_1(\Omega_{\epsilon',\mathbb{K}_1})^{1/2}\|_{p,q}\|(\Omega_{\epsilon',\mathbb{K}_1})^{-1}\Omega_{\epsilon,\mathbb{K}_0}\|_{p,p}^{1/2}.\tag{67}$$

By taking the supremum of both sides with respect to the $(\Lambda_i)_{1\le i\le m-1}$ matrices, we deduce that

$$\vartheta_m^{\Omega_{\epsilon,\mathbb{K}_0}}\le\|(\Omega_{\epsilon',\mathbb{K}_1})^{-1}\Omega_{\epsilon,\mathbb{K}_0}\|_{p,p}^{1/2}\,\vartheta_m^{\Omega_{\epsilon',\mathbb{K}_1}}.\tag{68}$$

On the other hand,

$$(\Omega_{\epsilon',\mathbb{K}_1})^{-1}\Omega_{\epsilon,\mathbb{K}_0}=\mathrm{Diag}\left(\frac{\sigma^2_{\epsilon,\mathbb{K}_0,1}}{\sigma^2_{\epsilon',\mathbb{K}_1,1}},\ldots,\frac{\sigma^2_{\epsilon,\mathbb{K}_0,N_0}}{\sigma^2_{\epsilon',\mathbb{K}_1,N_0}}\right),\tag{69}$$

where, by using the fact that $\Omega_{\epsilon,\mathbb{K}_0}\preceq\Omega_{\epsilon',\mathbb{K}_1}$,

$$(\forall\ell\in\{1,\ldots,N_0\})\quad\frac{\sigma_{\epsilon,\mathbb{K}_0,\ell}}{\sigma_{\epsilon',\mathbb{K}_1,\ell}}\le 1.\tag{70}$$

Since $(\Omega_{\epsilon',\mathbb{K}_1})^{-1}\Omega_{\epsilon,\mathbb{K}_0}$ is a diagonal matrix with elements lower than or equal to 1, $\|(\Omega_{\epsilon',\mathbb{K}_1})^{-1}\Omega_{\epsilon,\mathbb{K}_0}\|_{p,p}\le 1$ and it follows from (68) that

$$\vartheta_m^{\Omega_{\epsilon,\mathbb{K}_0}} \leq \vartheta_m^{\Omega_{\epsilon',\mathbb{K}_1}}. \tag{71}$$

(iv): Let $(\epsilon, \epsilon') \in]0,1]^2$ with $\epsilon < \epsilon'$. We have $\Omega_{\epsilon,\mathbb{K}_0} \preceq \Omega_{\epsilon',\mathbb{K}_0}$ and, according to (iii),

$$\vartheta_m^{\Omega_{\epsilon,\mathbb{K}_0}} \leq \vartheta_m^{\Omega_{\epsilon',\mathbb{K}_0}}. \tag{72}$$

(v): If $\mathbb{K}_0 \subset \mathbb{K}_1$, then $\Omega_{\epsilon,\mathbb{K}_0} \preceq \Omega_{\epsilon,\mathbb{K}_1}$ and the result follows from (iii).
(vi): We have

$$\sum_{\substack{\mathbb{K}\subset\{1,\ldots,N_0\}\\ \operatorname{card}\mathbb{K}=K}} \Omega_{\epsilon,\mathbb{K}}^{1/2} = \left(\binom{N_0-1}{K-1} + \left(\binom{N_0}{K} - \binom{N_0-1}{K-1}\right)\epsilon\right) \mathrm{Id}_{N_0}. \tag{73}$$

By using the relation

$$\binom{N_0}{K} = \frac{N_0}{K}\binom{N_0-1}{K-1}, \tag{74}$$

we deduce that

$$\sum_{\substack{\mathbb{K}\subset\{1,\ldots,N_0\}\\ \operatorname{card}\mathbb{K}=K}} \Omega_{\epsilon,\mathbb{K}}^{1/2} = \omega_{K,\epsilon}\,\mathrm{Id}_{N_0}. \tag{75}$$

For every $i \in \{1,\ldots,m-1\}$, let $\Lambda_i \in \mathcal{D}_{N_i}(\{2\alpha_i - 1, 1\})$. Then,

$$\begin{aligned}\|W_m\Lambda_{m-1}\cdots\Lambda_1 W_1\|_{p,q} &= \frac{1}{\omega_{K,\epsilon}}\left\|W_m\Lambda_{m-1}\cdots\Lambda_1 W_1\Big(\sum_{\substack{\mathbb{K}\subset\{1,\ldots,N_0\}\\ \operatorname{card}\mathbb{K}=K}} \Omega_{\epsilon,\mathbb{K}}^{1/2}\Big)\right\|_{p,q}\\ &\leq \frac{1}{\omega_{K,\epsilon}} \sum_{\substack{\mathbb{K}\subset\{1,\ldots,N_0\}\\ \operatorname{card}\mathbb{K}=K}} \|W_m\Lambda_{m-1}\cdots\Lambda_1 W_1\Omega_{\epsilon,\mathbb{K}}^{1/2}\|_{p,q}.\end{aligned} \tag{76}$$

We deduce that

$$\begin{aligned}\vartheta_m &= \sup_{\substack{\Lambda_1\in\mathcal{D}_{N_1}(\{2\alpha_1-1,1\}),\\ \vdots\\ \Lambda_{m-1}\in\mathcal{D}_{N_{m-1}}(\{2\alpha_{m-1}-1,1\})}} \|W_m\Lambda_{m-1}\cdots\Lambda_1 W_1\|_{p,q}\\ &\leq \frac{1}{\omega_{K,\epsilon}} \sum_{\substack{\mathbb{K}\subset\{1,\ldots,N_0\}\\ \operatorname{card}\mathbb{K}=K}} \sup_{\substack{\Lambda_1\in\mathcal{D}_{N_1},\\ \vdots\\ \Lambda_{m-1}\in\mathcal{D}_{N_{m-1}}}} \|W_m\cdots\Lambda_1 W_1\Omega_{\epsilon,\mathbb{K}}^{1/2}\|_{p,q}\\ &= \frac{1}{\omega_{K,\epsilon}} \sum_{\substack{\mathbb{K}\subset\{1,\ldots,N_0\}\\ \operatorname{card}\mathbb{K}=K}} \vartheta_m^{\Omega_{\epsilon,\mathbb{K}}}.\end{aligned} \tag{77}$$

Furthermore, according to (ii) and (iv), for every $\mathbb{K} \subset \{1, \ldots, N_0\}$,

$$\vartheta_m^{\Omega_{\epsilon,\mathbb{K}}} \leq \vartheta_m^{\Omega_{1,\mathbb{K}}} = \vartheta_m. \tag{78}$$

This yields

$$\max_{\substack{\mathbb{K}\subset\{1,\ldots,N_0\}\\ \mathrm{card}\,\mathbb{K}=K}} \vartheta_m^{\Omega_{\epsilon,\mathbb{K}}} \leq \vartheta_m. \tag{79}$$

(vii): The proof is similar to that of (vi) by noticing that, if $\mathcal{P}$ is a partition of $\{1, \ldots, N_0\}$, then

$$\sum_{\mathbb{K}\in\mathcal{P}} \Omega_{\epsilon,\mathbb{K}}^{1/2} = \omega_{\mathcal{P},\epsilon} \, \mathrm{Id}_{N_0} \,. \tag{80}$$

(viii): For every $\varnothing \neq \mathbb{K} \subset \{1, \ldots, N_0\}$, $\vartheta_m^{\Omega_{\epsilon,\mathbb{K}}} = \vartheta_m^{\Omega_{0,\mathbb{K}}} + o(\epsilon)$. It is thus sufficient to prove this inequality in the limit case when $\epsilon \to 0$. For every $i \in \{1, \ldots, m-1\}$, let $\Lambda_i \in \mathcal{D}_{N_i}(\{2\alpha_i - 1, 1\})$. Let $x \in \mathbb{R}^{N_0}$ and let $x_{\mathbb{K}}$ be the projection of x onto the space of vectors whose components indexed by $\{1, \ldots, N_0\} \setminus \mathbb{K}$ are zero. We have thus $x_{\mathbb{K}_0} = x_{\mathbb{K}_1} + x_{\mathbb{K}_2}$. Then,

$$\begin{aligned}
&\|W_m\Lambda_{m-1}\cdots\Lambda_1 W_1 \Omega_{0,\mathbb{K}_0}^{1/2} x\|_q \\
&= \|W_m\Lambda_{m-1}\cdots\Lambda_1 W_1 x_{\mathbb{K}_0}\|_q \\
&\leq \|W_m\Lambda_{m-1}\cdots\Lambda_1 W_1 x_{\mathbb{K}_1}\|_q + \|W_m\Lambda_{m-1}\cdots\Lambda_1 W_1 x_{\mathbb{K}_2}\|_q \\
&= \|W_m\Lambda_{m-1}\cdots\Lambda_1 W_1 \Omega_{0,\mathbb{K}_1}^{1/2} x_{\mathbb{K}_1}\|_q + \|W_m\Lambda_{m-1}\cdots\Lambda_1 W_1 \Omega_{0,\mathbb{K}_2}^{1/2} x_{\mathbb{K}_2}\|_q \\
&\leq \|W_m\Lambda_{m-1}\cdots\Lambda_1 W_1 \Omega_{0,\mathbb{K}_1}^{1/2}\|_{p,q}\|x_{\mathbb{K}_1}\|_p + \|W_m\Lambda_{m-1}\cdots\Lambda_1 W_1 \Omega_{0,\mathbb{K}_2}^{1/2}\|_{p,q}\|x_{\mathbb{K}_2}\|_p.
\end{aligned} \tag{81}$$

By using Hölder's inequality, we deduce that

$$\begin{aligned}
&\|W_m\Lambda_{m-1}\cdots\Lambda_1 W_1 \Omega_{0,\mathbb{K}_0}^{1/2} x\|_q \\
&\qquad \leq \Big(\|W_m\cdots\Lambda_1 W_1 \Omega_{0,\mathbb{K}_1}^{1/2}\|_{p,q}^{p^*} + \|W_m\cdots\Lambda_1 W_1 \Omega_{0,\mathbb{K}_2}^{1/2}\|_{p,q}^{p^*}\Big)^{1/p^*} (\|x_{\mathbb{K}_1}\|_p^p + \|x_{\mathbb{K}_2}\|_p^p)^{1/p}.
\end{aligned} \tag{82}$$

Since $\|x_{\mathbb{K}_1}\|_p^p + \|x_{\mathbb{K}_2}\|_p^p = \|x_{\mathbb{K}_0}\|_p^p$, it follows that

$$\|W_m\Lambda_{m-1}\cdots\Lambda_1 W_1 \Omega_{0,\mathbb{K}_0}^{1/2}\|_{p,q} \leq \Big(\|W_m\cdots\Lambda_1 W_1 \Omega_{0,\mathbb{K}_1}^{1/2}\|_{p,q}^{p^*} + \|W_m\cdots\Lambda_1 W_1 \Omega_{0,\mathbb{K}_2}^{1/2}\|_{p,q}^{p^*}\Big)^{1/p^*}. \tag{83}$$

Taking the supremum with respect to $(\Lambda_i)_{1\leq i\leq m-1}$ and majorizing the supremum of the sum in the right-hand side by the sum of the suprema yield (53).