Supplementary Material: Generating function method for calculating the potentials of inhomogeneous polyhedra

1 DERIVATION OF THE CONNECTION BETWEEN THE POTENTIAL AND THE GENERATING FUNCTION

Let us derive Eq. (6) that expresses potential $\varphi(\mathbf{R})$ via generating function $G(\mathbf{R}, \mathbf{k})$. We start from the Maclaurin series for $G(\mathbf{R}, \mathbf{k})$, Eq. (7), where

$$G^{(0)}(\mathbf{R}) = \iiint\limits_V \tfrac{1}{|\mathbf{r}-\mathbf{R}|}\, \mathrm{d}^3\mathbf{r},$$

$$G^{(1)}_\alpha(\mathbf{R}) = \iiint\limits_V \tfrac{r_\alpha - R_\alpha}{|\mathbf{r}-\mathbf{R}|}\, \mathrm{d}^3\mathbf{r},$$

$$G^{(2)}_{\alpha\beta}(\mathbf{R}) = \tfrac{1}{2!} \iiint\limits_V \tfrac{(r_\alpha - R_\alpha)(r_\beta - R_\beta)}{|\mathbf{r}-\mathbf{R}|}\, \mathrm{d}^3\mathbf{r},$$

and so on. It is evident from Eq. (7) that

$$G\left(\mathbf{R}, \frac{\partial}{\partial \mathbf{R}}\right) \rho(\mathbf{R}) = G^{(0)}(\mathbf{R})\rho(\mathbf{R}) + G^{(1)}_\alpha(\mathbf{R})\rho_{,\alpha}(\mathbf{R}) + G^{(2)}_{\alpha\beta}(\mathbf{R})\rho_{,\alpha\beta}(\mathbf{R}) + \ldots, \tag{S1}$$

where we use a shorthand notation $\rho_{,\alpha}(\mathbf{R}) = \partial\rho(\mathbf{R})/\partial R_\alpha$, $\rho_{,\alpha\beta}(\mathbf{R}) = \partial^2\rho(\mathbf{R})/\partial R_\alpha \partial R_\beta$, and so on. Substituting the expressions for $G^{(0)}(\mathbf{R})$, $G^{(1)}_\alpha(\mathbf{R})$, etc. into Eq. (S1), one can obtain

$$G\left(\mathbf{R}, \frac{\partial}{\partial \mathbf{R}}\right) \rho(\mathbf{R}) = \iiint\limits_V \left[\rho(\mathbf{R}) + \rho_{,\alpha}(\mathbf{R})\,(r_\alpha - R_\alpha) + \frac{1}{2!}\rho_{,\alpha\beta}(\mathbf{R})\,(r_\alpha - R_\alpha)(r_\beta - R_\beta) + \ldots\right] \frac{\mathrm{d}^3\mathbf{r}}{|\mathbf{r}-\mathbf{R}|}$$

$$= \iiint\limits_V \rho(\mathbf{r}) \frac{\mathrm{d}^3\mathbf{r}}{|\mathbf{r}-\mathbf{R}|} = \varphi(\mathbf{R}), \tag{S2}$$

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2 EXPRESSING THE GENERATING FUNCTION VIA CONTRIBUTIONS OF FACES, EDGES AND VERTICES

For simplicity, in this Section we consider the generating function at $\mathbf{R} = 0$. Let us denote it as $G_0(\mathbf{k})$:

$$G_0(\mathbf{k}) = G(0, \mathbf{k}) = \iiint\limits_V \frac{e^{\mathbf{k}\cdot\mathbf{r}}}{|\mathbf{r}|}\, \mathrm{d}^3\mathbf{r}. \tag{S3}$$

This does not lead to any loss of generality, because equality $\mathbf{R} = 0$ just means that we choose point $\mathbf{R}$ as an origin of coordinates. In the end of this Section, we will return to the generating function in its original form, Eq. (5).

We will perform a transformation of the expression for $G_0(\mathbf{k})$ in four steps. The first step consists in reducing the volume integral in Eq. (S3) to a surface integral via Gauss's theorem. For this purpose we find such a vector function $\mathbf{u}(\mathbf{r})$ that

$$\nabla \cdot \mathbf{u} = \frac{e^{\mathbf{k}\cdot\mathbf{r}}}{|\mathbf{r}|}. \tag{S4}$$

It is convenient to choose this function in the form

$$\mathbf{u}(\mathbf{r}) = \frac{\mathbf{r}}{|\mathbf{r}|} f(\mathbf{k}\cdot\mathbf{r}). \tag{S5}$$

Then it follows from Eq. (S4) that

$$f(\mathbf{k}\cdot\mathbf{r}) = \frac{e^{\mathbf{k}\cdot\mathbf{r}}(\mathbf{k}\cdot\mathbf{r}-1)+1}{(\mathbf{k}\cdot\mathbf{r})^2}. \tag{S6}$$

Here the last term in the numerator is a constant of integration, its value is chosen such that function f remains finite at $\mathbf{k}=0$. Applying Gauss's theorem to the r.h.s. of Eq. (S3), one can obtain a representation of G_0 as a two-dimensional integral over the surface ∂V of the body:

$$G_0(\mathbf{k}) = \iint\limits_{\partial V} \frac{\mathbf{n}\cdot\mathbf{r}}{|\mathbf{r}|} f(\mathbf{k}\cdot\mathbf{r})\, \mathrm{d}S, \tag{S7}$$

where $\mathbf{n}$ is the outward normal to the surface (a unit vector, see Fig. S1), and $\mathrm{d}S$ is a surface element. Since the surface is a polyhedron, the integral in Eq. (S7) is a sum of contributions of polyhedron's faces. Let us denote these contributions as G_f, where subscript f labels a face. Therefore

$$G_0(\mathbf{k}) = \sum_f G_f(\mathbf{k}). \tag{S8}$$

The second step is a transformation of quantity G_f using a two-dimensional version of Gauss's theorem (or, equivalently, Stokes' theorem). It is convenient to choose the coordinate system K_f such that axis z is directed perpendicular to face f, along the outward normal to face f, and axis y is directed along the projection of vector k to face f (Fig. S1a). Then, $\mathbf{n}\cdot\mathbf{r} = z$, and

$$\mathbf{k}\cdot\mathbf{r} = k_{\|} y + k_z z, \tag{S9}$$

where $k_{\|}$ is the absolute value of the projection of $\mathbf{k}$ to face f. In system K_f, expression for G_f acquires the form

$$G_f = \iint\limits_{\text{face } f} \frac{z\, f(k_{\|} y + k_z z)}{r}\, \mathrm{d}x\, \mathrm{d}y, \tag{S10}$$

where $r = \sqrt{x^2+y^2+z^2}$. Let us define a vector $\mathbf{v}(x,y)$ on face f as follows:

$$v_x = \frac{x}{r} z\beta(y), \quad v_y = \frac{y}{r} z\beta(y), \tag{S11}$$

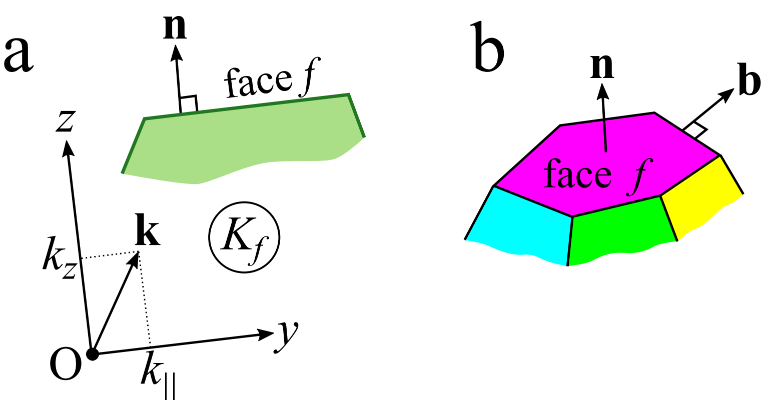

Figure S1. (a) Coordinate system K_f associated with face f. In this system, axis z is directed perpendicular to face f, along vector $\mathbf{n}$. Axis y lies in the plane spanned by vectors $\mathbf{n}$ and $\mathbf{k}$. Unit vector $\mathbf{n}$ is the outward normal to face f. The origin of coordinates is denoted as O. (b) Unit vector $\mathbf{b}$ that appears in Eq. (S14). This vector lies in the plane of face f and is directed out of this face, perpendicular to its boundary.

where function $\beta(y)$ obeys equation

$$\frac{\mathrm{d}(y\beta)}{\mathrm{d}y} = f(k_{\|}y + k_z z). \tag{S12}$$

Then

$$\frac{\mathrm{d}v_x}{\mathrm{d}x} + \frac{\mathrm{d}v_y}{\mathrm{d}y} = \frac{z\, f(k_{\|}y + k_z z)}{r} + \frac{z^3\beta(y)}{r^3}. \tag{S13}$$

Expressing the integrand of the r.h.s of Eq. (S10) via Eq. (S13) and applying the two-dimensional Gauss's theorem, we obtain

$$G_f = -\iint\limits_{\text{face } f} \frac{z^3\beta(y)}{r^3}\,\mathrm{d}x\,\mathrm{d}y + \oint\limits_{\partial f} \mathbf{b}\cdot\mathbf{v}\,\mathrm{d}l, \tag{S14}$$

where ∂f stands for the boundary of face f; $\mathbf{b}$ is the outward normal vector to the boundary of face f lying in the plane of this face (see Fig. S1b); and $\mathrm{d}l$ is the line element of the face perimeter.

The third step aims at expressing the quantity G_f through a line integral and a solid angle. Again, we use the two-dimensional Gauss's theorem for this. Let us define vector function $\mathbf{w}(x, y)$ on face f:

$$w_x = \frac{x}{r}\gamma(y), \quad w_y = \frac{k_{\|}\lambda}{r}, \tag{S15}$$

where

$$\gamma(y) = \frac{\beta(y) + k_{\|}\lambda y + \mu}{y^2 + z^2}, \tag{S16}$$

and quantities λ and μ do not depend on coordinates x and y. We will choose the values of λ and μ in such a way that function $\gamma(y)$ has no poles. It follows from Eqs. (S15) and (S16) that

$$\frac{\mathrm{d}w_x}{\mathrm{d}x} + \frac{\mathrm{d}w_y}{\mathrm{d}y} = \frac{\beta(y) + \mu}{r^3}. \tag{S17}$$

Deriving from here the integrand $z^3\beta(y)/r^3$ of the first term in the r.h.s. of Eq. (S14), taking into account that z is constant on the face, and applying Gauss's theorem, one can obtain the following representation for G_f:

$$G_f = \iint\limits_{\text{face } f} \frac{z^3\mu}{r^3}\,\mathrm{d}x\,\mathrm{d}y + \oint\limits_{\partial f} \mathbf{b}\cdot(\mathbf{v} - z^3\mathbf{w})\,\mathrm{d}l. \tag{S18}$$

The first integral here is related to the *solid angle* Ω_f subtended by face f at the origin of coordinates:

$$\Omega_f = -z_f \iint\limits_{\text{face } f} \frac{1}{r^3}\,\mathrm{d}x\,\mathrm{d}y, \tag{S19}$$

where $z_f = \mathbf{n}\cdot\mathbf{r}$ is z-coordinate of face f. It is evident from comparison of the two latter equations, that the first integral in Eq. (S18) is equal to $-z_f^2\mu\Omega_f$. The second integral is a sum of contributions of edges surrounding face f, since the boundary ∂f consists of these edges. Let us denote the contribution of edge e to G_f as $\mathcal{L}_{fe}$. Therefore, Eq. (S18) takes on the following form:

$$G_f = -z_f^2\mu\Omega_f + \sum_e \mathcal{L}_{fe}, \tag{S20}$$

where summation is over edges adjacent to face f, and

$$\mathcal{L}_{fe} = \int\limits_{\text{edge } e} \mathbf{b}\cdot(\mathbf{v} - z^3\mathbf{w})\,\mathrm{d}l. \tag{S21}$$

Equations (S20) and (S21) contain functions $\beta(y)$, $\gamma(y)$ and constants λ, μ that are not yet specified. Let us calculate them. The solution of Eq. (S12), that remains finite at $y = 0$, is

$$\beta(y) = \frac{1}{k_\parallel y}\left(\frac{e^{k_\parallel y + k_z z} - 1}{k_\parallel y + k_z z} - \frac{e^{k_z z} - 1}{k_z z}\right). \tag{S22}$$

In order to get rid of poles of function $\gamma(y)$, it is necessary that the numerator of the r.h.s of Eq. (S16) is equal to zero at $y = \pm iz$:

$$\beta(\pm iz) \pm ik_\parallel\lambda z + \mu = 0, \tag{S23}$$

whence

$$\lambda = -\frac{\beta(iz) - \beta(-iz)}{2ik_\parallel z} \tag{S24}$$

and

$$\mu = -\frac{\beta(iz) + \beta(-iz)}{2}. \tag{S25}$$

Function $\gamma(y)$ is now fully defined by Eq. (S16), where $\beta(y)$, λ and μ are to be substituted from Eqs. (S22), (S24) and (S25).

The last step is the calculation of line integrals $\mathcal{L}_{fe}$ in Eq. (S21). For convenience, we find expressions for these integrals in a different coordinate system $\tilde{K}_{fe}$ associated with face f and edge e, see Fig. S2. In this system, axis x is directed along edge e, axes y and z are directed along vectors $\mathbf{b}$ and $\mathbf{n}$ defined above.

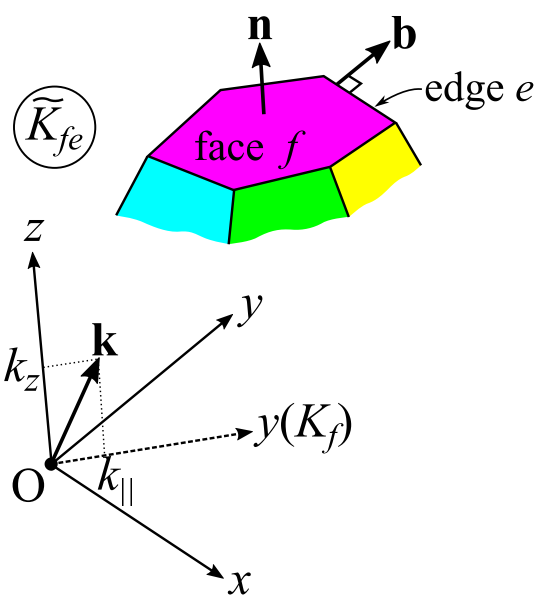

Figure S2. Coordinate system $\tilde{K}_{fe}$ associated with face f and edge e. In this system, axis z is directed perpendicular to face f. Axis y is directed parallel to face f and perpendicular to edge e. Axis x is directed along edge e. Unit vectors $\mathbf{n}$ and $\mathbf{b}$ show directions of axes z and y, correspondingly. $y(K_f)$ is axis y in coordinate system K_f. The origin of coordinates is denoted as O.

In the transition from K_f to $\tilde{K}_{fe}$, the argument y of functions $\beta(y)$ and $\gamma(y)$ transforms as

$$y \to \frac{k_x x + k_y y}{k_{\|}}, \tag{S26}$$

and the integrand of Eq. (S21) is expressed in system $\tilde{K}_{fe}$ as follows (see Supplemental Section 5):

$$\mathbf{b} \cdot (\mathbf{v} - z^3 \mathbf{w}) = \frac{-z^3 k_y \lambda + yz\beta + \frac{z^3 k_x (k_y x - k_x y)}{k_{\|}^2}\gamma}{\sqrt{x^2 + y^2 + z^2}}. \tag{S27}$$

The numerator of the r.h.s of the latter equation is a function of coordinate x along the edge. Denoting this function as $\mathcal{F}(x)$, one can obtain from Eq. (S21) that

$$\mathcal{L}_{fe} = \int_{x_1}^{x_2} \frac{\mathcal{F}(x)}{\sqrt{x^2 + r_\perp^2}}\, \mathrm{d}x, \tag{S28}$$

where x_1 and x_2 are coordinates of the end points of edge e, and $r_\perp^2 \equiv y^2 + z^2$ is the squared distance from the coordinate origin to the line that contains edge e. Since function $\mathcal{F}(x)$ is *entire* (i. e. has no singularities at finite x), there is a Taylor series expansion:

$$\mathcal{F}(x) = \mathcal{F}_0 + \mathcal{F}_1 x + \mathcal{F}_2 x^2 + \dots \tag{S29}$$

Based on this expansion, one can reduce the integral in Eq. (S28) to a much simpler integral

$$L_e = \int\limits_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{x^2 + r_\perp^2}}. \tag{S30}$$

Quantity L_e can be understood as a potential at the origin of coordinates, created by uniformly charged edge e with the unit linear charge density. Using identities from Nenashev and Dvurechenskii (2017)

$$\int\limits_{x_1}^{x_2} \frac{x^n}{\sqrt{x^2 + r_\perp^2}}\,\mathrm{d}x = -\frac{n-1}{n} r_\perp^2 \int\limits_{x_1}^{x_2} \frac{x^{n-2}}{\sqrt{x^2 + r_\perp^2}}\,\mathrm{d}x + \frac{x_2^{n-1}}{n}\sqrt{x_2^2 + r_\perp^2} - \frac{x_1^{n-1}}{n}\sqrt{x_1^2 + r_\perp^2}, \tag{S31}$$

where $n \geq 2$, and

$$\int\limits_{x_1}^{x_2} \frac{x}{\sqrt{x^2 + r_\perp^2}}\,\mathrm{d}x = \sqrt{x_2^2 + r_\perp^2} - \sqrt{x_1^2 + r_\perp^2}, \tag{S32}$$

one can obtain (see details in Supplemental Section 6) that

$$\mathcal{L}_{fe} = B L_e + C(x_2)\sqrt{x_2^2 + r_\perp^2} - C(x_1)\sqrt{x_1^2 + r_\perp^2}, \tag{S33}$$

where

$$B = \sum_{n=0}^{\infty} \mathcal{F}_{2n} (-r_\perp^2)^n \, \frac{(2n-1)!!}{(2n)!!}, \tag{S34}$$

$$C(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathcal{F}_{m+2n+1} x^m (-r_\perp^2)^n \, \frac{(m+2n)!!(m-1)!!}{m!!(m+2n+1)!!}. \tag{S35}$$

Note that factors $\sqrt{x_1^2 + r_\perp^2}$ and $\sqrt{x_2^2 + r_\perp^2}$ in Eq. (S33) are distances from the origin of coordinates to the ends of edge e.

Combining equations (S8), (S20) and (S33), one can get the following expression for generating function $G_0(\mathbf{k})$:

$$G_0(\mathbf{k}) = \sum_f A_f(\mathbf{k})\,\Omega_f + \sum_{(f,e)} B_{fe}(\mathbf{k})\,L_e + \sum_{(f,e,v)} C_{fev}(\mathbf{k})\,|\mathbf{r}_v|. \tag{S36}$$

Here, the first sum of the r.h.s. is over faces of the polyhedron. The second sum is over all pairs (f, e), where f is a face, and e is an edge adjacent to face f. Similarly, the third sum is over all triples (f, e, v), where f is a face, e is an edge adjacent to face f, and v is one of two vertices that are connected by edge e. A_f stands for $-z_f^2\mu$, where μ is defined by Eq. (S25); B_{fe} is defined by Eq. (S34); C_{fev} is either $C(x_2)$ or $-C(x_1)$, where function $C(x)$ is introduced in Eq. (S35); and $\mathbf{r}_v$ is the radius-vector of vertex v.

It is natural to interpret Eq. (S36) as follows: generating function G_0 is a sum of contributions of all faces, edges, and vertices of the polyhedral body. The contribution of face f contains factor Ω_f, a solid angle subtended by this face at the origin of coordinates. The contribution of edge e contains factor L_e that has the meaning of potential of uniformly charged edge e at the origin. And the contribution of vertex v contains the distance $|\mathbf{r}_v|$ between the vertex and the origin.

Finally, we generalize these results to function $G(\mathbf{R},\mathbf{k})$. The generalization is rather straightforward: we just replace the origin of coordinates with point $\mathbf{R}$ in Eq. (S36):

$$G(\mathbf{R},\mathbf{k}) = \sum_{f} A_f(\mathbf{R},\mathbf{k})\,\Omega_f(\mathbf{R}) + \sum_{(f,e)} B_{fe}(\mathbf{R},\mathbf{k})\,L_e(\mathbf{R}) + \sum_{(f,e,v)} C_{fev}(\mathbf{R},\mathbf{k})\,|\mathbf{r}_v - \mathbf{R}|. \tag{S37}$$

$\Omega_f(\mathbf{R})$ here is the solid angle subtended by face f at point $\mathbf{R}$, and $L_e(\mathbf{R})$ is the potential of uniformly charged edge e at point $\mathbf{R}$. Quantities A_f, B_{fe} and C_{fev} become dependent on $\mathbf{R}$, since coordinates x_1, x_2, y, z are counted from point $\mathbf{R}$.

Functions $A_f = -z^2\mu$ for different faces f can be reduced to one and the same function. Indeed, A_f depends on three parameters: coordinate z and components k_z, $k_\parallel$ of vector $\mathbf{k}$. One can express these parameters through the outward normal vector $\mathbf{n}_f$ to face f and a radius-vector $\mathbf{r}_f$ of an arbitrarily chosen point on face f (see Fig. 2):

$$z = \mathbf{n}_f \cdot (\mathbf{r}_f - \mathbf{R}), \tag{S38}$$

$$k_z = \mathbf{n}_f \cdot \mathbf{k}, \tag{S39}$$

$$k_\parallel^2 = k^2 - k_z^2. \tag{S40}$$

Therefore

$$A_f(\mathbf{R},\mathbf{k}) = \mathcal{A}(\mathbf{r}_f - \mathbf{R}, \mathbf{k}, \mathbf{n}_f), \tag{S41}$$

where function $\mathcal{A}$ is universal, i. e. is the same for all faces and even for all polyhedra. Similarly, B_{fe} determines by coordinates y, z and components k_x, k_y, k_z of vector $\mathbf{k}$. These parameters, in their turn, depend on the outward normal vector $\mathbf{n}_f$ to face f, the outward normal vector $\mathbf{b}_{fe}$ to edge e lying in the plane of face f, and a radius-vector $\mathbf{r}_e$ of an arbitrarily chosen point on edge e (see Fig. 2):

$$y = \mathbf{b}_{fe} \cdot (\mathbf{r}_e - \mathbf{R}), \tag{S42}$$

$$z = \mathbf{n}_f \cdot (\mathbf{r}_e - \mathbf{R}), \tag{S43}$$

$$k_y = \mathbf{b}_{fe} \cdot \mathbf{k}, \tag{S44}$$

$$k_z = \mathbf{n}_f \cdot \mathbf{k}, \tag{S45}$$

$$k_x^2 = k^2 - k_y^2 - k_z^2. \tag{S46}$$

One can thus reduce all functions B_{fe} to the same universal function $\mathcal{B}$:

$$B_{fe}(\mathbf{R},\mathbf{k}) = \mathcal{B}(\mathbf{r}_e - \mathbf{R}, \mathbf{k}, \mathbf{n}_f, \mathbf{b}_{fe}). \tag{S47}$$

In the same manner, coordinates x, y, z and components k_x, k_y, k_z, that appear in function C_{fev}, can be expressed via normal vectors $\mathbf{n}_f$ and $\mathbf{b}_{fe}$, the unit vector $\mathbf{l}_{ev}$ directed along edge e toward vertex v from the opposite vertex, and radius-vector $\mathbf{r}_v$ of vertex v (see Fig. 2):

$$x = \mathbf{l}_{ev} \cdot (\mathbf{r}_v - \mathbf{R}), \qquad k_x = \mathbf{l}_{ev} \cdot \mathbf{k}, \tag{S48}$$

$$y = \mathbf{b}_{fe} \cdot (\mathbf{r}_v - \mathbf{R}), \qquad k_y = \mathbf{b}_{fe} \cdot \mathbf{k}, \tag{S49}$$

$$z = \mathbf{n}_f \cdot (\mathbf{r}_v - \mathbf{R}), \qquad k_z = \mathbf{n}_f \cdot \mathbf{k}. \tag{S50}$$

This allows us to reduce all functions C_{fev} to one universal function $\mathcal{C}$:

$$C_{fev}(\mathbf{R},\mathbf{k}) = \mathcal{C}(\mathbf{r}_v - \mathbf{R}, \mathbf{k}, \mathbf{n}_f, \mathbf{b}_{fe}, \mathbf{l}_{ev}). \tag{S51}$$

Substitution of expressions for A_f, B_{fe}, and C_{fev} from Eqs. (S41), (S47) and (S51) into Eq. (S37) gives rise to equation (12) of the main paper.

3 DERIVATION OF SERIES REPRESENTATIONS (13) – (15) FOR FUNCTIONS $\mathcal{A}$, $\mathcal{B}$ AND $\mathcal{C}$

According to the results of Supplemental Section 2,

$$\mathcal{A} = -z^2\mu = z^2\,\frac{\beta(iz)-\beta(-iz)}{2}, \tag{S52}$$

where function β is defined by Eq. (S22). Introducing function $\psi(t)$

$$\psi(t) = \frac{e^t-1}{t} = \sum_{q=0}^{\infty}\frac{t^q}{(q+1)!}, \tag{S53}$$

one can express function β as follows:

$$\beta(\xi) = \frac{\psi(k_\parallel\xi + k_z z)-\psi(k_z z)}{k_\parallel \xi} = \sum_{m=0}^{\infty}\frac{(k_\parallel\xi)^m}{(m+1)!}\,\frac{\mathrm{d}^{m+1}\psi(k_z z)}{\mathrm{d}(k_z z)^{m+1}} = \sum_{m=0}^{\infty}\sum_{q=0}^{\infty}\frac{(k_\parallel\xi)^m}{(m+1)!}\,\frac{(k_z z)^q}{q!(m+q+2)}. \tag{S54}$$

Inserting this representation for β into Eq. (S52), one can see that only even values of m remain. Denoting $m=2p$, one obtains

$$\mathcal{A} = z^2\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-k_\parallel^2 z^2)^p}{(2p+1)!}\,\frac{(k_z z)^q}{q!(2p+q+2)}, \tag{S55}$$

that is identical to Eq. (13) up to relabeling of mute variables. Hence, Eq. (13) is now justified.

In order to derive the formulas for $\mathcal{B}$ and $\mathcal{C}$, we recall that, according to the results of Section 2,

$$\mathcal{B} = \sum_{n=0}^{\infty}\mathcal{F}_{2n}(-r_\perp^2)^n\,\frac{(2n-1)!!}{(2n)!!}, \tag{S56}$$

$$\mathcal{C} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{F}_{m+2n+1}x^m(-r_\perp^2)^n\,\frac{(m+2n)!!(m-1)!!}{m!!(m+2n+1)!!}, \tag{S57}$$

where $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$ are coefficients of series expansion (S29) of function $\mathcal{F}(x)$, and $\mathcal{F}(x)$ is the numerator of the right-hand side of Eq. (S27). One can write this numerator as follows:

$$\mathcal{F}(x) = \mathcal{F}^{(1)} + \mathcal{F}^{(2)}(x) + \mathcal{F}^{(3)}(x) + \mathcal{F}^{(4)}(x) \tag{S58}$$

with

$$\mathcal{F}^{(1)} = -z^3 k_y \lambda, \tag{S59}$$

$$\mathcal{F}^{(2)}(x) = yz\,\beta\left(\frac{k_x x + k_y y}{k_\|}\right), \tag{S60}$$

$$\mathcal{F}^{(3)}(x) = -\frac{z^3 k_x^2 y}{k_\|^2}\,\gamma\left(\frac{k_x x + k_y y}{k_\|}\right), \tag{S61}$$

$$\mathcal{F}^{(4)}(x) = \frac{z^3 k_x k_y}{k_\|^2}\,x\gamma\left(\frac{k_x x + k_y y}{k_\|}\right). \tag{S62}$$

Correspondingly,

$$\mathcal{B} = \mathcal{B}^{(1)} + \mathcal{B}^{(2)} + \mathcal{B}^{(3)} + \mathcal{B}^{(4)} \tag{S63}$$

and

$$\mathcal{C} = \mathcal{C}^{(1)} + \mathcal{C}^{(2)} + \mathcal{C}^{(3)} + \mathcal{C}^{(4)}. \tag{S64}$$

Since $\mathcal{F}^{(1)}$ does not depend on x, its series expansion contains only the zeroth term. Therefore

$$\mathcal{B}^{(1)} = \mathcal{F}^{(1)} = -z^3 k_y \lambda \tag{S65}$$

and

$$\mathcal{C}^{(1)} = 0. \tag{S66}$$

The series expansion for quantity $\mathcal{B}^{(1)}$ can be obtained from Eqs. (S24), (S54) and (S65). Only odd values of subscript m remain, so it is convenient to represent them as $m = 2p + 1$. As a result,

$$\mathcal{B}^{(1)} = z^3 k_y \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-k_\|^2 z^2)^p}{(2p+2)!}\,\frac{(k_z z)^q}{q!(2p+q+3)}. \tag{S67}$$

In order to find $\mathcal{B}^{(2)}$ and $\mathcal{C}^{(2)}$, it is necessary to substitute value $\xi = (k_x x + k_y y)/k_\|$ of argument ξ into Eq. (S54), and then expand factors $(k_\|\xi)^m$:

$$(k_\|\xi)^m = (k_x x + k_y y)^m = \sum_{s=0}^{m}\binom{m}{s}(k_x x)^s (k_y y)^{m-s}, \tag{S68}$$

where

$$\binom{m}{s} = \frac{m!}{s!(m-s)!}. \tag{S69}$$

Eqs. (S54), (S60) and (S68), after introducing summation index $u = m - s$ instead of m, give rise to

$$\mathcal{F}^{(2)}(x) = yz\sum_{u=0}^{\infty}\sum_{q=0}^{\infty}\sum_{s=0}^{\infty}\binom{s+u}{u}\frac{(k_x x)^s\,(k_y y)^u}{(s+u+1)!}\,\frac{(k_z z)^q}{q!\,(s+u+q+2)}. \tag{S70}$$

This equation can be rewritten as

$$\mathcal{F}^{(2)}(x) = \mathcal{F}_0^{(2)} + \mathcal{F}_1^{(2)} x + \mathcal{F}_2^{(2)} x^2 + \dots, \tag{S71}$$

where

$$\mathcal{F}_s^{(2)} = yz \sum_{u=0}^{\infty} \sum_{q=0}^{\infty} \binom{s+u}{u} \frac{k_x^s \,(k_y y)^u \,(k_z z)^q}{(s+u+1)!\, q!\, (s+u+q+2)}. \tag{S72}$$

These values of $\mathcal{F}_s^{(2)}$, being inserted into Eqs. (S56) and (S57), provide the following expressions for $\mathcal{B}^{(2)}$ and $\mathcal{C}^{(2)}$:

$$\mathcal{B}^{(2)} = yz \sum_{n=0}^{\infty} \sum_{u=0}^{\infty} \sum_{q=0}^{\infty} \binom{2n+u}{u} \frac{(-k_x^2 r_\perp^2)^n \,(k_y y)^u \,(k_z z)^q}{(2n+u+1)!\, q!\, (2n+u+q+2)} \, \frac{(2n-1)!!}{(2n)!!}, \tag{S73}$$

$$\begin{aligned} \mathcal{C}^{(2)} = k_x yz \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{u=0}^{\infty} \sum_{q=0}^{\infty} &\binom{m+2n+u+1}{u} \\ &\times \frac{(k_x x)^m \,(-k_x^2 r_\perp^2)^n \,(k_y y)^u \,(k_z z)^q}{(m+2n+u+2)!\, q!\, (m+2n+u+q+3)} \, \frac{(m+2n)!!(m-1)!!}{m!!(m+2n+1)!!}. \end{aligned} \tag{S74}$$

Quantities $\mathcal{B}^{(3)}$, $\mathcal{B}^{(4)}$, $\mathcal{C}^{(3)}$ and $\mathcal{C}^{(4)}$ depend on function γ. Therefore, our next goal is to find out the series expansion for this function. According to Eqs. (S16), (S24) and (S25),

$$\gamma(\xi) = \frac{1}{\xi^2 + z^2} \left[\beta(\xi) - \frac{\beta(iz) - \beta(-iz)}{2iz} \xi - \frac{\beta(iz) + \beta(-iz)}{2} \right]. \tag{S75}$$

Expanding function β into the Maclaurin series

$$\beta(\xi) = \sum_{m=0}^{\infty} \beta_m \xi^m, \tag{S76}$$

where

$$\beta_m = \sum_{q=0}^{\infty} \frac{k_\parallel^m \,(k_z z)^q}{(m+1)!\, q!\, (m+q+2)}, \tag{S77}$$

one can obtain the following representation for function γ:

$$\gamma(\xi) = \sum_{m=0}^{\infty} \beta_m \gamma_m(\xi), \tag{S78}$$

where functions $\gamma_m(\xi)$ are defined by Eq. (S75), in which $\beta(\xi)$ is replaced with ξ^m. For even values of m

$$\gamma_m(\xi) = \frac{\xi^m - (iz)^m}{\xi^2 - (iz)^2} = \xi^{m-2} + \xi^{m-4}(iz)^2 + \xi^{m-6}(iz)^4 + \ldots + \xi^0 (iz)^{m-2}$$

$$= \sum_{t=0}^{(m-2)/2} \xi^{m-2t-2}(-z^2)^t, \quad \text{(S79)}$$

and for odd values of m

$$\gamma_m(\xi) = \frac{\xi^m - (iz)^{m-1}\xi}{\xi^2 - (iz)^2} = \xi^{m-2} + \xi^{m-4}(iz)^2 + \xi^{m-6}(iz)^4 + \ldots + \xi^1 (iz)^{m-3}$$

$$= \sum_{t=0}^{(m-3)/2} \xi^{m-2t-2}(-z^2)^t. \quad \text{(S80)}$$

Note that $\gamma_0(\xi) = \gamma_1(\xi) = 0$ for all ξ. Eqs. (S78) – (S80) can be summarized as follows:

$$\gamma(\xi) = \sum_{m=2}^{\infty} \sum_{t=0}^{[(m-2)/2]} \beta_m \xi^{m-2t-2}(-z^2)^t, \quad \text{(S81)}$$

where square brackets denote the integer part. Introducing parameter $p = m - 2t - 2$, one can rewrite Eq. (S81) in a more convenient form:

$$\gamma(\xi) = \sum_{p=0}^{\infty} \sum_{t=0}^{\infty} \beta_{p+2t+2} \xi^p (-z^2)^t. \quad \text{(S82)}$$

Substituting Eq. (S77) here, we obtain the final expression for function γ as a power series:

$$\gamma(\xi) = k_\parallel^2 \sum_{p=0}^{\infty} \sum_{t=0}^{\infty} \sum_{q=0}^{\infty} \frac{(k_\parallel \xi)^p \, (-k_\parallel^2 z^2)^t \, (k_z z)^q}{(p+2t+3)! \, q! \, (p+2t+q+4)}. \quad \text{(S83)}$$

This expression is to be put into Eqs. (S61) and (S62) in order to obtain series expansions for $\mathcal{F}^{(3)}(x)$ and $\mathcal{F}^{(4)}(x)$. After transformations similar to Eqs. (S68) – (S70), function $\mathcal{F}^{(3)}(x)$ takes the form

$$\mathcal{F}^{(3)}(x) = -z^3 k_x^2 y \sum_{u=0}^{\infty} \sum_{t=0}^{\infty} \sum_{q=0}^{\infty} \sum_{s=0}^{\infty} \binom{s+u}{u} \frac{(k_x x)^s \, (k_y y)^u \, (-k_\parallel^2 z^2)^t \, (k_z z)^q}{(s+u+2t+3)! \, q! \, (s+u+2t+q+4)}. \quad \text{(S84)}$$

Similarly,

$$\mathcal{F}^{(4)}(x) = z^3 k_y \sum_{u=0}^{\infty} \sum_{t=0}^{\infty} \sum_{q=0}^{\infty} \sum_{s=0}^{\infty} \binom{s+u}{u} \frac{(k_x x)^{s+1} \, (k_y y)^u \, (-k_\parallel^2 z^2)^t \, (k_z z)^q}{(s+u+2t+3)! \, q! \, (s+u+2t+q+4)}. \quad \text{(S85)}$$

From Eq. (S84), one can obtain contributions $\mathcal{B}^{(3)}$ and $\mathcal{C}^{(3)}$ via Eqs. (S56) and (S57), correspondingly:

$$\mathcal{B}^{(3)} = -z^3 k_x^2 y \sum_{n=0}^{\infty}\sum_{u=0}^{\infty}\sum_{t=0}^{\infty}\sum_{q=0}^{\infty} \binom{2n+u}{u} \times \frac{(-k_x^2 r_\perp^2)^n \, (k_y y)^u \, (-k_\parallel^2 z^2)^t \, (k_z z)^q}{(2n+u+2t+3)! \, q! \, (2n+u+2t+q+4)} \, \frac{(2n-1)!!}{(2n)!!}, \quad \text{(S86)}$$

$$\mathcal{C}^{(3)} = -z^3 k_x^3 y \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{u=0}^{\infty}\sum_{t=0}^{\infty}\sum_{q=0}^{\infty} \binom{m+2n+u+1}{u} \times \frac{(k_x x)^m \, (-k_x^2 r_\perp^2)^n \, (k_y y)^u \, (-k_\parallel^2 z^2)^t \, (k_z z)^q}{(m+2n+u+2t+4)! \, q! \, (m+2n+u+2t+q+5)} \, \frac{(m+2n)!! \, (m-1)!!}{m!! \, (m+2n+1)!!}. \quad \text{(S87)}$$

Similarly, it follows from Eq. (S85) that

$$\mathcal{B}^{(4)} = -z^3 k_x^2 k_y r_\perp^2 \sum_{n=0}^{\infty}\sum_{u=0}^{\infty}\sum_{t=0}^{\infty}\sum_{q=0}^{\infty} \binom{2n+u+1}{u} \times \frac{(-k_x^2 r_\perp^2)^n \, (k_y y)^u \, (-k_\parallel^2 z^2)^t \, (k_z z)^q}{(2n+u+2t+4)! \, q! \, (2n+u+2t+q+5)} \, \frac{(2n+1)!!}{(2n+2)!!}, \quad \text{(S88)}$$

$$\mathcal{C}^{(4)} = z^3 k_x k_y \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{u=0}^{\infty}\sum_{t=0}^{\infty}\sum_{q=0}^{\infty} \binom{m+2n+u}{u} \times \frac{(k_x x)^m \, (-k_x^2 r_\perp^2)^n \, (k_y y)^u \, (-k_\parallel^2 z^2)^t \, (k_z z)^q}{(m+2n+u+2t+3)! \, q! \, (m+2n+u+2t+q+4)} \, \frac{(m+2n)!! \, (m-1)!!}{m!! \, (m+2n+1)!!}. \quad \text{(S89)}$$

Summing all the contributions $\mathcal{B}^{(1)}$, $\mathcal{B}^{(2)}$, $\mathcal{B}^{(3)}$, $\mathcal{B}^{(4)}$ given by Eqs. (S67), (S73), (S86) and (S88), one can obtain Eq. (14) for function $\mathcal{B}$. Similarly, Eq. (15) for function $\mathcal{C}$ can be obtained as a sum of contributions $\mathcal{C}^{(2)}$, $\mathcal{C}^{(3)}$, $\mathcal{C}^{(4)}$ from Eqs. (S74), (S87) and (S89). Therefore, representations (14) and (15) for functions $\mathcal{B}$ and $\mathcal{C}$ have been derived.

4 CONVERGENCE OF SERIES THAT REPRESENT FUNCTIONS $\mathcal{A}$, $\mathcal{B}$ AND $\mathcal{C}$

In this section, we apply the direct comparison test for proving the convergence of infinite series (13) – (15) that represent functions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$.

Let us consider the expression to the right of summation signs in Eq. (13). There is an upper estimate for the absolute value of this expression:

$$\left| \frac{(k_z z)^s (-k_\parallel^2 z^2)^u}{s!(2u+1)!(s+2u+2)} \right| < \frac{|k_z z|^s}{s!} \, \frac{|k_\parallel^2 z^2|^u}{u!}. \quad \text{(S90)}$$

Consequently,

$$|\mathcal{A}| < |z^2| \sum_{s=0}^{\infty} \sum_{u=0}^{\infty} \frac{|k_z z|^s}{s!} \frac{|k_\parallel^2 z^2|^u}{u!}. \tag{S91}$$

The right-hand side of Eq. (S91) is a converging series:

$$|z^2| \sum_{s=0}^{\infty} \sum_{u=0}^{\infty} \frac{|k_z z|^s}{s!} \frac{|k_\parallel^2 z^2|^u}{u!} = |z^2| \exp\left(|k_z z| + |k_\parallel^2 z^2|\right). \tag{S92}$$

Therefore series (13), that defines quantity $\mathcal{A}$, also converges by virtue of the direct comparison test.

Quantity $\mathcal{B}$, as defined by Eq. (14), is a sum of four contributions $\mathcal{B}^{(1)}$, $\mathcal{B}^{(2)}$, $\mathcal{B}^{(3)}$ and $\mathcal{B}^{(4)}$ given by Eqs. (S67), (S73), (S86) and (S88), correspondingly. Their convergence can be tested separately.

The convergence proof for $\mathcal{B}^{(1)}$ is essentially the same as that for $\mathcal{A}$:

$$|\mathcal{B}^{(1)}| < |z^3 k_y| \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{|k_\parallel^2 z^2|^p}{p!} \frac{|k_z z|^q}{q!} = |z^3 k_y| \exp\left(|k_\parallel^2 z^2| + |k_z z|\right). \tag{S93}$$

Similarly,

$$|\mathcal{B}^{(2)}| < |yz| \sum_{n=0}^{\infty} \sum_{u=0}^{\infty} \sum_{q=0}^{\infty} \frac{|k_x^2 r_\perp^2|^n}{n!} \frac{|k_y y|^u}{u!} \frac{|k_z z|^q}{q!} = |yz| \exp\left(|k_x^2 r_\perp^2| + |k_y y| + |k_z z|\right), \tag{S94}$$

$$\begin{aligned} |\mathcal{B}^{(3)}| < |z^3 k_x^2 y| \sum_{n=0}^{\infty} \sum_{u=0}^{\infty} \sum_{t=0}^{\infty} \sum_{q=0}^{\infty} \frac{|k_x^2 r_\perp^2|^n}{n!} \frac{|k_y y|^u}{u!} \frac{|k_\parallel^2 z^2|^t}{t!} \frac{|k_z z|^q}{q!} \\ = |z^3 k_x^2 y| \exp\left(|k_x^2 r_\perp^2| + |k_y y| + |k_\parallel^2 z^2| + |k_z z|\right), \end{aligned} \tag{S95}$$

and the upper estimate for $\mathcal{B}^{(4)}$ is the same as that for $\mathcal{B}^{(4)}$, up to the factor before the exponent:

$$|\mathcal{B}^{(4)}| < |z^3 k_x^2 k_y r_\perp^2| \exp\left(|k_x^2 r_\perp^2| + |k_y y| + |k_\parallel^2 z^2| + |k_z z|\right). \tag{S96}$$

Hence, representation (14) of quantity $\mathcal{B}$ is a converging series.

Similar considerations apply to quantity $\mathcal{C}$. Its representation as an infinite series, Eq. (15), is a sum of contributions $\mathcal{C}^{(2)}$, $\mathcal{C}^{(3)}$ and $\mathcal{C}^{(4)}$ defined by Eqs. (S74), (S87) and (S89), correspondingly. Then,

$$\begin{aligned} |\mathcal{C}^{(2)}| < |k_x yz| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{u=0}^{\infty} \sum_{q=0}^{\infty} \frac{|k_x x|^m}{m!} \frac{|k_x^2 r_\perp^2|^n}{n!} \frac{|k_y y|^u}{u!} \frac{|k_z z|^q}{q!} \\ = |k_x yz| \exp\left(|k_x x| + |k_x^2 r_\perp^2| + |k_y y| + |k_z z|\right), \end{aligned} \tag{S97}$$

$$|\mathcal{C}^{(3)}| < |z^3 k_x^3 y| \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{u=0}^{\infty}\sum_{t=0}^{\infty}\sum_{q=0}^{\infty} \frac{|k_x x|^m}{m!}\frac{|k_x^2 r_\perp^2|^n}{n!}\frac{|k_y y|^u}{u!}\frac{|k_\parallel^2 z^2|^t}{t!}\frac{|k_z z|^q}{q!}$$
$$= |z^3 k_x^3 y| \exp\Big(|k_x x| + |k_x^2 r_\perp^2| + |k_y y| + |k_\parallel^2 z^2| + |k_z z|\Big), \quad \text{(S98)}$$

and

$$|\mathcal{C}^{(4)}| < |z^3 k_x k_y| \exp\Big(|k_x x| + |k_x^2 r_\perp^2| + |k_y y| + |k_\parallel^2 z^2| + |k_z z|\Big). \quad \text{(S99)}$$

This proves that $\mathcal{C}^{(2)}$, $\mathcal{C}^{(3)}$ and $\mathcal{C}^{(4)}$ are converging series. Consequently, the power-series representation (15) for quantity $\mathcal{C}$ converges.

5 TRANSITION FROM COORDINATE SYSTEM K_F TO $\tilde{K}_{FE}$: DERIVATION OF EQUATIONS (S26) AND (S27)

For clarity, we indicate near each equation of this section, whether it is related to coordinate system K_f or to $\tilde{K}_{fe}$.

In system K_f, x- and y-components of vector $\mathbf{k}$ are equal to 0 and $k_\parallel$, respectively (see Fig. S1a). Therefore

$$\mathbf{k}\cdot\mathbf{r} = k_\parallel y + k_z z, \quad (K_f) \quad \text{(S100)}$$

while in system $\tilde{K}_{fe}$

$$\mathbf{k}\cdot\mathbf{r} = k_x x + k_y y + k_z z. \quad (\tilde{K}_{fe}) \quad \text{(S101)}$$

Taking into account that quantities $\mathbf{k}\cdot\mathbf{r}$, k_z and z are the same in both coordinate systems, one can obtain transformation rule (S26) from Eqs. (S100) and (S101).

Then, let us consider the left-hand side of Eq. (S27). In system $\tilde{K}_{fe}$, vector $\mathbf{v}$ is equal to

$$\mathbf{v} = \frac{z\beta}{r}(x, y, 0). \quad (\tilde{K}_{fe}) \quad \text{(S102)}$$

The unit vector $\mathbf{b}$ in this system is directed along axis y (see Fig. S2). Hence,

$$\mathbf{b}\cdot\mathbf{v} = v_y = \frac{yz\beta}{r}. \quad (\tilde{K}_{fe}) \quad \text{(S103)}$$

Vectors $\mathbf{b}$ and $\mathbf{w}$ lie in the xy-plane in system K_f. Then, according Eq. (S15),

$$\mathbf{b}\cdot\mathbf{w} = b_x w_x + b_y w_y = \frac{x b_x \gamma + k_\parallel b_y \lambda}{r} \quad (K_f) \quad \text{(S104)}$$

and

$$\mathbf{r}\cdot\mathbf{b} = x b_x + y b_y. \quad (K_f) \quad \text{(S105)}$$

Deriving $x b_x$ from Eq. (S105) and substituting it into Eq. (S104), one can obtain

$$\mathbf{b}\cdot\mathbf{w} = \frac{\mathbf{r}\cdot\mathbf{b}\gamma - y b_y \gamma + k_\parallel b_y \lambda}{r} \quad (K_f) \quad \text{(S106)}$$

Let us perform the transition from K_f to $\tilde{K}_{fe}$ in Eq. (S106). In system K_f, $\mathbf{k} = (0, k_\parallel, k_z)$ and $\mathbf{b} = (b_x, b_y, 0)$, whence

$$\mathbf{k} \cdot \mathbf{b} = k_\parallel b_y. \quad (K_f) \tag{S107}$$

On the other hand, $\mathbf{b} = (0, 1, 0)$ in system $\tilde{K}_f$. Consequently

$$\mathbf{k} \cdot \mathbf{b} = k_y, \quad (\tilde{K}_{fe}) \tag{S108}$$

$$\mathbf{r} \cdot \mathbf{b} = y. \quad (\tilde{K}_{fe}) \tag{S109}$$

Eqs. (S107) and (S108) provide the transformation rule for quantity b_y from K_f to $\tilde{K}_{fe}$:

$$b_y \to \frac{k_y}{k_\parallel}. \tag{S110}$$

Applying transformation rules (S26) and (S110) to Eq. (S106) and using Eq. (S109), one can express quantity $\mathbf{b} \cdot \mathbf{w}$ in system $\tilde{K}_{fe}$ as follows:

$$\mathbf{b} \cdot \mathbf{w} = \frac{y\gamma - \frac{(k_x x + k_y y) k_y \gamma}{k_\parallel^2} + k_y \lambda}{r}. \quad (\tilde{K}_{fe}) \tag{S111}$$

Finally, gathering Eqs. (S103) and (S111) together, writing r as $\sqrt{x^2 + y^2 + z^2}$ and taking into account that

$$k_\parallel^2 = k_x^2 + k_y^2, \quad (\tilde{K}_{fe}) \tag{S112}$$

one can easily derive Eq. (S27).

6 REDUCING INTEGRAL $\mathcal{L}_{FE}$ TO INTEGRAL L_E

It is convenient to introduce dimensionless quantities $\hat{x}_1 = x_1/r_\perp$, $\hat{x}_2 = x_2/r_\perp$, $\hat{r}_1 = \sqrt{x_1^2 + r_\perp^2}/r_\perp$, $\hat{r}_2 = \sqrt{x_2^2 + r_\perp^2}/r_\perp$, and

$$L_n = r_\perp^{-n} \int\limits_{x_1}^{x_2} \frac{x^n \, \mathrm{d}x}{\sqrt{x^2 + r_\perp^2}}. \tag{S113}$$

In these notations, $L_e \equiv L_0$ and

$$\mathcal{L}_{fe} = \sum_{n=0}^{\infty} \mathcal{F}_n r_\perp^n L_n. \tag{S114}$$

Relations (S31) and (S32) acquire the following form:

$$L_n = -\frac{n-1}{n} L_{n-2} + \frac{\hat{x}_2^{n-1} \hat{r}_2 - \hat{x}_1^{n-1} \hat{r}_1}{n}, \quad (n \geq 2) \tag{S115}$$

$$L_1 = \hat{r}_2 - \hat{r}_1. \tag{S116}$$

Applying these relations repeatedly, one can express all integrals L_n with $n = 1, 2, 3, \dots$ via quantities L_0, $\hat{x}_1$, $\hat{x}_2$, $\hat{r}_1$ and $\hat{r}_2$:

$$
\begin{aligned}
L_1 &= \hat{r}_2 - \hat{r}_1, \\
L_2 &= -\frac{1}{2}L_0 + \frac{1}{2}(\hat{x}_2\hat{r}_2 - \hat{x}_1\hat{r}_1), \\
L_3 &= -\frac{2}{3}(\hat{r}_2 - \hat{r}_1) + \frac{1}{3}(\hat{x}_2^2\hat{r}_2 - \hat{x}_1^2\hat{r}_1), \\
L_4 &= \frac{1\cdot 3}{2\cdot 4}L_0 - \frac{3}{2\cdot 4}(\hat{x}_2\hat{r}_2 - \hat{x}_1\hat{r}_1) + \frac{1}{4}(\hat{x}_2^3\hat{r}_2 - \hat{x}_1^3\hat{r}_1),
\end{aligned}
$$

and so on. These results can be summarized as follows:

$$
L_n = a_n L_0 + \sum_{p=0}^{n-1} b_{np}(\hat{x}_2^p\hat{r}_2 - \hat{x}_1^p\hat{r}_1), \tag{S117}
$$

where a_n and b_{np} are some numerical coefficients. Our next goal is to find their values.

It is easy to find out from Eqs. (S115) and (S116) that

$$
a_n = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ (-1)^{\frac{n}{2}}\frac{(n-1)!!}{n!!}, & \text{if } n \text{ is even}. \end{cases} \tag{S118}
$$

This equality holds also at $n = 0$, if we imply that $(-1)!! = 0!! = 1$.

Let us prove that

$$
b_{np} = (-1)^{\frac{n-p-1}{2}} \frac{(n-1)!!\,(p-1)!!}{n!!\,p!!} \quad \text{if } 0 \le p \le n-1 \text{ and } n-p \text{ is odd}, \tag{S119}
$$

$$
b_{np} = 0 \quad \text{otherwise}. \tag{S120}
$$

Indeed, Eqs. (S119) and (S120) provide correct results for $n = 0$ and $n = 1$: all b_{0p} and b_{1p} are equal to zero, except for $b_{10} = 1$. For $n \ge 2$, coefficients b_{np} can be determined by means of recurrence relation

$$
b_{np} = \begin{cases} -\frac{n-1}{n}b_{(n-2)p}, & \text{if } p < n-1, \\ \frac{1}{n}, & \text{if } p = n-1, \\ 0, & \text{if } p > n-1 \end{cases} \tag{S121}
$$

that follows from Eq. (S115). One can check that coefficients b_{np} defined by Eqs. (S119) and (S120) obey relation (S121). This proves that Eqs. (S119) and (S120) are valid.

Finally, after substituting Eqs. (S117) – (S120) into Eq. (S114) and some renaming of subscript indexes, integral $\mathcal{L}_{fe}$ acquires the form expressed by Eqs. (S33) – (S35).

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