Equational reasoning: A Systematic Review of the Cuisenaire-Gattegno Approach: Supplemental Material

Introduction

In this resource we describe how we conducted the meta-analysis and compare Cui with Davydov's report of his contemporary experiments in early algebra.

In Appendix A we give an overview of the literature, highlighting the evolution of the Cui approach and scholarly assessment of it. Appendix B is a primary source taken from "An experiment in introducing elements of algebra in elementary school," reproduced from Sovietskia Pedagogika, number 8, as translated (Davydov, 1962) (permission applied for).

Appendix A: Qualitative Review

The goal of the meta-analysis is to evaluate the effectiveness of the Cuisenaire-Gattegno interventions on measures of mathematical performance. We assessed a long list of 37 records for eligibility for which abstracts were available (with full-text examination if necessary to determine inclusion). These are summarised in Table 1. The direction of the reported effect is shown as Cui = Control, Cui>Control or Cui<Control. Peer reviewed findings (marked *) were equally balanced between Cui and conventional teaching. Other studies were more favorable to Cui.

Several results can be highlighted from the literature:

- Cui materials were generally found to be superior to the traditional approach in Grade 1 (Fennema, 1972a).
- For Grades 1 and 2 Cui pupils acquire concepts and skills not taught in a traditional programme. These skills are obtained without no apparent loss of traditional computational and reasoning skills (Hollis, 1965; Nasca, 1966).
- By Grade 3 a 1-3 grade post-test found that "The children in the modified curriculum utilizing Cuisenaire rods were facile in manipulating the rods and in verbalizing many abstract terms... However, this was not carried over to the written, computational procedures" Passy (1963b).
- Cuisenaire rods can aid learning about place value and number skills in Grades 1-3 (Allen, 1978).

Table 1: *Cuisenaire and Cuisenaire-Gattegno Studies*

Study	**n**	**Grade**	**Days**	**Design**	**Effect**	**Evidence for Fidelity**
Beard (1964)	5	pre-K	60	OB	$>$	Mathematics with Numbers in Colour (MNC) A (1961)
Aurich (1963)	90	1	180	QEX	$>$	Arithmetic with Numbers in Colour (ANC) (1960)
*Hollis (1965)	80	1-2	360	QEX	$>$	MNC A,B (1961)
Sweeney (1968)	354	1	180	QEX	$=$	ANC (1960)
*Rawlinson (1965)	354	1	140	QEX	$=$	ANC (1960)
Crowder (1965)	425	1	143	QEX	$>$	Assert faithful
Fedon (1966)	26	1	180	QEX	$=$	ANC 1-3 (1957)
Ellis (1964)	1066	1-2	180-360	QEX	$>$	ANC 1-7 (1957-9)
Lin (2013)	666	1-2	NA	QEX	$>$	Place value only
Huang (2019)	NA	1-2	NA	QEX	$>$	Unknown
Gell (1963)	26	2	180	QEX	$=$	MNC A–D (1961)
*Nasca (1966)	45	2	180	QEX	$>$	Cuisenaire & Gattegno (1960)
*Fennema (1972b)	95	2	14	QEX	$<$	No textbook. Gattegno (2011a)
Egan (1990)	81	2	180	QEX	$=$	Davidson (1978), MNC (1964)
Dairy (1969)	98	K-2	540	QEX	$>$	Teacher worksheets
Haynes (1963)	106	3	30	QEX	$=$	Multiplication. MNC Book A,B (1958–59). ANC Books 7-10.
*Passy (1963a)	1865	1-3	540	QEX	$<$	ANC (1957-9)
*Lucow (1962)	254	3	30	QEX	$>$	Growth in $*$ and $\div$ tested
*Rodman (1964)	73	1-3	540	EX	$>$	
*Brownell (1967a, 1968)	478	1-3	345-546	OB	$>$	Scotland (faithful)
*Brownell (1967a, 1968)	628	1-3	401-443	OB	$<$	England (problematic). 3-way
*Brownell (1967b)	1109	1-3	540	QEX	$>$	Assume MNC 1, 2 (1963)
Steencken (2001)	25	4	60	OB	*NA*	Fractions only
Yankelewitz (2009)	25	4	17	OB	*NA*	Fractions as magnitudes
Steiner (1964)	102	4	180	QEX	$>$	Teacher's introduction Gattegno (2011b)
Keagle & Brummett (1993)	57	4	4	QEX	$<$	Fractions only
du Bon Pasteur (1966)	373	2–4	360-720	QEX	$>$	Cuisenaire & Gattegno (1962)
Bellemare (1967)	373	2–4	360-720	QEX	$>$	Cuisenaire & Gattegno (1962)
Robinson (1978)	119	3,4	5	EX	$=$	Decimal fractions only
Bulgar (2002)	38	4,5	50	OB	*NA*	Fractions only
Lamon & Scott (1970)	74	4,6	10	QEX	$=$	Custom isomorphism test
Wallace (1974)	154	4–6	15	EX	$>$	Fractions only MNC (1966)
Romero (1977)	240	1–6	160	QEX	$>$	Not reported
Allen (1978)	100	7	30	QEX	$>$	30 custom lesson designs for number bonds to 20
Rich (1972)	122	7	7	EX	*NA*	Fractions only
Marchese (2009)	14	8	2	OB	*NA*	Volume and surface area
Adom & Adu (2020)	250	9	15	QEX	$>$	Fractions only

- At Grade 7 the Cui approach is "was significantly better than the traditional program approach for improving the basic skills of addition and subtraction" (Allen, 1978).

Most non published studies concluded that the Cui approach was as effective (=) or superior (>) to the conventional approach. A typical assessment at the time was made by Lorna Dairy, a Kindergarten teacher who wrote "My enthusiasm for the use of Cuisenaire rods in the teaching of mathematics has grown tremendously each of the three years I have used the rods. I never cease to be amazed at the interest of the children nor the rapidity with which they grasp the concepts as they use the rods. I have watched highly intelligent children advance on their own without any adult pressure to achieve. I have also watched culturally deprived children come to life intellectually with the understanding of rods" (Dairy, 1969, p. 3).

The influence of Cui waned in the 1980s-90s. Wright (1992) studied the reversion from the Cuisenaire approach in Australia back to a counting first curriculum. He attributed this change to the influence of the U.S. 'back to basics' movement. He wrote,"The earlier used 'study of a number' approach involved studying number facts with several operations (e.g. addition, subtraction, multiplication and halving) for individual numbers in tum, and was a key feature of the Cuisenaire approach. When the counting-based approach to introducing operations replaced the purely Cuisenaire approach, the legacy of the latter approach (i.e. study of a number) was initially retained but is now being replaced by a teaching sequence which focuses on the operations in turn. In this approach, addition is studied in detail and for an extended period, then subtraction and so on. This was the prevailing approach in New South Wales, prior to the introduction of Cuisenaire."

Reflecting on hearing Gattegno speak in the 1980s John Mason noted "I began to get a taste of what it is like when an experienced 'grey-beard' assembles their to-them-coherent-and-comprehensive framework or theory. Whereas when the fragments were being worked on and described there is often considerable interest amongst colleagues, once the whole is assembled, people don't really want to know" (Mason, 2010, p. 5).

While Cuisenaire rods have gone in and out of fashion in general teaching they have become a fixed component in remedial classes. The idea of coloured cuboids as a model for number has been given renewed impetus with the recent invention of 'Numberblocks' – the BAFTA award winning UK television animation series aimed at a pre-school audience (BBC, 2017). They will play a major role in Covid-19 catchup.

Appendix B: Gattegno, Piaget and El'konin-Davydov early algebra

Cui was developed by Gattegno in collaboration with Jean Paiget, a development psychologist and Jean Dieudonné, an author of the Bourbaki reforms to mathematics education. A similar initiative taken by Davydov and his colleagues in the Soviet Union is receiving renewed attention in the contemporary literature.

Piaget's stages of development theory reinforces a traditional school mathematics curriculum that starts with counting-on and back in arithmetic. One consequence has been that primary mathematics has been resistant to change for more than half a century: algebraic writing and reasoning are deferred to the higher grades (Sime, 1973; Copeland, 1970). Piaget based his theory on clinical diagnostic tests of mathematical concepts with manipulatives, such as a one to one correspondence between rows of counters, or class inclusion, where he found repeatable patterns of failure in groups of young learners. He wrote, "It is quite impossible to gain positive results through the use of a learning procedure involving strategies of which the child at his particular stage is incapable. These are boundaries, or limits, which cannot be crossed...There is a second problem however... this is the ordinal succession not in general development but in the development of the individual. This I must confess is a problem that I have unfortunately never studied, because I have no interest whatsoever in the individual " (Piaget, 1971, p. 211).

Gattegno had a different perspective on learning. He parted company with Piaget on the mental powers attributed to young children (Gattegno, 1984, p. 33). He felt that "the historic development of culture, if it has something to bring to our understanding of the present moment, can be entirely foreign to what a mind stimulated in a new way can or could do, unforeseen in the former experience of the group" (Gattegno, 1958, p. 17). He and his collaborators challenged Piaget's 'counting first' dogma. They carefully documented pedagogy, early algebra curricula and experience with the rods. This made it possible for teachers and researchers to replicate and extend their work (Gattegno, 1970; Goutard, 2017; Dawson, 1991; Benson, 2014; Ainsworth, 2017; Cane, 2017).

An influential independent development of an early algebra curriculum which shares some significant similarities with Cui took place in Khrushchev's Soviet Union (see Appendix B for a description of school context and curriculum content) (Dieudonne, 1990; Schmittau & Morris, 2004; Radford, 2021). There B. D. El'Konin and V. V. Davydov's experiments "convincingly show that algebra can be taught more adequately, and at an even earlier age than it is now" (Freudenthal, 1974).

Davydov's work was possible because Soviet mathematicians led by Kolmogorov were able to successfully defend their professional autonomy both from the influence of local political ideology and Western dogma. Citing Vygotsky Davydov wrote "Vygotsky wrote in criticism of Piaget's views: 'For Piaget the indicator of the level of the child's thinking is not what the child knows nor what he is capable of learning, but how he thinks, in a field about which he has no knowledge. Instruction and development, knowledge and thought are opposed here in the sharpest way.' As we see it the conjunction 'and' in the problem of 'instruction and development' is neither disjunctive nor contrastive but, on the contrary, copulative. Apart from instruction there is not and cannot be mental development at all… In setting up elementary instruction material and testing experimentally the possibilities for learning this new material, the potentialities for the mental development of children of early school age are in fact being investigated." (Davydov, 1975, p. 52)

Like Gattegno, Davydov and his colleagues built on the work of Jean Dieudonné noting that "Bourbaki's ideas about the "architecture of Mathematics" are quite tempting to teachers, logicians, and psychologists. One begins to envision the study of mathematics as being based on general (simple) structures and the academic subject being developed through the interrelations and interweaving among them... The ideas inherent in the experimental study of structuring mathematics are of primary significance for they establish the pre-requisites for a substantial and justifiable revision of the ideas of traditional education, for working out a new interpretation of the nature of abstraction and generalization, for the connection between the general and the particular, for ways of developing the child's though process, and so forth."(Davydov, 1975, p. 81)

Writing of this period in Soviet education Borovik et al. (2021) notes "We think it is important to emphasise that Kolmogorov and his comrades-in-arms did more than critique the state of mathematics education: they had created new educational structures – such as mathematical circles and mathematics competitions, focused on the mathematics of qualitative analytic thinking. They also created a new, previously never existing, cultural system: the advanced level 'outreach mathematics,' and the community which shared its values." The 'Measure Up' curriculum in the USA adopts a Davydov approach. A recent special issue of Educational Studies in Mathematics is devoted to his ideas (Coles, 2021).

El'konin-Davydov early algebra

Contemporary with Gattegno's work in the anglophone and francophone worlds, mathematicians in the Soviet Union, under the leadership of V. V. Davydov and D. B. El'konin conducted an influential series of experiments with an early algebra programme. The following is taken from Davydov's and El'konin's accounts (Davydov, 1962; El'konin, 1961).

"The foundation of mathematical knowledge is laid in elementary school. But unfortunately both the mathematics teachers themselves and the methods specialists and psychologists have been paying very little attention to the content of elementary mathematics. It is sufficient to say that the arithmetic program in elementary school (1st to 4th grades) took shape in its main aspects some 50 to 60 years ago and naturally reflects the system of mathematical, methodological and psychological conceptions of that time.

"Let us examine the characteristic features of the arithmetic program now used in elementary school. Its basic content is "whole numbers and operations, with them, studied in a definite sequence" (Uchpedgiz, 1961, p. 70). At first the children study the four operations within the range of 10 to 20 (first grade), then come oral calculations within the limits of 100 (first and second grades), then oral and written calculations within 1000 (2nd and 3rd grades), and finally with millions and billions (3rd and 4th grades). In the 4th grade the chidren study some dependencies between the data and results of arithmetic operations, as well as the simplest fractions. This work (along with training in solving corresponding problems) takes 620 out of the 792 hours allocated to arithmetic for the entire four years. At the same time the program envisages the study of metric measures and time measures, training in use of them for measurement (37 hours), knowledge of some elements of visual geometry – drawing of rectangle and square, calculation of dimensions (36 hours). In addition, it is recommended that 80 hours is used to review the material.

"The knowledge and skills required must be applied to solving problems and performing the simplest calculations. Throughout the course the children solve problems parallel with studying numbers and operations – this takes half of all the time allocated to the course. Solving problems helps the children to understand the concrete meaning of the operations, to get a clear idea of the various cases of their application, to establish the dependencies between the values, and to acquire elementary habits in analysis and synthesis. From the 1st through 4th grade the children are given the following main types of problems (simple and composite) to solve: finding the sum and remainder, product and quotient, increasing and diminishing given numbers, differential and multiple comparison, the simple rule of three, proportional division, finding an unknown by two differences, calculating the arithmetical mean, and some other kinds of problems..

"The program is organised so as to allot a maximum amount of time to operating with numbers, to studying tables and methods for building them, and not to studying quantitative relationships and dependencies of values (measuring, for instance, is given but 37 hours, although it is here that the children get an introduction to this sphere). (We speak here) of the role and place of (procedures) in the process of the development of mathematical thinking in children. We consider this role to be exaggerated, that the methods of teaching caculation habits are not effective, since they are not supported by the child's knowledge of quantitative regularities (suffice it to say that the laws of arithmetic are not studied until the 5th grade).

...

"The following principles should, in our opinion, underlie a critical analysis of the existing program in arithmetic: 1) the concept of number is not identical to that of the quantitiative characterization of objects; 2) a number is not the initial form of expressing quantitiative relationships. ...

"It is well known that some quantitiative relationships can be fully expressed without numbers and before numbers, for instance, in segments, volumes etc (the relationship 'greater than', 'less than','equal'). Initial general mathematical conceptions in modern manuals are presented in symbols that do not assume the necessity of expressing objects in numbers. Thus in Gonin (1959) the basic mathematical objects are, from the very beginning, expressed by letters adn special symbols: for instance $A \cup B$ denotes the joining of sets; $A \cap B$ stands for intersection of sets; $A \backslash B$ – for the difference in sets. A notable feature is that particular kinds of numbers or numerical dependencies are cited only as examples, illustrations of the property of sets, but not as the only possible and only existing form of expressing them. From this point of view it is wrong to assume that letter denominations serve only as substitutes for figures. Actually, basic mathematical conceptions may be adequately expressed and substantiated precisely using letter symbols, which 'cover' in the objects the ties and relationships that are not recorded in numerical form. It is also noteworthy that some mathematical definitions are presented in a graphical form, through the relationship of segments, areas etc. All the basic properties of sets and values may be found and substantiated without resorting to numerical systems; moreover, the latter themselves may be substantiated on the basis of general mathematical conceptions.

...

"It is interesting that Academician A. N. Kolmogorov, in characterising the specific features of mathematical creativity, stresses the following point: " " Most mathematical disoveries are based on some simple idea: a visual geometric construction, a new elementary inequality, and the like. This simple idea has but to be properly applied to the solution of a problem which appears to be inaccessible at first sight." (Kolmogorov, 1961, p. 7) .. "Every child educated in (compulsory schooling) must be given *a system of scientific knowledge in mathematics*. The teaching of this study must naturally begin the the 1st grade. Only under such conditions can we consisently and purposefully develop in all children the habits of mathematical thinking which are indispensible to all those who take part in social production in the age of electronics and cybernetics. Real mathematics must be introduced from the child's very first day at school, and only on this basis should he be taught all the calculation techniques that are essential for practical work. This requires an integrated program of mathematics instruction for the 1st to 8th grades, and subsequently to the 11th grade, be developed, one that will reflect the present state of mathematics, psychology, logic and teaching methods.

"It would be desirable today to have the most diverse ideas concerning the structure and methods of constructing such a programme. The efforts of mathematicians, psychologists, logicians and methods specialists should be enlisted in the work of constructing it. However, all of its concrete variants must, in our opinion, meet the following requirements: 1) to overcome the existing gap between the content of mathematics in elementary and secondary school; 2) to provide a system of knowledge of the chief laws of quantitative relationships in the objective world; the properties of numbers as a special form of expressing quantity must become a special but not the main section of the program; 3) to cultivate in pupils thinking methods and not only calculating habits; this involves building a system of problems which is based on a deeper study of the sphere of dependencies of real magnitudes (the connections of mathematics with physics, chemistry, biology, and other sciences dealing with specific magnitudes); 4) to simplify decisively the entire calculating technique, reducing to a minimum the amount of work that cannot be done without the relevant tables, manuals and other auxiliary means.

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