Appendix A: Secondary-centered Jacobi integral

In the following, Eq 3 will be derived from Eq 1, i.e. the Jacobi integral will be expressed in terms of Keplerian elements around the secondary body. In a first step, the transformation from barycentric, synodic coordinates x, y, z to secondary-centered, inertial, Cartesian coordinates, ξ, η, ζ has to be performed. The synodic frame rotates with angular velocity n with respect to the inertial frame. With reference to Figure 1, the coordinates x, y, z are computed in the synodic, barycentric frame, whereas ξ, η, ζ are computed in the inertial frame, centered in m_2. Therefore, the transformation can be written as follows:

$$\begin{pmatrix} x-(1-\mu) \\ y \\ z \end{pmatrix} = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \tag{7}$$

where we have abbreviated $c := \cos nt$ and $s := \sin nt$. First, the kinetic term in Eq 1 shall be transformed. Taking the time derivative of Eq 7 yields:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = n \begin{pmatrix} -s & c & 0 \\ -c & -s & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} + \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix} \tag{8}$$

This allows writing the velocity terms of the Jacobi integral as follows:

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 + \dot{z}^2 &= \left(-ns\xi + nc\eta + c\dot{\xi} + s\dot{\eta}\right)^2 \\ &\quad + \left(-nc\xi - s\eta n - s\dot{\xi} + c\dot{\eta}\right)^2 + \dot{\zeta}^2 \\ &= \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 + n^2\left(\xi^2 + \eta^2\right) + 2n\left(\eta\dot{\xi} - \xi\dot{\eta}\right) \end{aligned} \tag{9}$$

In the last line some straightforward algebra has been skipped. Next, we rewrite the centrifugal potential terms of Eq 1 by considering the expression:

$$\begin{aligned} (1-\mu)r_1^2 + \mu r_2^2 &= (1-\mu)\left[(x+\mu)^2 + y^2 + z^2\right] \\ &\quad + \mu\left[(x-(1-\mu))^2 + y^2 + z^2\right] \\ &= x^2 + y^2 + z^2 + \mu(1-\mu) \end{aligned} \tag{10}$$

where Eq 2 has been used. Considering that $z = \zeta$, Eq 10 can be rewritten as:

$$x^2 + y^2 = (1-\mu)r_1^2 + \mu r_2^2 - \zeta^2 - \mu(1-\mu) \tag{11}$$

Inserting Eqs. 9, 11 into Eq 1 and using $\xi^2 + \eta^2 + \zeta^2 = r_2^2$, one obtains the Jacobi integral in secondary-centered, inertial, Cartesian coordinates:

$$\begin{aligned} C_J &= n^2(1-\mu)\left(r_1^2 - r_2^2\right) + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - \left(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right) \\ &\quad + 2n\left(\xi\dot{\eta} - \eta\dot{\xi}\right) - n^2\mu(1-\mu) \end{aligned} \tag{12}$$

In a second step, the Cartesian coordinates are replaced by Keplerian elements, a_s, e_s and i_s around the secondary body using definitions of the angular momentum and energy in the two-body problem:

$$\xi\dot{\eta} - \eta\dot{\xi} = h_z = \sqrt{\mu a_s\left(1 - e_s^2\right)}\cos i_s \tag{13}$$

$$\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 = v_{sc}^2 = \frac{2\mu}{r_2} - \frac{\mu}{a_s} \tag{14}$$

Inserting these into Eq 12 yields:

$$\begin{aligned} C_J &= n^2(1-\mu)\left(r_1^2 - r_2^2\right) + \frac{2(1-\mu)}{r_1} + \frac{\mu}{a_s} + 2n\sqrt{\mu a_s\left(1-e_s^2\right)}\cos i_s \\ &\quad - n^2\mu(1-\mu) \end{aligned} \tag{15}$$

which is identical to Eq 3. The dependence of the Jacobi integral on r_1 and r_2 cannot be removed, but these parameters can be approximated by constant values in the vicinity of the secondary, e.g. by choosing $r_1 = 1$ and $r_2 = 0$.

Appendix B: Primary-centered Jacobi integral

In the following, Eq 4 will be derived, i.e. an expression of the Jacobi integral as a function of primary-centered Keplerian elements will be computed. The derivation is completely analogous to the previous section and an intermediate result can readily be obtained by simply performing the transformation

$$\begin{aligned} \mu &\leftrightarrow 1-\mu \\ r_1 &\leftrightarrow r_2 \\ a_s &\rightarrow a_p \\ e_s &\rightarrow e_p \\ i_s &\rightarrow i_p \end{aligned} \tag{16}$$

on Eq 15. This yields:

$$\begin{aligned} C_J &= n^2\mu\left(r_2^2 - r_1^2\right) + \frac{2\mu}{r_2} + \frac{(1-\mu)}{a_p} + 2n\sqrt{(1-\mu)a_p\left(1-e_p^2\right)}\cos i_p \\ &\quad - n^2\mu(1-\mu) \end{aligned} \tag{17}$$

The usefulness of Eq 17 is, however, limited because r_1 and r_2 cannot be simply approximated by a constant value, as previously, for the orbits under consideration. Therefore, some additional approximations need to be made. The sum of the first and last term in Eq 17 can be transcribed using Eq 10:

$$\begin{aligned} \mu\left(r_2^2 - r_1^2\right) - \mu(1-\mu) &= x^2 + y^2 + z^2 - r_1^2 \\ &= -2x\mu - \mu^2 \\ &\approx 0 \end{aligned} \tag{18}$$

where in the second line, Eq 2 has been used. The third line assumes $\mu \ll 1$ and $x \lesssim 1$. Moreover, the second term in Eq 17 can be neglected compared to the third term because $\mu \ll 1$. These approximations are valid if the secondary is much lighter than the primary and if the spacecraft is not too close to the secondary. Overall, Eq 17 becomes:

$$C_J \approx \frac{1-\mu}{a_p} + 2n\sqrt{(1-\mu)a_p\left(1-e_p^2\right)}\cos i_p \tag{19}$$

which is identical to the first line of Eq 4.

In order to obtain an expression for the Jacobi integral as a function on the flyby infinite velocity (at the secondary body), consider the relationship between the infinite velocity and the velocity vectors of the secondary body and the spacecraft in Figure 15.

Since in adimensional units the secondary body velocity has length 1, the cosine theorem applied to velocity triangle reads:

$$\begin{aligned} v_\infty^2 &= 1 + v_{sc}^2 - 2v_{sc}\cos\beta \\ &= 1 + v_{sc}^2 - 2v_{sc}\cos\gamma\cos i_p \end{aligned} \tag{20}$$

In the second line the relationship between the angle β the flight-path angle, γ and inclination i_p of the spacecraft with respect to the orbital plane of the primaries has been used. The second term in Eq. 20 can be replaced with the vis-viva equation for the spacecraft velocity, v_{sc}, at encounter which happens at radius 1:

$$v_{sc}^2 = 2(1-\mu) - \frac{1-\mu}{a_p} \tag{21}$$

The $v_{sc}\cos\gamma$ term in Eq 20 can be replaced using the definition of the angular momentum

$$v_{sc}\cos\gamma = \sqrt{(1-\mu)a_p\left(1-e_p^2\right)} \tag{22}$$

Inserting both into Eq 20 yields:

$$v_\infty^2 = 1 + 2(1-\mu) - \frac{1-\mu}{a_p} - 2\sqrt{(1-\mu)a_p\left(1-e_p^2\right)}\cos i_p \tag{23}$$

Up to a constant, this is identical to the right-hand side of Eq. 19 (remember that $n = 1$ in adimensional units). Therefore, the Jacobi integral can also be written as

$$C_J \approx 3 - 2\mu - v_\infty^2 \tag{24}$$

which is identical to Eq 4.

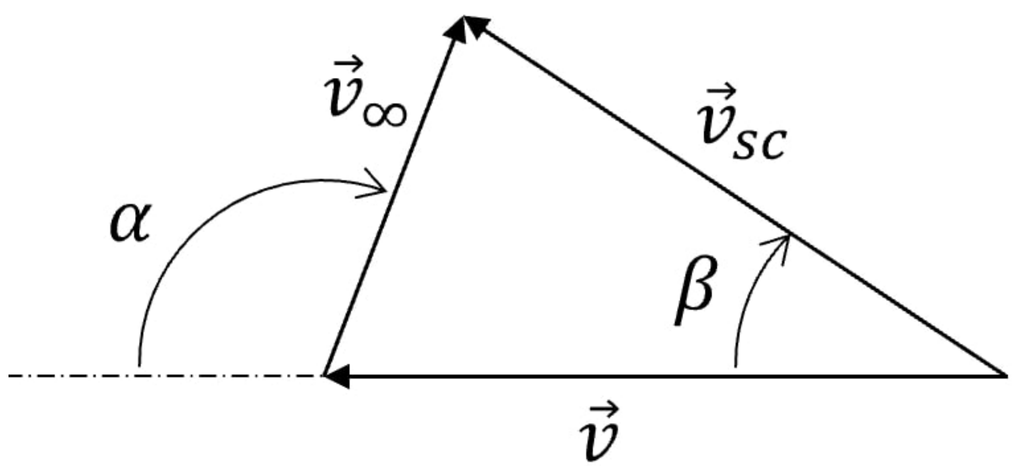

FIGURE 15
Definition of the pump angle, α, and auxiliary angle, β for an encounter at the secondary body in the CR3BP.