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Supplementary materials: derivation of Equation (1)

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In this appendix, a Darcy-type equation, which describes the boundary conditions of the experimental set-up, is derived. First, the Darcy Law and continuity equations are presented. Then, the physics of flow towards a well is introduced. After this necessary background information, the model to calculate the hydraulic conductivity with the new set-up is presented. This model is derived step by step by increasing the complexity. First, the radius of the experimental columns is assumed to be equal to infinity and the case is studied for the situation without internal sinks and with internal sinks. Then, the equation is adjusted for a column with finite radius. This situation, with finite radius and internal sinks, represents the conditions of the performed experiment.

1.1. BACKGROUND: DARCY LAW AND CONTINUITY EQUATION

In a general form, Darcy equation can be written as:

$$J_w = -\frac{K}{\mu}\nabla P \tag{1.1}$$

where J_w (m/s) is the water flux as measured in the laboratory, K (m^2) is the intrinsic permeability (*Fredlund et al.*, 2012), μ (kg/ms) is the viscosity of the suspending fluid (i.e., water from lake Markermeer) and ∇P is the applied gradient pressure. In soil mechanics, it is common to use the hydraulic conductivity k in m/s (*Terzaghi and Peck*, 1967), which in the present paper is defined as:

$$k = K\frac{\rho g}{\mu} \tag{1.2}$$

The gradient in pressure is a vector that is opposite to the vector of the water flow, as the flow of water goes from regions of high pressure P_{high} to regions of low pressure P_{low}. To ensure that we have $K > 0$ a minus sign in front of K is necessary. For unidirectional flows, as studied by Darcy, one gets:

$$J_w = \frac{K}{\mu}\frac{P_{high} - P_{low}}{L} \tag{1.3}$$

where the pressure gradient $(P_{high} - P_{low})/L$ is the applied (macroscopic) pressure gradient on the sample of length L.

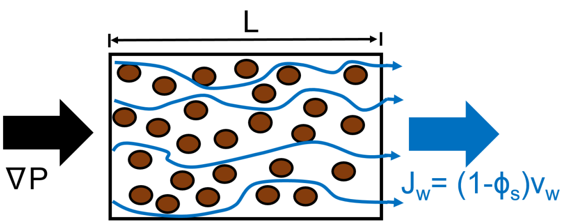

Figure 1.1: Water, under a pressure gradient ∇P, is forced into a porous medium. J_w (m/s) is the measured flux of water coming out of the porous medium, where v_w is the velocity of water inside the soil pores.

The flux J_w represents a macroscopic flux and is defined as the volume of water exiting the porous medium per unit of surface (see Figure 1.1). Its units are therefore

$(\mathrm{m}^3/\mathrm{s})/\mathrm{m}^2 = \mathrm{m}/\mathrm{s}$. It can be linked to the microscopic flux of water v_w that flows inside the incompressible soil pores by realising that:

$$J_w = \frac{dV_w}{dV} v_w = \phi_w v_w = (1-\phi_s) v_w \tag{1.4}$$

where dV_w is the small element of volume of water at a height z, dV a little element of the total volume and ϕ_w and ϕ_s are the volumetric concentrations of water and soil respectively. By the definition of hydrostatic pressure, one can write:

$$P(z) = \rho_w g z + P_{atm} \tag{1.5}$$

where z is the distance between free water and the point where the pressure is measured. The pressure P_{atm} is the atmospheric pressure.

1.2. BACKGROUND: THE CONTINUITY EQUATION AND FLOW TOWARDS A WELL

The local form of the continuity equation (valid for a position (r,z) in cylindrical coordinates or (x,y,z) in Cartesian coordinates) is given by:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \cdot v) = \rho \cdot (q_{source} - q_{sink}) \tag{1.6}$$

where ρ is the density of the medium. The velocity of the fluid is given by $v = J$. Note that the dimensions of q_{source}; q_{sink} are in (1/s) and that they represent microscopic flows as they are defined at a given position (r,z). One can therefore see these sinks and sources as flows per (microscopic) volume, i.e. $(\mathrm{m}^3/\mathrm{s})/\mathrm{m}^3 = 1/\mathrm{s}$. It may be assumed that, inside the volume element considered, sources and sinks (q_{source}; q_{sink}) exist. In these sources and sinks, matter (and mass) can be appearing or disappearing: for instance, roots that can take up water, or a leaking pipe can release water. The first term of this continuity equation accounts for a compressible medium.

However, for the case vertical well is placed in the soil (Figure 1.2), having an uptake rate of Q_0 (m^3/s) over the height H, the general continuity equation cannot be used and adaptations are needed. The volume element is defined as the tube of height H and thickness $(R - r_0)$ where r_0 is the radius of the well.

The continuity equation in cylindrical coordinates gives:

$$\nabla \cdot v = -q_{sink} \tag{1.7}$$

and therefore:

$$\frac{J_r}{r} + \frac{\partial J_r}{\partial r} = -q_{sink} \tag{1.8}$$

Here, J_r is the flux of water in the radial direction. The general solution to Equation 1.8 is:

$$J_r = \frac{A(z)}{r} - q_{sink}\frac{r}{2} \tag{1.9}$$

where $A(z)$ is an integration constant that only depends on z and represents the initial boundary conditions. In the following, it assumed that A does not depend on z and

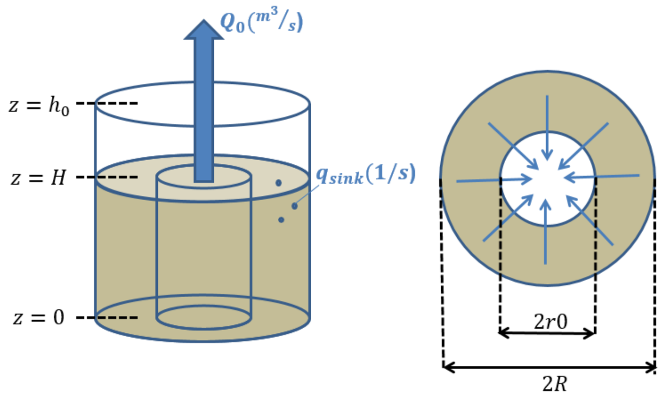

Figure 1.2: Schematic representation of the volume element (grey): it is a porous tube in which water flows radially towards the well that is placed in its centre. The tube is in contact with water at $z = H$, and the air/water interface is at $z = h_0$. This water level in the drainage pipe is kept constant.

the flows are identical at each height z in the column. This is a necessary hypothesis since in the experiment only the flow over the whole length of the column was measured. Therefore, there is no measured information about the flow at a given layer, only the flow integrated over H (the height of the column). Thus, it is not know how A varies over z in reality.

1.3. VERTICAL WELL, RADIUS R IS INFINITY AND WITHOUT INTERNAL SINKS IN THE SEDIMENT

In this section, an equation for the simplest case of a vertical well placed in the soil with and the radius R is infinity and no internal sinks (no roots or other uptakes) is derived. In this case, only the measured total Q_0 (m^3/s) in the drainage pipe over the height H (see Figure 1.2) is accounted and q_{sink}=0. Because there is no sink term within the sediment:

$$J_r = \frac{A(z)}{r} \tag{1.10}$$

By integration of J_r over z, one finds (note that Q_0 is defined in the opposite direction of the unit vector):

$$\int_0^H J_r(r) 2\pi r dz = -Q_0 \tag{1.11}$$

$$A 2\pi H = -Q_0 \tag{1.12}$$

which implies:

$$A = \frac{-Q_0}{2\pi H} \tag{1.13}$$

and

$$J_r = \frac{-Q_0}{2\pi H r} \tag{1.14}$$

From the Darcy equation,

$$J_r = \frac{-K}{\mu}\left(\frac{dP}{dr}\right) = \frac{-Q_0}{2\pi Hr}; \frac{dP}{dr} = \frac{-\mu Q_0}{K2\pi Hr} \tag{1.15}$$

Integrating P in the r direction yields:

$$P(r,z) = \frac{-A\mu}{K}\ln\left(\frac{r}{r_0}\right) + B(z) \tag{1.16}$$

where B(z) is an integration constant that only depends on z. The last boundary condition is: $P(r = r_0, z) = P_0$, where P_0 is a given hydrostatic pressure, and therefore:

$$P(r) = P_0 + \frac{Q_0\mu}{2\pi KH}\ln\left(\frac{r}{r_0}\right) \tag{1.17}$$

Note that the logarithm is always positive when $r > r_0$ which indicates that the pressures becomes lower towards to centre of the circle – which is in agreement with the fact that water always flows from regions of high pressures to regions of low pressures. This yields:

$$\Delta P = P(r = R) - P(r = r_0) = \frac{\mu Q_0}{2\pi KH}\ln\left(\frac{R}{r_0}\right) \tag{1.18}$$

This last equation agrees with literature studying the flow towards wells (e.g *Todd and Mays*, 2005). Equation 1.18 is obtained with no restriction about the water flow at R. Indeed, the flow of water is then non-zero:

$$J_r = \frac{Q_0}{2\pi R} \tag{1.19}$$

However, for large R the flux goes to zero.

1.3.1. VERTICAL WELL, RADIUS R IS INFINITY AND UPTAKE OF WATER BY INTERNAL SINKS

In this section, Equation 1.18 is adapted for the case that there are both a well and internal sinks. Examples of an internal sink are the roots of plants, that are assumed to be homogeneously distributed in the soil. Evaporation is accounted for as an internal sink, which is a simplification. The total water that is taken up by evaporation (in the control column) and by the roots and evaporation (in the sediment column with plants) is Q_{sink} (m^3/s). Q_{sink} is linked to q_{sink} (1/s) by dividing by the volume V=$\pi(R^2$-$r_0^2)H$:

$$q_{sink} = \frac{Q_{sink}}{\pi(R^2 - r_0^2)H} \tag{1.20}$$

The sink term only works within the soil: the water taken by the plants will not flow in the inner column. Therefore at the inner column one has:

$$\int_0^H \frac{A}{r_0} 2\pi r_0 dz = -Q_0 \tag{1.21}$$

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which, from the continuity Equation 1.9, yields:

$$J_r = \frac{-Q_0}{2\pi r H} - \frac{Q_{sink}}{\pi(R^2 - {r_0}^2)H}\frac{r}{2} \tag{1.22}$$

1.3.2. VERTICAL WELL, FINITE SIZE COLUMN AND UPTAKE OF WATER BY INTERNAL SINKS

This is the case of the experiment performed. In this case, the problem is similar to the one discussed above, except that the boundary condition at the exterior r=R is changed. For the column, the boundary becomes that there is no flow through the column wall, i.e. an outer impervious boundary condition, hence:

$$J_r(r = R) = 0 \tag{1.23}$$

Therefore, from Equation 1.22:

$$Q_0 = -\frac{R^2}{(R^2 - r_0^2)}Q_{sink} \tag{1.24}$$

which gives:

$$J_r = \frac{Q_{sink}}{2\pi r H}\frac{R^2}{\left(R^2 - r_0^2\right)} - \frac{Q_{sink}}{\pi\left(R^2 - r_0^2\right)H}\frac{r}{2} \tag{1.25}$$

$$J_r = \frac{Q_{sink} r}{2\pi\left(R^2 - r_0^2\right)H}\left(\frac{R^2}{r^2} - 1\right) \tag{1.26}$$

The associated pressure gradient can be found using Darcy:

$$J_r = \frac{-K}{\mu}\frac{\partial P}{\partial r} \tag{1.27}$$

which gives:

$$P(r) = \frac{\mu Q_{sink}}{2\pi K\left(R^2 - r_0^2\right)H}\left(R^2 \ln(r) - \frac{r^2}{2}\right) + C \tag{1.28}$$

where C is an integration constant. The water inside the inner well is kept at constant height h_0. This implies that

$$P(r = r_0) = P_{atm} + \rho_w g(h_0 - z) \tag{1.29}$$

Therefore the integration constant is found and:

$$P(r,z) = \frac{\mu Q_{sink}}{2\pi K\left(R^2 - r_0^2\right)H}\left(R^2 \ln\left(\frac{r}{r_0}\right) - \frac{r^2}{2} + \frac{r_0^2}{2}\right) + P_{atm} + \rho_w g(h_0 - z) \tag{1.30}$$

yielding

$$\Delta P = P(R,z) - P(r = r_0, z) \tag{1.31}$$

$$\Delta P = \frac{\mu Q_{sink}}{2\pi K\left(R^2 - r_0^2\right) H}\left(R^2 \ln\left(\frac{R}{r_0}\right) - \frac{R^2}{2} + \frac{r_0^2}{2}\right) \tag{1.32}$$

and

$$\Delta P = \frac{\mu Q_{sink} R^2}{2\pi K(R^2 - {r_0}^2) H}\left(\ln\left(\frac{R}{r_0}\right) - \frac{1}{2} + \frac{{r_0}^2}{2R^2}\right) \tag{1.33}$$

The term between brackets is always positive, and in the limit of large R one finds:

$$\Delta P = \frac{\mu Q_{sink}}{2\pi K H} \ln\left(\frac{R}{r_0}\right) \tag{1.34}$$

which corresponds to the general case (i.e. Equation 1.18) except that here the flow is generated by the Q_{sink}.

Finally, from Equation 1.33 and the correlation between the intrinsic permeability K (m^2) and hydraulic conductivity k (m/s) (see Equation 1.2), the equation used to calculate the hydraulic conductivity in the present paper is obtained:

$$k = \frac{\rho g Q_{sink} R^2}{2\pi \Delta P(R^2 - {r_0}^2) H}\left(\ln\left(\frac{R}{r_0}\right) - \frac{1}{2} + \frac{{r_0}^2}{2R^2}\right) \tag{1.35}$$

The results are presented in the main body of the paper, see Figure 6 "Depth-averaged conductivity for the control and vegetated column". Note that, with the data measured (total flow in/out of the column), only the average hydraulic conductivity of the whole column can be calculated. In order to calculate vertical hydraulic conductivity profiles of presented in the other supplementary materials, the assumption that the flow is equally distributed over the whole length was made. This is a bigger assumption for the column with plants, since where the plants are more active the flows may be larger. In the same way, for the uppermost layers of the column without plants, the flows induced by evaporation are underestimated.

References

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