Appendix: Multi-Soliton Solutions

The Hirota transformation given as

$$\Psi = \sqrt{\frac{24S}{R}}\partial_\xi \left[\tan^{-1}\left(\frac{f}{g}\right)\right], \tag{1}$$

produces a couple of following bilinear equations for the mKdV equation (**??**)

$$D_\xi^2(f.f + g.g) = 0, \tag{2}$$

$$(SD_\xi^3 + D_\tau)(f.g) = 0. \tag{3}$$

Here, D_ξ and D_τ are the Hirota operators that act antisymmetrically on pair of functions as

$$D_\xi^a D_\tau^b (f.g) = \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \xi'}\right)^a \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'}\right)^b (f(\xi,\tau)\, g(\xi',\tau'))\Bigg|_{\xi=\xi',\tau=\tau'}. \tag{4}$$

The functions f and g are the arbitrary dependent variable functions, which may be expanded in terms of ϵ as

$$\begin{aligned} f &= f_0 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + ..., \\ g &= g_0 + \epsilon g_1 + \epsilon^2 g_2 + \epsilon^3 g_3 + ... \end{aligned} \tag{5}$$

Since, mKdV is completely integrable equation, so ϵ^N-order generates the N-soliton solution for the mKdV equation. We assume $f_1 = F_1 e^{\eta_1}$ and $g_1 = G_1 e^{\eta_1}$, where $\eta_1 = k_1\xi + \Omega_1\tau + \eta_0$, k_1 is propagation vector, Ω_1 is the frequency, and η_0 is initial phase constant which may be taken to be zero. Plugging these values in Eq. (2), we get the following general expression for the arbitrary dependent variable functions

$$f_0 F_1 + g_0 G_1 = 0. \tag{6}$$

This expression gives the conditions on functions which determine that whether the solitary wave would be compressive or rarefactive.

0.1 Single Soliton Solution

One of the conditions on arbitrary functions given by Eq. (6) is

$$f_0 = G_1 = 0. \tag{7}$$

Putting in Eq. (3) gives the following dispersion relation

$$\Omega_1 = -Sk_1^3. \tag{8}$$

For single soliton, the functions f and g become

$$\begin{aligned} f &= e^{k_1(\xi - Sk_1^2\tau)}, \\ g &= 1. \end{aligned} \tag{9}$$

These functions yield the single compressive soliton solution for mKdV equation upon substitution in Eq. (1), given as follows

$$\Psi = \sqrt{\frac{24S}{R}}\partial_\xi \left[\tan^{-1}\left(e^{k_1(\xi - Sk_1^2\tau)}\right)\right]. \tag{10}$$

The other possible condition given by Eq. (6) is

$$g_0 = F_1 = 0. \tag{11}$$

For this condition, Eq. (2) gives the same frequency as defined in Eq. (8). However, the functions f and g are now obtained as

$$f = 1, \tag{12}$$

$$g = e^{k_1(\xi - Sk_1^2\tau)}. \tag{13}$$

Therefore, we get single rarefactive soliton solution of mKdV equation of the form

$$\Psi = \sqrt{\frac{24S}{R}}\partial_\xi \left[\tan^{-1}\left(\frac{1}{e^{k_1(\xi - Sk_1^2\tau)}}\right)\right]. \tag{14}$$

0.2 Two soliton Solution

To derive the two soliton solution for compressive case given by Eq. (7), we let

$$f_1 = e^{\eta_1} + e^{\eta_2}. \tag{15}$$

where $\eta_i = k_i\xi + \Omega_i\tau + \eta_0$ with $i = 1, 2$. For the condition of compressive solitons, substituting Eqs. (15) and (5) in Eq. (3) give us the following two dispersion relations

$$\Omega_i = -Sk_i^3, \tag{16}$$

The dependent variable functions then turn out to be

$$f = e^{k_1(\xi - Sk_1^2\tau)} + e^{k_2(\xi - Sk_2^2\tau)}, \tag{17}$$

$$g = 1 + a_{12}e^{k_1(\xi - Sk_1^2\tau) + k_2(\xi - Sk_2^2\tau)}. \tag{18}$$

Here a_{12} is referred to as a parameter that describes the interaction of two solitons given as follows

$$a_{12} = -((k_1 - k_2)^2/(k_1 + k_2)^2). \tag{19}$$

Substituting these values in Eq. (1), the compressive two soliton solution for mKdV equation becomes

$$\Psi = \sqrt{\frac{24S}{R}}\partial_\xi \left[\tan^{-1}\left(\frac{e^{k_1(\xi - Sk_1^2\tau)} + e^{k_2(\xi - Sk_2^2\tau)}}{1 + a_{12}e^{k_1(\xi - Sk_1^2\tau) + k_2(\xi - Sk_2^2\tau)}}\right)\right]. \tag{20}$$

For the rarefactive case, given by Eq. (11), we take

$$g_1 = e^{\eta_1} + e^{\eta_2}, \tag{21}$$

which gives the rarefactive two soliton solution as

$$\Psi = \sqrt{\frac{24S}{R}} \partial_\xi \left[\tan^{-1} \left(\frac{1 + a_{12} e^{k_1(\xi - S k_1^2 \tau) + k_2(\xi - S k_2^2 \tau)}}{e^{k_1(\xi - S k_1^2 \tau)} + e^{k_2(\xi - S k_2^2 \tau)}} \right) \right], \tag{22}$$

by adopting the same above procedure.