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Novel decision aid model for green supplier selection based on extended EDAS approach under pythagorean fuzzy Z-numbers

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The main objective of this study is to identify the green suppliers that would most effectively assist manufacturing producers in implementing green manufacturing production while including uncertainty and reliability in their decision-making. For this firstly, we justify and manifest the idea of Pythagorean Fuzzy Z-numbers (PyFZNs). It has significant implications for improving the effectiveness of decision-making processes in several theories of uncertainty. It can more flexibly explain real-world data and human cognition due to its capacity to express imprecise and reliable information. Thus it is a more accurate mathematical tool for addressing accuracy and uncertainty. Secondly, we defined the Pythagorean fuzzy Z-number arithmetic aggregation operators and geometric aggregation operators. Thirdly, based on the proposed operators and EDAS (Evaluation based on distance from average solution) approach, a fast decision model is designed to deal with the issue of multi-criteria decision-making. Finally, using PyFZN data we also provide a numerical example to demonstrate the usability of the created multicriteria decision-making (MDM) approach. Moreover, a case study also proves its efficacy.

KEYWORDS

pythagorean fuzzy z-number, pythagorean fuzzy z-number weighted arithmetic operator, pythagorean fuzzy z-number weighted geometric operator, decision making, EDAS method

1 Introduction

Multi attribute group decision making is a problem in several disciplines, such as management, engineering, and economics. For a very long time, it has been considered that such data which accesses the options in terms of requirement and weight is expressed in actual numbers. The majority of the desired values however are contaminated by uncertainty, which making a decision difficult for the decision-makers to correct judgment as the system becomes more complex every day. To deal with the uncertainties, Zadeh (Zadeh, 1965) introduced the fuzzy set theory notion in 1965, which described the degree of membership. It performed a very significant role in decision making (Ashraf et al., 2022a; Ashraf et al., 2022b; Zhang et al., 2022). In 2011, Zadeh (Zadeh, 2011) further proposed the idea of Z-numbers to better highlight the limitations and reliability of the evaluation by an ordered pair of fuzzy numbers in unpredictable conditions. Compared to the traditional fuzzy number, it is a more extensive concept that is intimately linked to reliability. Given this, the Z-number suggests that a pair of fuzzy numbers in the order of restriction and reliability could be better equipped to explain human knowledge and judgments. Theoretically, the Z-numbers reflect the numeric values of internal machine-mind processes that facilitate adapted real-world understanding (Banerjee and Pal, 2015). The four basic mathematical operations in linear programming addition, subtraction, multiplication, and division as well as algebraic operations like maximum, minimum, square, and square root of continuous Z-numbers were established in 2016 (Aliev et al., 2016). On the basis of the extension principle applied to Z-numbers, a technique for the creation of functions was proposed. This methodology is very helpful in the reduction of uncertainty, while computing the values of Z-valued functions. Z-numbers can be arranged using the proposed total utility of Z-number, which can also be used to make multi criteria judgments under uncertain conditions (Kang et al., 2018b). Ronald proposes a Dempster-Shafer such as belief structure representation of Z-values that incorporates type-2 fuzzy logic (Yager, 2012). Ronald R discusses mixing Z-number and the Dempster-Shafer (D-S) evidence theory, where the Z-number is used to simulate the fuzziness and dependability of sensor data and the D-S evidence theory is used to combine the complex information from Z-numbers (Jiang et al., 2016). Numerous scholars created the TODIM technique based on the Choquet integral for circumstances involving multi-criteria decision-making using linguistic Z-numbers (Wang et al., 2017). Using the Z-number application, the supplier selection problem's idea was illustrated (Jabbarova, 2017). The BWM technique and the Z-number extension were studied in order to handle the informational uncertainty in a multi-criteria decision system (Aboutorab et al., 2018). He created the QUALIFLEX method's linguistic Z-QUALIFLEX extension, which uses linguistic Z-numbers to solve the LGEDM problem (Ding et al., 2020). In (Kang et al., 2018a), reliable methods that can more accurately and flexibly simulate the process of human competition and cooperation are presented for study analysis based on the usefulness of the Z-number in evolutionary games. They present a generalized Z-number that is more in line with human expression tendencies and a multi-criteria decision-making approach based on the Dempster-Shafer (DS) theory and generalized Z-numbers (Ren et al., 2020). To demonstrate the suggested framework and demonstrate its effectiveness in environmental evaluations, the provides an environmental evaluation framework based on the Dempster-Shafer theory and Z-numbers (Kang et al., 2020).

A class of non-standard Pythagorean fuzzy subsets with pairs of membership grades (a, b) meeting the requirement $0 \leq a^2 + b^2 \leq 1$ was introduced using the Pythagorean complement and alternative definitions of complement operations. They provided a number of aggregation techniques and investigated multicriteria decision making in the scenario for these Pythagorean fuzzy set (Yager, 2013). According to (Garg, 2016), some of the aggregator operators discussed include generalized Pythagorean fuzzy Einstein weighted averaging, generalized Pythagorean fuzzy Einstein ordered weighted averaging, and Pythagorean fuzzy Einstein weighted averaging. The Einstein sum, product, and exponentiation as well as geometric aggregation operators and the intuitionistic fuzzy Einstein weighted and ordered weighted geometric operators were only a few of the operations on intuitionistic fuzzy sets that were discussed (Wang and Liu, 2011). The geometric aggregation operators, the intuitionistic fuzzy Einstein weighted and ordered weighted geometric operators, as well as the Einstein sum, product, and exponentiation, were all discussed (Rahman et al., 2017). The Pythagorean Dombi fuzzy aggregation operators introduced by (Akram et al., 2019), ELECTRI-I approach for it by (Akram et al., 2020) and TOPSIS approach by (Akram et al., 2021b). Garg and Sharaf (Garg et al., 2022) presented the spherical fuzzy EDAS approach. Generalized aggregation operators are proposed in (Ashraf and Abdullah, 2019; Chinram et al., 2020; Ashraf et al., 2023). Many scholars worked on its hybrid structures such as cubic Pythagorean linguistic fuzzy numbers (Naeem et al., 2021), rough Pythagorean fuzzy bipolar soft information (Akram and Ali, 2020) and sine trigonometric Pythagorean fuzzy information (Ashraf et al., 2021). It performs better role than fuzzy sets in decision making (Khan et al., 2019; Akram et al., 2021a; Zeng et al., 2023). Saeed et al. (Saeed et al. (2023) presented the refined Pythagorean fuzzy sets and interval-valued complex Pythagorean fuzzy set based methodology is developed in (Yazbek et al., 2023). We refer some decision making techniques (Ashraf et al., 2022c; Garg and Sharaf, 2022) for more details.

We looked at the connection between Z-numbers and linguistic summaries, For instance Dempster-Shafer belief systems and type-2 fuzzy logic were used to describe Z values (Kang et al., 2020). Theoretical components of mathematical operations including addition, subtraction, multiplication, division, and computing the square root of a Z-number over discrete Z-numbers have been proposed (Aliev et al., 2015b). They suggested a Z-number-based computing with words (CWW) algorithm, defined a Z-number-based operator for assessing the degree of requirement satisfaction, described CWW simulation experiments using Z-numbers, examined the benefits and drawbacks of Z-numbers, and offered potential resolution approaches (Pal et al., 2013). The authors developed Z-TOPSIS, a novel variant of the TOPSIS approach that streamlines multi-criteria decision-making problems based on the idea of Z-numbers, in order to support the concept of ranking alternatives using Z-numbers. Additionally, the authors offered a link to some existing information in fuzzy sets (Yaakob and Gegov, 2016). The researchers advise researching the fully Z-number based LP (Z-LP) model and utilizing a method to address Z-LP problems that combines differential evolution optimization and Z-number arithmetic created by the authors in order to better fit real-world problems within the LP framework. Using a benchmark LP problem, the suggested model and solution approach for Z-LP are then shown (Aliev et al., 2015a). For replicating the effects of Pilates exercises on students' motivation, concentration, anxiety, and academic success, they have presented an innovative way. Due to the ambiguity of data relating to cognitive assessment of psychological features and their partial reliability, the use of "Z-if . . . then rules" for modelling the considered relationship has been pushed for the first time (Aliev and Memmedova, 2015). The topic (Kang et al., 2016) is split into two parts: the first explains how to use the fuzzy expectation to convert a Z-number into a concept of fuzzy number, and the second explains how to use the genetic algorithm to determine the best priority weight for supplier selection. This method of calculating the priority weight of the judgment matrix is quick and flexible. The first recommendation in this study is an enhanced ranking approach for generalized fuzzy numbers that take into account the weight of centroid points, fuzziness levels, and fuzzy number spreads. This approach is particularly effective for evaluating symmetric fuzzy numbers and crisp numbers, among other things, and can alleviate some of the drawbacks of existing approaches (Jiang et al., 2017). For the first time, a multi-layer method of grading Z-numbers is proposed in this work. This method has two layers: fuzzy number ranking and Z number conversion (Bakar and Gegov, 2015).

Few articles, to our knowledge, seek to explore the rationality and certainty of information in efficient and effective decision-making in a systematic and comprehensive manner. For handling uncertainty effectively in real-world applications, fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets, picture fuzzy sets, hesitant fuzzy sets, etc. all are unable to deal with the reliability of their membership levels. While reliability or certainty is the core component to deal with making decisions efficiently and trustworthy. In order to eliminate this constraint, this research uses the strategic illustration, conceptual evolution map, and major route analysis to explain the development of the novel Pythagorean fuzzy Z-number (PyFZN) field comprehensively. We suggest a PyFZN by considering Pythagorean fuzziness in terms of both membership and non-membership degrees with reliability. We defined some fundamental properties and aggregation operators, which perform a very significant role in making decisions. Furthermore, we suggested a multicriteria decision-making (MDM) approach, which aims to reduce the overall uncertainty of the decision matrix to increase the credibility of the conclusions. To demonstrate the effectiveness of this novel concept we illustrated a real-life example based on it. Moreover, we also presented a comparative analysis with the existing studies to show its supremacy. The primary goals of this study are: 1. To investigate a number of distinct interactive averaging and geometric AOs concerning PyFZNs to circumvent the constraints. 2. Getting some key characteristics and unique cases of the newly defined AOs. 3. To create a novel MDM strategy using the interactive AOs in a fuzzy Pythagorean context that has been suggested. 4. To present the benefits and viability of the developed MDM strategy.

The main goals of this study are to address the aforementioned research topics and close a knowledge gap.

The rest of this article is organized as follows. Preliminary definitions and concepts are presented in Section 2 and are required to support our primary findings. The concept of Pythagorean fuzzy Z-number, their properties, and score function is explained in Section 3. A thorough investigation of this idea is done in Section 4 where we defined a Pythagorean fuzzy Z-number with arithmetic and geometric aggregation operators and also demonstrate some theorems related to these ideas. In Section 5 we defined a Pythagorean fuzzy Z-number ordered weighted arithmetic and geometric aggregation operators and also demonstrate some theorems related to these ideas. The MDM technique based on intended operators clarified in Section 6, which also provided a numerical example for selecting the riskiest companies to invest money as business partners. We provide an EDAS approach for PyFZNs in Section 7 and also illustrate it with the aid of an example. Finally, in Section 8, we presented a comparative analysis and in Section 9, the conclusion and future research directions for this paper are depicted.

2 Preliminaries

Definition 1. (Zadeh, 1965) Let X be a nonempty set, a fuzzy set A in X is characterized by a membership function

where μ_A: X → [0, 1], the function defines the degree of membership of the element, x ∈ X.

That is: A fuzzy set A in X is an object having the form:

$$A = \{\langle x, \mu_A(x)\rangle | x \in X\}.$$

Definition 2. (Yager, 2013) Assume that B is the Pythagorean fuzzy set and here M is a universal set described as

$$B = \{l, \mu_B(l), \nu_B(l) | l \in M\},$$

where the function $\mu_B(l)\colon M \to [0,1]$ and $\nu_B(l)\colon M \to [0,1]$ are the degree of membership and the degree of nonmembership respectively, which satisfies the following requirement:

$$0 \le (\mu_B(l))^2 + (\nu_B(l))^2 \le 1.$$

Definition 3. *The concept of Z-number first introduced by zadeh in 2011 (Zadeh, 2011). The Z-number is discussed as, taking order pair of fuzzy numbers $Z = (S, T)$, where S is a fuzzy ristriction on the values of M and T gives the reliability for S, here M being a universal set.*

3 Pythagorean fuzzy Z-number

Now we define Pythagorean fuzzy Z-number (*PyFZN*) as given below.

Definition 4. *Assume that Gz is a Pythagorean fuzzy Z-number ($PyFZN$) and M be the universal set:*

$$Gz = \{l, \mu(S,T)(l), \nu(S,T)(l) | l \in M\}$$

where the funtion $\mu(S,T)(l)\colon M \to [0,1]$ and $\nu(S,T)(l)\colon M \to [0,1]$ are constructed as follows:

$$Gz = \{(\mu(S,T)), \nu(S,T)\} = \{(\mu_S, \mu_T), (\nu_S, \nu_T)\}$$

It meets the following requirements:

$$0 \le (\mu(S)(l))^2 + (\nu(S)(l))^2 \le 1$$
$$0 \le (\mu(T)(l))^2 + (\nu(T)(l))^2 \le 1.$$

Now we will discussing the properties of Pythagorean fuzzy Z-numbers which already discuss in Definition 4.

Definition 5. *Let $Gz_1 = \{(\mu_1(S,T)), \nu_1(S,T)\} = \{(\mu_{S_1}, \mu_{T_1}), (\nu_{S_1}, \nu_{T_1})\}$ and $Gz_2 = \{(\mu_2(S,T)), \nu_2 S, T\} = \{(\mu_{S_2}, \mu_{T_2}), (\nu_{S_2}, \nu_{T_2})\}$ be two Pythagorean fuzzy Z-numbers (P_yFZN_S) and ⱡ >0 which satisfies the following characteristics:*

(1) $Gz_1 \supseteq Gz_2$ if and only if $\mu_{S_1} \ge \mu_{S_2}$, $\mu_{T_1} \ge \mu_{T_2}$ and $\nu_{S_1} \le \nu_{S_2}, \nu_{T_1} \le \nu_{T_2}$.

(2) $Gz_1 = Gz_2$ if and only if $Gz_1 \supseteq Gz_2$ and $Gz_1 \subseteq Gz_2$,

(3) $Gz_1 \cup Gz_2 = \{(\mu_{S_1} \vee \mu_{S_2}, \mu_{T_1} \vee \mu_{T_2}), (\nu_{S_1} \wedge \nu_{S_2}, \nu_{T_1} \wedge \nu_{T_2})\},$

(4) $Gz_1 \cap Gz_2 = \left\{(\mu_{S_1} \wedge \mu_{S_2}, \mu_{T_1} \wedge \mu_{T_2}), (\nu_{S_1} \vee \nu_{S_2}, \nu_{T_1} \vee \nu_{T_2})\right\}$,

(5) $(Gz_1)^c = \left\{(\nu_{S_1}, \nu_{T_1}), (\mu_{S_1}, \mu_{T_1})\right\}$,

(6) $Gz_1 \oplus Gz_2 = \left\{\left(\sqrt{\mu_{S_1}^2 + \mu_{S_2}^2 - \mu_{S_1}^2\mu_{S_2}^2}, \sqrt{\mu_{T_1}^2 + \mu_{T_2}^2 - \mu_{T_1}^2\mu_{T_2}^2}\right) (\nu_{S_1}\nu_{S_2}, \nu_{T_1}\nu_{T_2})\right\}$,

(7) $Gz_1 \otimes Gz_2 = \left\{(\mu_{S_1}\mu_{S_2}, \mu_{T_1}\mu_{T_2}), \left(\sqrt{\nu_{S_1}^2 + \nu_{S_2}^2 - \nu_{S_1}^2\nu_{S_2}^2}, \sqrt{\nu_{T_1}^2 + \nu_{T_2}^2 - \nu_{T_1}^2\nu_{T_2}^2}\right)\right\}$,

(8) $Ⱡ\, Gz_1 = \left\{\left(\sqrt{1-(1-\mu_{S_1}^2)^{Ⱡ}}, \sqrt{1-(1-\mu_{T_1}^2)^{Ⱡ}}\right), (\nu_{S_1}^{Ⱡ}, \nu_{T_1}^{Ⱡ})\right\}$,

(9) $G^{Ⱡ}z_1 = \left\{(\mu_{S_1}^{Ⱡ}\mu_{T_1}^{Ⱡ}), \left(\sqrt{1-(1-\nu_{S_1}^2)^{Ⱡ}}, \sqrt{1-(1-\nu_{T_1}^2)^{Ⱡ}}\right)\right\}$.

Definition 6. *Let $Gz_1 = \left\{(\mu_{S_1}, \mu_{T_1}), (\nu_{S_1}, \nu_{T_1})\right\}$ and $Gz_2 = \left\{(\mu_{S_2}, \mu_{T_2}), (\nu_{S_2}, \nu_{T_2})\right\} \in PyFZN_S$. Then the score funtion is defined as*

$$J(Gz_d) = \frac{1 + \mu_{S_d}\mu_{T_d} - \nu_{S_d}\nu_{T_d}}{2}. \tag{3.1}$$

where $J(Gz_d) \in [0,1]$. If the score of $J(Gz_1) \geq J(Gz_2)$, then $Gz_1 \geq Gz_2$.

Example 1. *Consider two pythagorean fuzzy z-number as $Gz_1 = \{(0.6, 0.8), (0.1, 0.3)\}$ and $Gz_2 = \{(0.5, 0.7), (0.2, 0.4)\}$. Therefore, the base of score funtion the ranking of given pythagorean fuzzy z-number is defined as:Using Eq. 3.1*

$$J(Gz_1) = \left(\frac{1 + (0.6 \times 0.8) - (0.1 \times 0.3)}{2}\right) = 0.725$$

$$J(Gz_2) = \left(\frac{1 + (0.5 \times 0.7) - (0.2 \times 0.4)}{2}\right) = 0.595.$$

Hence, the score of $J(Gz_1) \geq J(Gz_2)$, then $Gz_1 \geq Gz_2$.

4 Pythagorean fuzzy Z-numbers weighted aggregation operators

We may propose the weighted aggregation operators for *PyFZNs* in this part based on actions (6) to (9) in Definition 2.

4.1 PyFZNW operator

We can talk about the *PyFZNW* operator of *PyFZNs* in relation to the basis operations (6) and (8) in Definition 2.

Definition 7. *Let $Gz_d = \left\{(\mu_{S_d}, \mu_{T_d}), (\nu_{S_d}, \nu_{T_d})\right\} (d = 1, 2, \ldots ń)$ be a group of PyFZNsand $PyFZNW: \Omega^{ń} \to \Omega$. Then the PyFZNW operator is defined as*

$$PyFZNW(Gz_1, Gz_2, \ldots Gz_{ń}) = \sum_{d=1}^{ń} Ⱡ_d Gz_d \tag{4.1}$$

where $Ⱡ_d (d = 1, 2, \ldots ń)$ is the weight vector with $0 \leq L = {}_d \leq 1$ and $\sum_{d=1}^{ń} Ⱡ_d = 1$.

Theorem 1. *Let $Gz_d = \left\{(\mu_{S_d}, \mu_{T_d}), (\nu_{S_d}, \nu_{T_d})\right\} (d = 1, 2, \ldots ń)$ be a group of PyFZNs. Then, the collected value of the PyFZNW operator is a PyFZN, which is obtained by the following formula:*

$$PyFZNW(Gz_1, Gz_2, \ldots Gz_{ń}) = \sum_{d=1}^{ń} Ⱡ_d Gz_d = \left\{\begin{pmatrix}\sqrt{1 - \Pi_{d=1}^{ń}\left(1-\mu_{S_1}^2\right)^{Ⱡ_d}}, \\ \sqrt{1 - \Pi_{d=1}^{ń}\left(1-\mu_{T_1}^2\right)^{Ⱡ_d}} \end{pmatrix}, \left(\Pi_{d=1}^{ń}\nu_{S_1}^{Ⱡ_d}, \Pi_{d=1}^{ń}\nu_{T_1}^{Ⱡ_d}\right)\right\} \tag{4.2}$$

where $Ⱡ_d (d = 1, 2, \ldots ń)$ is the weight vector with $0 \leq L = {}_d \leq 1$ and $\sum_{d=1}^{ń} Ⱡ_d = 1$.

Proof. Using mathematical induction to prove the above Theorem 1. In Definition 4 using operation (6) and (8), if $ń = 2$ we obtain the following result:

$$PyFZNW(Gz_1, Gz) = Ⱡ_1 Gz_1 \oplus Ⱡ_2 Gz_2$$

$$= \left\{\begin{matrix}\left(\begin{matrix}\sqrt{1-\left(1-\mu_{S_1}^2\right)^{Ⱡ_1}} + \sqrt{1-\left(1-\mu_{S_1}^2\right)^{Ⱡ_2}} - \\ \sqrt{1-\left(1-\mu_{S_1}^2\right)^{Ⱡ_1}}\sqrt{1-\left(1-\mu_{S_1}^2\right)^{Ⱡ_2}}\end{matrix}\right), \\ \left(\begin{matrix}\sqrt{1-\left(1-\mu_{T_1}^2\right)^{Ⱡ_1}} + \sqrt{1-\left(1-\mu_{T_2}^2\right)^{Ⱡ_2}} - \\ \sqrt{1-\left(1-\mu_{T_1}^2\right)^{Ⱡ_1}}\sqrt{1-\left(1-\mu_{T_2}^2\right)^{Ⱡ_2}}\end{matrix}\right), \\ \left(\nu_{S_1}^{Ⱡ_1}\nu_{S_2}^{Ⱡ_2}, \nu_{T_1}^{Ⱡ_1}\nu_{T_2}^{Ⱡ_2}\right).\end{matrix}\right\}$$

$$= \left\{\begin{matrix}\left(\sqrt{1-\Pi_{d=1}^{2}\left(1-\mu_{S_d}^2\right)^{Ⱡ_d}}, \sqrt{1-\Pi_{d=1}^{2}\left(1-\mu_{T_d}^2\right)^{Ⱡ_d}}\right), \\ \left(\Pi_{d=1}^{2}\nu_{S_d}^{Ⱡ_d}, \Pi_{d=1}^{2}\nu_{T_d}^{Ⱡ_d}\right)\end{matrix}\right\}. \tag{4.3}$$

(2). If $ń = m$, using Eq. 5.2 we get the following form:

$$PyFZNW(Gz_1, Gz_2, \ldots Gz_m) = \sum_{d=1}^{m} Ⱡ_d Gz_d$$

$$= \left\{\begin{matrix}\left(\sqrt{1-\Pi_{d=1}^{m}\left(1-\mu_{S_1}^2\right)^{Ⱡ_d}}, \sqrt{1-\Pi_{d=1}^{m}\left(1-\mu_{T_1}^2\right)^{Ⱡ_d}}\right), \\ \left(\Pi_{d=1}^{m}\nu_{S_1}^{Ⱡ_d}, \Pi_{d=1}^{m}\nu_{T_1}^{Ⱡ_d}\right)\end{matrix}\right\}. \tag{4.4}$$

(3). If $ń = m + 1$, using the operation (6) and (8) in Definition 2, and also use Eqs 5.3, 5.4 we have the following result:

$$PyFZNW(Gz_1, Gz_2, \ldots Gz_m Gz_{m+1}) = \sum_{d=1}^{m} Ⱡ_d Gz_d \oplus Ⱡ_{m+1} Gz_{m+1}$$

$$= \left\{\begin{matrix}\left(\sqrt{1-\Pi_{d=1}^{m}\left(1-\mu_{S_d}^2\right)^{Ⱡ_d}}, \sqrt{1-\Pi_{d=1}^{m}\left(1-\mu_{T_d}^2\right)^{Ⱡ_d}}\right), \\ \left(\Pi_{d=1}^{m}\nu_{S_d}^{Ⱡ_d}, \Pi_{d=1}^{m}\nu_{T_d}^{Ⱡ_d}\right)\end{matrix}\right\} \oplus Ⱡ_{m+1} Gz_{m+1}.$$

$$= \left\{\begin{matrix}\left(\sqrt{1-\Pi_{d=1}^{m+1}\left(1-\mu_{S_d}^2\right)^{Ⱡ_d}}, \sqrt{1-\Pi_{d=1}^{m+1}\left(1-\mu_{Td}^2\right)^{Ⱡ_d}}\right), \\ \left(\Pi_{d=1}^{m+1}\nu_{S_d}^{Ⱡ_d}, \Pi_{d=1}^{m+1}\nu_{T_d}^{Ⱡ_d}\right)\end{matrix}\right\}$$

It is true for all $ń$, then the verification is completed.

Theorem 2. *The PyFZNW operator implies the following properties:*

(1) Idempotency: Let $Gz_d = \left\{(\mu_{S_d}, \mu_{T_d}), (\nu_{S_d}, \nu_{T_d})\right\}$ $(d = 1, 2, \ldots ń)$ PyFZNs. If $Gz_d = Gz (d = 1, 2, \ldots ń)$, there is $PyFZNW(Gz_1, Gz_2, \ldots Gz_{ń}) = Gz$

(2) Boundedness: Let $Gz_d = \left\{(\mu_{S_d}, \mu_{T_d}), (\nu_{S_d}, \nu_{T_d})\right\} (d = 1, 2, \ldots ń)$ as a group of PyFZN and let

$$Gz_{\min} = \{\min(\mu_d(S,T)), \max(\nu_d(S,T))\}$$
$$= \left\{(\min_d(\mu_{sd}), \min_d(\mu_{Td})), (\max(\nu_{sd}), \max \nu(Td))\right\}$$
$$Gz_{\max} = \{\max(\mu_d(S,T)), \min \nu_d(S,T)\}$$
$$= \left\{(\max_d(\mu_{sd}), \max_d(\mu_{Td})), (\min_d(\nu_{sd}), \min_d(\nu_{Td}))\right\}$$

Then,

$$Gz_{\min} \leq PyFZNW(Gz_1, Gz_2, \ldots Gz_{ń}) \leq Gz_{\max}$$

can keep

(3) Monotonoicity: set $Gz_d = \{(\mu_d(S,T)), \nu_d(S,T)\} = \left\{(\mu_{S_d}, \mu_{T_d}), (\nu_{S_d}, \nu_{T_d})\right\} (d = 1, 2, \ldots ń)$ and $G^{\star}z_d = \{(\mu_d^{\star}(S,T)), \nu_d^{\star}(S,T)\} = \left\{(\mu_{S_d}^{\star}, \mu_{T_d}^{\star}), (\nu_{S_d}^{\star}, \nu_{T_d}^{\star})\right\} (d = 1, 2, \ldots ń)$ as

two groups of $PyFZN_S$.When $Gz_d \le G^{\star}z_d$, there is $PyFZNW(Gz_1,Gz_2,...Gz_{\acute{n}}) \le PyFZNW(G^{\star}z_1,G^{\star}z_2,...G^{\star}z_{\acute{n}})$.

Proof. (1) If $Gz_d = Gz\,(d = 1, 2, \ldots \acute{n})$, then the result of equatuin (3) is given by

$$\begin{aligned} PyFZNW(Gz_1,Gz_2,...Gz_{\acute{n}}) &= \sum_{d=1}^{\acute{n}} Ɫ_d Gz_d \\ &= \left\{ \begin{pmatrix} \sqrt{1-\Pi_{d=1}^{\acute{n}}\left(1-\mu_{S_d}^2\right)^{Ɫ_d}}, \\ \sqrt{1-\Pi_{d=1}^{\acute{n}}\left(1-\mu_{T_d}^2\right)^{Ɫ_d}} \end{pmatrix}, \left(\Pi_{d=1}^{\acute{n}}\nu_{S_d}^{Ɫ_d}, \Pi_{d=1}^{\acute{n}}\nu_{T_d}^{Ɫ_d}\right) \right\}, \\ &= \left\{ \begin{pmatrix} \sqrt{1-(1-\mu_S^2)^{\Pi_{d=1}^{\acute{n}}Ɫ_d}}, \\ \sqrt{1-(1-\mu_T^2)^{\Pi_{d=1}^{\acute{n}}Ɫ_d}} \end{pmatrix}, \left((\nu_S)^{\Pi_{d=1}^{\acute{n}}Ɫ_d}, (\nu_T)^{\Pi_{d=1}^{\acute{n}}Ɫ_d}\right) \right\}, \\ &= \left\{\left(\sqrt{1-(1-\mu_S^2)}, \sqrt{1-(1-\mu_T^2)}\right), (\nu_S, \nu_T)\right\}, \\ &= \{(\mu_S, \mu_T), (\nu_S, \nu_T)\} = G_Z. \end{aligned}$$

(2) Since $Gz_{\min}$ and $Gz_{\max}$ are given by the minimum *PyFZN* and the maximum *PyFZN*, then the inequality $Gz_{\min} \le Gz \le Gz_{\max}$ exists. Thus, there is $\sum_{d=1}^{\acute{n}} Ɫ_d Gz_{\min} \le \sum_{d=1}^{\acute{n}} Ɫ_d Gz_d \le \sum_{d=1}^{\acute{n}} Ɫ_d Gz_{\max}$.

Based on the above property (1) $Gz_{\min} \le \sum_{d=1}^{\acute{n}} Ɫ_d Gz_d \le Gz_{\max}$ can exists, i,e, and $Gz_{\min} \le PyFZNW(Gz_1,Gz_2,...Gz_{\acute{n}}) \le Gz_{\max}$.

(3) $Gz_d \le G^{\star}z_d$, there is $\sum_{d=1}^{\acute{n}} Ɫ_d Gz_d \le \sum_{d=1}^{\acute{n}} Ɫ_d G^{\star}z_d$, *d.e.*, $PyFZNW(Gz_1,Gz_2,...Gz_{\acute{n}}) \le PyFZNW(G^{\star}z_1,G^{\star}z_2,...G^{\star}z_{\acute{n}})$. All the properties of given theorem is complete.

4.2 PyFZNWG operator

Using the operation (7) and (9) in Definition 4, we give the *PyFZNWG* operator of PyFZNs.

Definition 8. *Let $Gz_d = \left\{(\mu_{S_d}, \mu_{T_d}), (\nu_{S_d}, \nu_{T_d})\right\}(d = 1, 2, \ldots \acute{n})$ be a group of PyFZNs.Then the PyFZNWG: $\Omega^{\acute{n}} \to \Omega$ operator is defined as:*

$$PyFZNWG(Gz_1,Gz_2,...Gz_{\acute{n}}) = \Pi_{d=1}^{\acute{n}} G^{Ɫ_d} z_d \tag{4.5}$$

where $Ɫ_d\,(d = 1, 2, \ldots \acute{n})$ with $0 \le \text{Ł} = {}_d \le 1$ and $\sum_{d=1}^{\acute{n}} Ɫ_d = 1$.

Theorem 3. *Let $Gz_d = \left\{(\mu_{S_d}, \mu_{T_d}), (\nu_{S_d}, \nu_{T_d})\right\}(d = 1, 2, \ldots \acute{n})$ be a group of PyFZNs. Then, the collected value of the PyFZNWG operator is a PyFZN, which is obtained by the following formula:*

$$\begin{aligned} &= PyFZNW(Gz_1,Gz_2,...Gz_{\acute{n}}) = \Pi_{d=1}^{\acute{n}} G^{Ɫ_d} z_d \\ &= \left\{ \left(\Pi_{d=1}^{\acute{n}}\mu_{S_d}^{Ɫ_d}, \Pi_{d=1}^{\acute{n}}\mu_{T_d}^{Ɫ_d}\right), \begin{pmatrix} \sqrt{1-\Pi_{d=1}^{\acute{n}}\left(1-\nu_{S_d}^2\right)^{Ɫ_d}}, \\ \sqrt{1-\Pi_{d=1}^{\acute{n}}\left(1-\nu_{T_d}^2\right)^{Ɫ_d}} \end{pmatrix} \right\} \end{aligned} \tag{4.6}$$

where $Ɫ_d\,(d = 1, 2, \ldots \acute{n})$ with $0 \le \text{Ł} = {}_d \le 1$ and $\sum_{d=1}^{\acute{n}} Ɫ_d = 1$. By the similar verification process of Theorem 1

Theorem 4. *The PyFZNWG operator of also implies the following properties*

(1) Idempotency:Set: Let $Gz_d = \{(\mu_d(S,T)), \nu_d(S,T)\} = \left\{(\mu_{S_d}, \mu_{T_d}), (\nu_{S_d}, \nu_{T_d})\right\}(d = 1, 2, \ldots \acute{n})$ be a group of PyFZNs. If $Gz_d = Gz\,(d = 1, 2, \ldots \acute{n})$, there is $PyFZNWG(Gz_1,Gz_2,...Gz_{\acute{n}}) = Gz$

(2) Boundedness: Set $Gz_d = \{(\mu_d(S,T)), \nu_d(S,T)\} = \left\{(\mu_{S_d}, \mu_{T_d}), (\nu_{S_d}, \nu_{T_d})\right\}(d = 1, 2, \ldots \acute{n})$ as a group of P_yFZN and let

$$\begin{aligned} Gz_{\min} &= \{\min(\mu_d(S,T)),\ \max(\nu_d(S,T))\} \\ &= \left\{\left(\min_d(\mu_{sd}), \min_d(\mu_{Td})\right),\ (\max(\nu_{sd}),\ \max \nu(Td))\right\} \\ Gz_{\max} &= \{\max(\mu_d(S,T)),\ \min \nu_d(S,T)\} \\ &= \left\{\left(\max_d(\mu_{sd}), \max_d(\mu_{Td})\right),\ \left(\min_d(\nu_{sd}), \min_d(\nu_{Td})\right)\right\} \end{aligned}$$

Then,$Gz_{\min} \le PyFZNWG(Gz_1,Gz_2,...Gz_{\acute{n}}) \le Gz_{\max}$ can keep.

(3) Monotonoicity: set $Gz_d = \{(\mu_d(S,T)), \nu_d(S,T)\} = \left\{(\mu_{S_d}, \mu_{T_d}), (\nu_{S_d}, \nu_{T_d})\right\}(d = 1, 2, \ldots \acute{n})$ and $G^{\star}z_d = \{(\mu_d^{\star}(S,T)), \nu_d^{\star}(S,T)\} = \left\{(\mu_{S_d}^{\star}, \mu_{T_d}^{\star}), (\nu_{S_d}^{\star}, \nu_{T_d}^{\star})\right\}(d = 1, 2, \ldots \acute{n})$ as two groups of P_yFZN_S.When $Gz_d \le G^{\star}z_d$, there is $PyFZNWG(Gz_1,Gz_2,...Gz_{\acute{n}}) \le PyFZNWG$ $(G^{\star}z_1,G^{\star}z_2,...G^{\star}z_{\acute{n}})$. Can also be confirmed Theorem 2 using the aforementioned properties corresponding to the PyFZNWG operator, which is not repeated here.

5 Pythagorean fuzzy Z-numbers ordered weighted aggregation operators

On the basis of the operations (6)–(9) in Definition 4, we may suggest the weighted aggregation operators of PyFZNs in this section.

5.1 PyFZNOW operator

We are able to provide *PyFZNW* operators for *PyFZNs*.

Definition 9. *Let $\alpha_d = \left\{(\mu_{S_{\alpha_d}}, \mu_{T_{\alpha_d}})(\nu_{S_{\alpha_d}}, \nu_{T_{\alpha_d}})\right\}(d = 1, 2, \ldots \acute{n})$ be acollection of PFZNs, then the Pythagorean fuzzy Z-number order weighted averaging aggregation operator is defined as:*

$$PyFZNOW(\alpha_1, \alpha_2, \ldots \alpha_{\acute{n}}) = Ɫ_1\alpha_{\sigma(1)} \oplus Ɫ_2\alpha_{\sigma(2)} \oplus \ldots \oplus Ɫ_{\acute{n}}\alpha_{\sigma(\acute{n})}, \tag{5.1}$$

where $Ɫ_d\,(d = 1, 2, \ldots \acute{n})$ is the weighted vector of $\alpha_d\,(d = 1, 2, \ldots \acute{n})$ with $0 \le Ɫ_d \le 1$ and $\sum_{d=1}^{\acute{n}} Ɫ_d = 1$.

Theorem 5. *Let $\alpha_d = \left\{(\mu_{S_{\alpha_d}}, \mu_{T_{\alpha_d}})(\nu_{S_{\alpha_d}}, \nu_{T_{\alpha_d}})\right\}(d = 1, 2, \ldots \acute{n})$ be acollection of PFZNs, then their aggregated value by using PyFZNOW operators as*

$$PyFZNOW(\alpha_1, \alpha_2, \ldots \alpha_{\acute{n}}) = \left\{ \begin{pmatrix} \sqrt{1-\Pi_{d=1}^{\acute{n}}\left(1-\mu_{S\alpha_{\sigma(d)}}^2\right)^{Ɫ_d}}, \\ \sqrt{1-\Pi_{d=1}^{\acute{n}}\left(1-\mu_{T_{\alpha_{\sigma(d)}}}^2\right)^{Ɫ_d}} \end{pmatrix}, \left(\Pi_{d=1}^{\acute{n}}\nu_{S\alpha_{\sigma(d)}}^{Ɫ_d}, \Pi_{d=1}^{\acute{n}}\nu_{T\alpha_{\sigma(d)}}^{Ɫ_d}\right) \right\} \tag{5.2}$$

where $Ɫ_d\,(d = 1, 2, \ldots \acute{n})$ is the weighted vector of $\alpha_d\,(d = 1, 2, \ldots \acute{n})$ with $0 \le Ɫ_d \le 1$ and $\sum_{d=1}^{\acute{n}} Ɫ_d = 1$.

Proof. Using mathematical induction to to prove the above Theorem 1. In Definition 4 using operation (6) and (8), if $\acute{n} = 2$ we obtain the following result:

$$PyFZNOW(\alpha_{\sigma_1},\alpha_{\sigma 2}) = Ⱡ_1\alpha_{\sigma_1} \oplus Ⱡ_2\alpha_{\sigma 2}$$
$$= \left\{\begin{array}{c} \left(\begin{array}{c}\sqrt{1-\left(1-\mu^2_{S_{\alpha_{\sigma(1)}}}\right)^{Ⱡ_1}} + \sqrt{1-\left(1-\mu^2_{S_{\alpha_{\sigma(2)}}}\right)^{Ⱡ_2}} - \\ \sqrt{1-\left(1-\mu^2_{S_{\alpha_{\sigma(1)}}}\right)^{Ⱡ_1}}\sqrt{1-\left(1-\mu^2_{S_{\alpha_{\sigma(2)}}}\right)^{Ⱡ_2}}\end{array}\right), \\ \left(\begin{array}{c}\sqrt{1-\left(1-\mu^2_{T_{\alpha_{\sigma(1)}}}\right)^{Ⱡ_1}} + \sqrt{1-\left(1-\mu^2_{T_{\alpha_{\sigma(2)}}}\right)^{Ⱡ_2}} - \\ \sqrt{1-\left(1-\mu^2_{T_{\alpha_{\sigma(1)}}}\right)^{Ⱡ_1}}\sqrt{1-\left(1-\mu^2_{T_{\alpha_{\sigma(2)}}}\right)^{Ⱡ_2}}\end{array}\right), \\ \left(\nu^{Ⱡ_1}_{S_{\alpha_{\sigma(1)}}}\nu^{Ⱡ_2}_{S_{\alpha_{\sigma(2)}}}, \nu^{Ⱡ_1}_{T_{\alpha_{\sigma(1)}}}\nu^{Ⱡ_2}_{T_{\alpha_{\sigma(2)}}}\right)\end{array}\right\}$$
$$= \left\{\begin{array}{c}\left(\sqrt{1-\Pi^2_{d=1}\left(1-\mu^2_{S_{\alpha_{\sigma(d)}}}\right)^{Ⱡ_d}}, \sqrt{1-\Pi^2_{d=1}\left(1-\mu^2_{T_{\alpha_{\sigma(d)}}}\right)^{Ⱡ_d}}\right), \\ \left(\Pi^2_{d=1}\nu^{Ⱡ_d}_{S_{\alpha_{\sigma(d)}}}, \Pi^2_{d=1}\nu^{Ⱡ_d}_{T_{\alpha_{\sigma(d)}}}\right)\end{array}\right\}. \quad (5.3)$$

(2). If $ń = m$, using Eq. 9.6 we get the following form:

$$PyFZNOW(\alpha_1,\alpha_2,\dots\alpha_m) = \sum_{d=1}^{m} Ⱡ_d\alpha_{\sigma d}$$
$$= \left\{\begin{array}{c}\left(\sqrt{1-\Pi^m_{d=1}\left(1-\mu^2_{S_{\alpha_{\sigma(d)}}}\right)^{Ⱡ_d}}, \sqrt{1-\Pi^m_{d=1}\left(1-\mu^2_{T_{\alpha_{\sigma(d)}}}\right)^{Ⱡ_d}}\right), \\ \left(\Pi^ń_{d=1}\nu^{Ⱡ_d}_{S_{\alpha_{\sigma(d)}}}, \Pi^ń_{d=1}\nu^{Ⱡ_d}_{T_{\alpha_{\sigma(d)}}}\right)\end{array}\right\}. \quad (5.4)$$

(3). If $ń = m+1$, using the operation (6) and (8) in Definition 2, and also use Eqs 9.7, 9.8 we have the following result:

$$PyFZNOW(\alpha_1,\alpha_2,\dots,\alpha_m,\alpha_{m+1})$$
$$= \sum_{d=1}^{m} Ⱡ_d\alpha_{\sigma d} \oplus Ⱡ_{m+1}\alpha_{\sigma d+m}$$
$$= \left\{\begin{array}{c}\left(\sqrt{1-\Pi^m_{d=1}\left(1-\mu^2_{S_{\alpha_{\sigma(d)}}}\right)^{Ⱡ_d}}, \sqrt{1-\Pi^m_{d=1}\left(1-\mu^2_{T_{\alpha_{\sigma(d)}}}\right)^{Ⱡ_d}}\right), \\ \left(\Pi^m_{d=1}\nu^{Ⱡ_d}_{S_{\alpha_{\sigma(d)}}}, \Pi^m_{d=1}\nu^{Ⱡ_d}_{T_{\alpha_{\sigma(d)}}}\right)\end{array}\right\} \oplus$$
$$Ⱡ_{m+1}\alpha_{\sigma d+m}$$
$$= \left\{\begin{array}{c}\left(\sqrt{1-\Pi^{m+1}_{d=1}\left(1-\mu^2_{S_{\alpha_{\sigma(d)}}}\right)^{Ⱡ_d}}, \sqrt{1-\Pi^{m+1}_{d=1}\left(1-\mu^2_{T_{\alpha_{\sigma(d)}}}\right)^{Ⱡ_d}}\right), \\ \left(\Pi^{m+1}_{d=1}\nu^{Ⱡ_d}_{S_{\alpha_{\sigma(d)}}}, \Pi^{m+1}_{d=1}\nu^{Ⱡ_d}_{T_{\alpha_{\sigma(d)}}}\right)\end{array}\right\}.$$

It is true for all $ń$, hence the verification is completed.

Theorem 6. *The PyFZNOW operator implies the following properties:*

(1) Idempotency: Let $\alpha_d = \left\{(\mu_{S_{\alpha_d}}, \mu_{T_{\alpha_d}})(\nu_{S_{\alpha_d}}, \nu_{T_{\alpha_d}})\right\}$ $(d = 1,2,\dots ń)$ *PyFZNs. If* $\alpha_d = \alpha\,(d = 1,2,\dots ń)$, *there is* $PyFZNOW\;(\alpha_1,\alpha_2,\dots\alpha_ń) = \alpha$

(2) Boundedness: Let $\alpha_d = \left\{(\mu_{S_{\alpha_d}}, \mu_{T_{\alpha_d}})(\nu_{S_{\alpha_d}}, \nu_{T_{\alpha_d}})\right\}$ $(d = 1,2,\dots ń)$ *be a collection of* P_yFZN *and*

$$\alpha_{\min} = \langle \min(\mu_{\alpha_d}(S,T)),\ \max(\nu_{\alpha_d}(S,T))\rangle$$
$$= \left\{\left(\min_d\left(\mu_{s_{\alpha_d}}\right), \min_d\left(\mu_{T_{\alpha_d}}\right)\right), \left(\max\left(\nu_{s_{\alpha_d}}\right),\ \max\left(\nu_{T_{\alpha_d}}\right)\right)\right\}$$
$$\alpha_{\max} = \left\{\max\left(\mu_{\alpha_d}(S,T)\right),\ \min \nu_{\alpha_d}(S,T)\right\}$$
$$= \left\{\max_d\left(\mu_{s_{\alpha_d}}\right), \max_d\left(\mu_{T_{\alpha_d}}\right), \left(\min_d\left(\nu_{s_{\alpha_d}}\right), \min_d\left(\nu_{Td}\right)\right)\right\}$$

Then,

can keep. $\alpha_{\min} \le PyFZNOW(\alpha_1,\alpha_2,\dots,\alpha_ń) \le \alpha_{\max}$

(3) Monotonoicity: set $\alpha_d = \left\{(\mu_{S_d}, \mu_{T_d}), (\nu_{S_d}, \nu_{T_d})\right\}$ $(d = 1,2,\dots ń)$ *and* $\alpha^*_d = \left\{(\mu^*_{S_d}, \mu^*_{T_d}), (\nu^*_{S_d}, \nu^*_{T_d})\right\}$ $(d = 1,2,\dots ń)$ *be a collection of* $PyFZN_S$*. When* $\alpha_d \le \alpha^*_d$, *there is* $PyFZNOW(\alpha_1,\alpha_2,\dots\alpha_ń) \le PyFZNOW(\alpha^*_1,\alpha^*_2,\dots\alpha^*_ń)$.

Proof. (1) If $\alpha_d = \alpha\,(d = 1,2,\dots ń)$, then the result of Eq. 9.2 is given by

$$PyFZNOW(\alpha_1,\alpha_2,\dots,\alpha_ń) = \sum_{d=1}^{ń} Ⱡ_d\alpha_d$$
$$= \left\{\begin{array}{c}\left(\begin{array}{c}\sqrt{1-\Pi^ń_{d=1}\left(1-\mu^2_{S_{\alpha_d}}\right)^{Ⱡ_d}}, \\ \sqrt{1-\Pi^ń_{d=1}\left(1-\mu^2_{T_{\alpha_d}}\right)^{Ⱡ_d}}\end{array}\right), \\ \left(\Pi^ń_{d=1}\nu^{Ⱡ_d}_{S_{\alpha_d}}, \Pi^ń_{d=1}\nu^{Ⱡ_d}_{T_{\alpha_d}}\right)\end{array}\right\},$$
$$= \left\{\begin{array}{c}\left(\begin{array}{c}\sqrt{1-\left(1-\mu^2_{S_\alpha}\right)^{\Pi^ń_{d=1}Ⱡ_d}}, \\ \sqrt{1-\left(1-\mu^2_{T_\alpha}\right)^{\Pi^ń_{d=1}Ⱡ_d}}\end{array}\right), \\ \left((\nu_{S_\alpha})^{\Pi^ń_{d=1}Ⱡ_d}, (\nu_{T_\alpha})^{\Pi^ń_{d=1}Ⱡ_d}\right)\end{array}\right\},$$
$$= \left\{\left(\sqrt{1-\left(1-\mu^2_{S_\alpha}\right)}, \sqrt{1-\left(1-\mu^2_{T_\alpha}\right)}\right), (\nu_{S_\alpha}, \nu_{T_\alpha})\right\},$$
$$= \left\{(\mu_{S_\alpha}, \mu_{T_\alpha}), (\nu_{S_\alpha}, \nu_{T_\alpha})\right\} = \alpha.$$

(2) Since $\alpha_{\min}$ and $\alpha_{\max}$ are given by the minimum *PyFZN* and the maximum *PyFV*, then the inequality $\alpha_{\min} \le \alpha \le \alpha_{\max}$ exists. Thus, there is $\sum^ń_{d=1}Ⱡ_d\alpha_{\min} \le \sum^ń_{d=1}Ⱡ_d\alpha_d \le \sum^ń_{d=1}Ⱡ_d\alpha_{\max}$. Based on the above property (1) $\alpha_{\min} \le \sum^ń_{d=1}Ⱡ_d\alpha_d \le \alpha_{\max}$ can exists, i,e., there is $\alpha_{\min} \le PyFZNOW(\alpha_1,\alpha_{2,\dots}\alpha_ń) \le \alpha_{\max}$.

(3) $Gz_d \le G^*z_d$, there is $\sum^ń_{d=1}Ⱡ_d\alpha_d \le \sum^ń_{d=1}Ⱡ_d\alpha^*_d$, *d.e.*, $PyFZNOW(\alpha_1,\alpha_{2,\dots}\alpha_ń) \le PyFZNOW\;(\alpha^*_1,\alpha^*_2,\dots,\alpha^*_ń)$. All the properties of given theorem is complete.

5.2 PyFZNOWG operator

Using the operation (7) and (9) in Definition 4, we give the *PyFZNOWG* operator of *PyFZNs*.

Definition 10. *Let* $\alpha_d = \left\{(\mu_{S_{\alpha_d}}, \mu_{T_{\alpha_d}})(\nu_{S_{\alpha_d}}, \nu_{T_{\alpha_d}})\right\}$ $(d = 1,2,\dots ń)$ *be a collection of PyFZNs and* $PyFZNOWG\colon \Omega^ń \to \Omega$. *Then the PyFZNOWG operator is defined as:*

$$PyFZNOWG(\alpha_1,\alpha_{2,\dots}\alpha_ń) = \Pi^ń_{d=1}\alpha^{Ⱡ_d}_{\sigma d}$$

where $Ⱡ_d\,(d = 1,2,\dots ń)$ *with* $0 \le \mathrm{L} = {}_d \le 1$ *and* $\sum^ń_{d=1}Ⱡ_d = 1$.

Theorem 7. *Let* $\alpha_d = \left\{(\mu_{S_{\alpha_d}}, \mu_{T_{\alpha_d}})(\nu_{S_{\alpha_d}}, \nu_{T_{\alpha_d}})\right\}$ $(d = 1,2,\dots ń)$ *be a collection of PyFZNs. Then, the collected value of the PyFZNOWG operator is a PyFZN, which is obtained by the following formula:*

$$PyFZNOWG(\alpha_{\sigma(1)},\alpha_{\sigma(2)},\dots,\alpha_{\sigma(ń)}) = \Pi^ń_{d=1}\alpha^{Ⱡ_d}_{\sigma d}$$
$$= \left\{\begin{array}{c}\left(\Pi^ń_{d=1}\mu^{Ⱡ_d}_{S\alpha_{\sigma(d)}}, \Pi^ń_{d=1}\mu^{Ⱡ_d}_{T_{\alpha_{\sigma(d)}}}\right), \\ \left(\sqrt{1-\Pi^ń_{d=1}\left(1-\nu^2_{S_{\alpha_{\sigma(d)}}}\right)^{Ⱡ_d}}, \sqrt{1-\Pi^ń_{d=1}\left(1-\nu^2_{T\alpha_{\sigma(d)}}\right)^{Ⱡ_d}}\right)\end{array}\right\}$$

where $Ⱡ_d\,(d = 1,2,\dots ń)$ *with* $0 \le Ⱡ_d \le 1$ *and* $\sum^ń_{d=1}Ⱡ_d = 1$. *By the similar verification process of* Theorem 5.

Theorem 8. *The PyFZNOWoperator implies the following properties:*

(1) Idempotency: Let $\alpha_d = \left\{(\mu_{S_{\alpha_d}}, \mu_{T_{\alpha_d}})(\nu_{S_{\alpha_d}}, \nu_{T_{\alpha_d}})\right\}$ $(d = 1,2,\dots ń)$ $PyFZNs$. If $\alpha_d = \alpha\,(d = 1,2,\dots ń)$, there is $PyFZNOW(\alpha_1,\alpha_2,\dots\alpha_ń) = \alpha$

(2) Boundedness: Let $\alpha_d = \left\{(\mu_{S_{\alpha_d}}, \mu_{T_{\alpha_d}})(\nu_{S_{\alpha_d}}, \nu_{T_{\alpha_d}})\right\}(d = 1, 2, \ldots \acute{n})$ be a collection of *PyFZNs* and let

$$\begin{aligned}\alpha_{\min} &= \langle \min\left(\mu_{\alpha_d}(S,T)\right),\ \max\left(\nu_{\alpha_d}(S,T)\right)\rangle \\ &= \left\{\left(\min_d\left(\mu_{s_{\alpha_d}}\right), \min_d\left(\mu_{T_{\alpha_d}}\right)\right), \left(\max_d\left(\nu_{s_{\alpha_d}}\right), \max_d\left(\nu_{T_{\alpha_d}}\right)\right)\right\} \\ \alpha_{\max} &= \left\{\max\left(\mu_{\alpha_d}(S,T)\right),\ \min \nu_{\alpha_d}(S,T)\right\} \\ &= \left\{\max_d\left(\mu_{s_{\alpha_d}}\right), \max_d\left(\mu_{T_{\alpha_d}}\right),\ \left(\min_d\left(\nu_{s_{\alpha_d}}\right), \min_d\left(\nu_{Td}\right)\right)\right\}\end{aligned}$$

Then,

$$\alpha_{\min} \le PyFZNOWG(\alpha_1, \alpha_2, \ldots, \alpha_{\acute{n}}) \le \alpha_{\max}$$

can keep

(3) Monotonoicity: set $\alpha_d = \left\{(\mu_{S_d}, \mu_{T_d}), (\nu_{S_d}, \nu_{T_d})\right\}(d = 1, 2, \ldots \acute{n})$ and $\alpha_d^* = \left\{(\mu_{S_d}^*, \mu_{T_d}^*), (\nu_{S_d}^*, \nu_{T_d}^*)\right\}(d = 1, 2, \ldots \acute{n})$ be a collection of $PyFZ\acute{n}_S$.When $\alpha_d \le \alpha_d^*$, there is

$$PyFZNOW(\alpha_1, \alpha_2, \ldots \alpha_{\acute{n}}) \le PyFZNOW(\alpha_1^*, \alpha_2^*, \ldots \alpha_{\acute{n}}^*).$$

Can also be confirmed Theorem 2 using the aforementioned properties corresponding to the *PyFZNWG* operator, which is not repeated here.

6 MDM approach using the *PyFZNW* and *PyFZNWG* operator and the score funtion

In order to handle MDM difficulties, this part develops an MDM methodology using assessment data for both Pythagorean values and Pythagorean reliability measures. This approach relates to PyFZNW and PyFZNW Goperators and the score function. Since a set of n criteria $\acute{n} = \{l_1, l_2, \ldots l_m,\}$ are used to evaluate a set of m options $Q = \{Q_1, Q_2, \ldots Q_m\}$ in an MDM issue. The weight L = $_d$, are define as the weight vector Ⱡ= (Ⱡ$_1$,Ⱡ$_2$, . . .Ⱡ$_{\acute{n}}$), which takes into account the significance of each criterion l_d $(d = 1, 2, \ldots \acute{n})$. Decision-makers are asked to evaluate each criterion's applicability for each alternative $Q_j = \{1, 2, \ldots m\}$ using both membership and non-membership fuzzy values as well as related accuracy measures. $Gz_{jd} = \left\{(\mu_{jd}(S,T)), \nu_{jd}(S,T)\right\} = \left\{(\mu_{S_{jd}}, \mu_{T_{jd}}), (\nu_{S_{jd}}, \nu_{T_{jd}})\right\}$, where $\mu_{S_{jd}}, \mu_{T_{jd}} \in [0,1]$ and $\nu_{S_{jd}}, \nu_{T_{jd}} \in [0,1]$. Now the decision matrix of PyZN is determined as $Gz = (Gz_{jd})_{m \times \acute{n}}$. The decision process is defined in MDM problem as step1: Using Eqs 5.2, 6.2, the *PyZN* is defined by:

$$\begin{aligned}Gz_j &= PyFZNWGz_{j1}, Gz_{j2,\ldots} Gz_{j\acute{n}} = \sum_{d=1}^{\acute{n}} Ⱡ_d Gz_{jd} \\ &= \left\{\begin{matrix}\left(\sqrt{1 - \Pi_{d=1}^{\acute{n}}\left(1 - \mu_{S_{jd}}^2\right)^{Ⱡ_d}}, \sqrt{1 - \Pi_{d=1}^{\acute{n}}\left(1 - \mu_{T_{jd}}^2\right)^{Ⱡ_d}}\right), \\ \left(\Pi_{d=1}^{\acute{n}} \nu_{Sjd}^{Ⱡ_d}, \Pi_{d=1}^{\acute{n}} \nu_{Tjd}^{Ⱡ_d}\right)\end{matrix}\right\}\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}Gz_j &= PyFZNWG\left(Gz_{jd,} Gz_{jd,\ldots} Gz_{j\acute{n}}\right) = \Pi_{d=1}^{\acute{n}} G^{Ⱡ_d} z_{jd} \\ &= \left\{\begin{matrix}\left(\Pi_{d=1}^{\acute{n}} \mu_{Sjd}^{Ⱡ_d}, \Pi_{d=1}^{\acute{n}} \mu_{Tjd}^{Ⱡ_d}\right), \\ \left(\sqrt{1 - \Pi_{d=1}^{\acute{n}}\left(1 - \nu_{Sjd}^2\right)^{Ⱡ_d}}, \sqrt{1 - \Pi_{d=1}^{\acute{n}}\left(1 - \nu_{T_{jd}}^2\right)^{Ⱡ_d}}\right)\end{matrix}\right\}\end{aligned} \tag{6.2}$$

Step 2: Using Eq. 3.1, the score values of $J(G_{Zj})(j = 1, 2, \ldots m)$ are calculated.

Step 3: The best option among the rated options is chosen based on the score values.

Step 4: End.

6.1 An illustrative example and relative comparative analysis

To illustrate the relevance and efficacy of the developed MDM technique with PyFZN information, this section gives an example concerning the challenge of choosing business partners or suppliers. An example of a complex piece of machinery is an aircraft, which has intricate manufacturing processes and strict supplier criteria. Due to the complexity of airplanes, it is challenging for the major manufacturers to complete production on their own time. Therefore, other suppliers work with the primary manufacturers to finish the production of airplanes with a high degree of personalization in the aircraft's features. For instance, the structure, a component of an aircraft, is crucial to the manufacturing process. An essential component of the skeleton and aerodynamic form of the aircraft body, an aircraft structural part is available in a wide range of complex designs and a number of materials. The weight and strength requirements must be fully taken into account throughout the design and production of aircraft structural elements. The fabrication of structural components for airplanes is a challenging process with stringent standards (Tong and Zhu, 2020a).

Suppose a manufacturer needs to select a reliable supplier from among potential suppliers. Expert panel presents a set of five suppliers or alternatives $Q = \{Q_1, Q_2, \ldots Q_5\}$, which must meet the assessment standards of the criteria: l_1 is the product cost; l_2 is the product quality and l_3 is the delivery lead time.The three criteria's weight vector is written as (0.2, 0.5, 0.3) to denote their relative importance. Then, the PyZNs that are made up of their membership and non-membership fuzzy values and the measurements of associated reliabilities encourage the experts/decision-makers to evaluate the four suppliers/alternatives over the three criteria. As a result, the following PyZN decision matrix can be used to create all PyZNs:

$Gz = \quad (Gz_{jd})_{5\times3}$

$\{(0.6, 0.4), (0.5, 0.3)\}$	$\{(0.7, 0.1), (0.3, 0.5)\}$	$\{(0.4, 0.1), (0.8, 0.2)\}$
$\{(0.3, 0.1), (0.4, 0.5)\}$	$\{(0.4, 0.3), (0.6, 0.1)\}$	$\{(0.2, 0.7), (0.1, 0.3)\}$
$\{(0.2, 0.8), (0.4, 0.1)\}$	$\{(0.6, 0.1), (0.6, 0.2)\}$	$\{(0.3, 0.5), (0.6, 0.2)\}$
$\{(0.5, 0.6), (0.7, 0.2)\}$	$\{(0.3, 0.6), (0.4, 0.1)\}$	$\{(0.6, 0.3), (0.1, 0.4)\}$
$\{(0.3, 0.4), (0.1, 0.5)\}$	$\{(0.6, 0.1), (0.7, 0.2)\}$	$\{(0.5, 0.4), (0.6, 0.1)\}$

On the other hand we can apply PyFZNW

Step 1: To find PyZNs G_{Zj} $(j = 1, 2, 3, 4, 5)$ using (7.1) equation is defined as:

$$\begin{aligned}Gz_1 &= \{(0.266926, 0.09527), (0.445945, 0.34294)\}, \\ Gz_2 &= \{(0.104892, 0.037337), (0.323212, 0.191836)\}, \\ Gz_3 &= \{(0.078034, 0.271836), (0.553265, 0.17411)\}, \\ Gz_4 &= \{(0.194665, 0.252877), (0.295155, 0.17411)\}, \\ Gz_5 &= \{(0.125337, 0.114306), (0.452892, 0.195123)\}.\end{aligned}$$

Step 2: The score values $J(Gz_j)$ of PyFZNW for the alternatives $Q_j = \{1, 2, 3, 4, 5\}$ are given below:

$$J(Gz_1) = 0.436249, J(Gz_2) = 0.470956,$$
$$J(Gz_3) = 0.462442, J(Gz_4) = 0.498918,$$
$$J(Gz_5) = 0.462979.$$

Step 3: According to the score values $J(Gz_4) \geq J(Gz_2) \geq J(Gz_5) \geq J(Gz_3) \geq J(Gz_1)$, the five alternatives are ranked as $Q_4 \geq Q_2 \geq Q_5 \geq Q_3 \geq Q_1$.Hence the best supplier is Q_4

Now we can apply PyFZNWGA, in MDM problem can be solved using the invented MDM approach using the PyFZNWG operator, which is illustrated by the following decision-making process:

Step 1: The overall collected PyFZN Gz_j $(j = 1, 2, 3, 4, 5)$ are obtained as follow:

$$Gz_1 = \{(0.573849, 0.131951), (0.198604, 0.109134)\},$$
$$Gz_2 = \{(0.306735, 0.31052), (0.138146, 0.141117)\},$$
$$Gz_3 = \{(0.391217, 0.245646), (0.17265, 0.029779)\},$$
$$Gz_4 = \{(0409072., 0.487351), (0.247477, 0.055679)\}.$$
$$Gz_5 = \{(0.494528, 0.2), (0.042637, 0.142772)\}.$$

Step 2: The score values $J(Gz_j)$ of PyFZNWG for the alternatives $Q_j = \{1, 2, 3, 4, 5\}$ are given below:

$$J(Gz_1) = 0.527023, J(Gz_2) = 0.537876,$$
$$J(Gz_3) = 0.54548, J(Gz_4) = 0.592791,$$
$$J(Gz_5) = 0.546409.$$

Step 3: According to the score values $J(Gz_4) \geq J(Gz_5) \geq J(Gz_3) \geq J(Gz_2) \geq J(Gz_1)$, the five alternatives are ranked as $Q_4 \geq Q_5 \geq Q_3 \geq Q_2 \geq Q_1$.Hence the best supplier is Q_4. Here we first order the given matrix with the help of score funtion then the origional matrix becomes: $Gz = \quad (Gz_{jd})_{5\times 3}$

$\{(0.6, 0.4), (0.5, 0.3)\}$	$\{(0.7, 0.1), (0.3, 0.5)\}$	$\{(0.4, 0.1), (0.8, 0.2)\}$
$\{(0.2, 0.7), (0.1, 0.3)\}$	$\{(0.4, 0.3), (0.6, 0.1)\}$	$\{(0.3, 0.1), (0.4, 0.5)\}$
$\{(0.2, 0.8), (0.4, 0.1)\}$	$\{(0.3, 0.5), (0.6, 0.2)\}$	$\{(0.6, 0.1), (0.6, 0.2)\}$
$\{(0.5, 0.6), (0.7, 0.2)\}$	$\{(0.3, 0.6), (0.4, 0.1)\}$	$\{(0.6, 0.3), (0.1, 0.4)\}$
$\{(0.5, 0.4), (0.6, 0.1)\}$	$\{(0.3, 0.4), (0.1, 0.5)\}$	$\{(0.6, 0.1), (0.7, 0.2)\}$

On the other hand we can apply PyFZNOW

Step 1: To find PyZNs G_{Zj} $(j = 1, 2, 3, 4, 5)$ using Eq. 7.1 equation is defined as:

$$Gz_1 = \{(0.266926, 0.09527), (0.445945, 0.34294)\},$$
$$Gz_2 = \{(0.072905, 0.23382), (0.371273, 0.20189)\},$$
$$Gz_3 = \{(0.074235, 0.311844), (0.553265, 0.17411)\},$$
$$Gz_4 = \{(0.194665, 0.252877), (0.295155, 0.17411)\},$$
$$Gz_5 = \{(0.194665, 0.129066), (0.256543, 0.275292)\}.$$

Step 2: The score values $J(Gz_j)$ of *PyFZNOW* for the alternatives $Q_j = \{1, 2, 3, 4, 5\}$ are given below:

$$J(Gz_1) = 0.436249, J(Gz_2) = 0.471045,$$
$$J(Gz_3) = 0.46341, J(Gz_4) = 0.498918,$$
$$J(Gz_5) = 0.47725.$$

Step 3: According to the score values $J(Gz_4) \geq J(Gz_5) \geq J(Gz_2) \geq J(Gz_3) \geq J(Gz_1)$, the five alternatives are ranked as $Q_4 \geq Q_5 \geq Q_2 \geq Q_3 \geq Q_1$.Hence the best supplier is Q_4

Now we can apply PyFZNOWG, in MDM problem can be solved using the invented MDM approach using the PyFZNOWG operator, which is illustrated by the following decision-making process:

Step 1: The overall collected PyFZN Gz_j $(j = 1, 2, 3, 4, 5)$ are obtained as follow:

$$Gz_1 = \{(0.573849, 0.131951), (0.198604, 0.109134)\},$$
$$Gz_2 = \{(0.319428, 0.255611), (0.040122, 0.086437)\},$$
$$Gz_3 = \{(0.340574, 0.338925), (0.17265, 0.029779)\},$$
$$Gz_4 = \{(0409072., 0.487351), (0.247477, 0.055679)\}.$$
$$Gz_5 = \{(0.409072, 0.263902), (0.190216, 0.035776)\}.$$

Step 2: The score values $J(Gz_j)$ of PyFZNOWG for the alternatives $Q_j = \{1, 2, 3, 4, 5\}$are given below:

$$J(Gz_1) = 0.527023, J(Gz_2) = 0.539091,$$
$$J(Gz_3) = 0.555144, J(Gz_4) = 0.592791,$$
$$J(Gz_5) = 0.550575.$$

Step 3: According to the score values $J(Gz_4) \geq J(Gz_3) \geq J(Gz_5) \geq J(Gz_2) \geq J(Gz_1)$, the five alternatives are ranked as $Q_4 \geq Q_3 \geq Q_5 \geq Q_2 \geq Q_1$. Hence the best supplier is Q_4

According to the created *MDM* approach that makes use of the *PyFZNW*, *PyFZNOW*, *PyFZNWG* and *PyFZNOWG* operators as well as the score function, we can observe that the four types of ranking orders mentioned above for the five options and the best option are the same. As a result, the developed MDM strategy works.

7 The extended EDAS method based on novel pythagorean fuzzy Z-number

Evaluation Based on Distance from Average Solution (EDAS), a brand-new and powerful MCDM technique, was created. This method estimates the desirableness of an option based on how far from the average answer they are. In order to verify the efficacy of the Pythagorean fuzzy z-number weighted geometric AOs, a novel extended EDAS approach is developed here to manage the complex uncertain data in real-life DS situations. Assume there are a number of “alternatives” $\{\varnothing_1, \varnothing_2, \ldots, \varnothing_l\}$, and a satisfactory rating $\{R_1, R_2, \ldots, R_m\}$ for each. Then, $\text{Ⱡ}_d = (\text{Ⱡ}_1, \text{Ⱡ}_2, \ldots \text{Ⱡ}_m)^T$ specifies the usefulness of various characteristics R_d $(d = 1, 2, \ldots, m)$, such that L $= {}_d > 0$ and $\Sigma_{d=1}^{m} \text{Ⱡ}_d = 1$. Let $Gz_{jd} = \{U_{Sjd}U_{Tjd}\ V_{Sjd}V_{Tjd}\}$ where $0 \leq (U_{Sjd})^2 + (U_{Tjd})^2 \leq 1$ be the permissible rating for each attribute for each option.

Step 1: Choose a series of attributes that can be applied to assess the issue: Through a review of the literature, prospective assessment characteristics are gathered, and an expert DM committee is formed to screen the characteristics in order to create a respectable set of evaluation R_d $(d = 1, 2, \ldots, m)$.

$$Gz_{jd} = \begin{pmatrix} (U_{S11}U_{T11} & V_{S11}V_{T11}) & (U_{S12}U_{T12} & V_{S12}V_{T12}) & \ldots & (U_{S1m}U_{T1m} & V_{S1m}V_{T1m}) \\ (U_{S21}U_{T21} & V_{S21}V_{T21}) & (U_{S22}U_{T22} & V_{S22}V_{T22}) & \ldots & (U_{S2m}U_{T2m} & V_{S2m}V_{T2m}) \\ \ldots & & \ldots & & \ldots & \ldots & \\ (U_{Sr1}U_{Tr1} & V_{Sr1}V_{Tr1}) & (U_{Sr2}U_{Tr2} & V_{Sr2}V_{Tr2}) & \ldots & (U_{Srm}U_{Trm} & V_{Srm}V_{Trm}) \end{pmatrix}$$

Step 2: The normalized decision matrix is created using normalization as follows:

$$Gz_{jd} = \begin{matrix} \left(U_{Sjd}U_{Tjd} \quad V_{Sjd}V_{Tjd}\right), \text{if C1} \\ \left(V_{Sjd}V_{Tjd} \quad U_{Sjd}U_{Tjd}\right), \text{if c11} \end{matrix}$$

if R_d $(d = 1, 2, \ldots, m)$ is a benefit criterion, the statement use CI, if R_d $(d = 1, 2, \ldots, m)$ is a cost criterion, the statement CII use.

Step 3: Aggregated Data: The skilled uncertain data of required situations are aggregated using established *PyFZNWG* operators.

$$PyFZNW\left(Gz_1, Gz_2, \ldots Gz_{\acute{n}}\right) = \Pi_{d=1}^{\acute{n}} G^{\text{Ⱡ}_d} z_d = \left\{ \begin{array}{c} \left(\Pi_{d=1}^{\acute{n}} \mu_{S_d}^{\text{Ⱡ}_d}, \Pi_{d=1}^{\acute{n}} \mu_{T_d}^{\text{Ⱡ}_d}\right), \\ \left(\sqrt{1 - \Pi_{d=1}^{\acute{n}} \left(1 - \nu_{S_d}^2\right)^{\text{Ⱡ}_d}}, \sqrt{1 - \Pi_{d=1}^{\acute{n}} \left(1 - \nu_{T_d}^2\right)^{\text{Ⱡ}_d}}\right) \end{array} \right\}$$

Step 4: Verify the average solution ($A_V S$), which is based on all the criteria given. $A_V S = [A_V S_d]_{1\times m} = \left\{\frac{\Sigma_{j=1}^{\acute{n}} Gz_{jd}}{\acute{n}}\right\}_{1\times m}$ Using Definition 3, we obtain $A_V S = [A_V S_d]_{1\times m} = \left\{\frac{\Sigma_{j=1}^{\acute{n}} Gz_{jd}}{\acute{n}}\right\}_{1\times m}$

$$= \left\{ \begin{array}{c} \left(\Sigma_{j=1}^{\acute{n}} \sqrt{1 - \left(1 - \mu_{S_j}^2\right)^{\text{Ⱡ}}}, \sqrt{1 - \left(1 - \mu_{T_j}^2\right)^{\text{Ⱡ}}}\right), \\ \left(\Sigma_{j=1}^{\acute{n}} \nu_{S_j}^{\text{Ⱡ}}, \nu_{T_j}^{\text{Ⱡ}}\right) \end{array} \right\}$$

Step 5: Using the $A_V S$ values, the positive distance from average (PDA_v) and the negative distance from average (NDA_v) can be calculated:

$$PDA_v = \frac{\max 0, \left(Gz_{jd} - A_V S\right)}{A_V S}$$

$$NDA_v = \frac{\max 0, \left(A_V S - Gz_{jd}\right)}{A_V S}$$

To compute the PDA and NDA, we can use the score function of $PyFZNs$ mentioned in Definition 3 as follows:

$$PDA_v = \frac{\max 0, \left(J\left(Gz_{jd}\right) - J\left(A_V S\right)\right)}{J\left(A_V S\right)}$$

$$NDA_v = \frac{\max 0, \left(J\left(A_V S\right) - J\left(Gz_{jd}\right)\right)}{J\left(A_V S\right)}$$

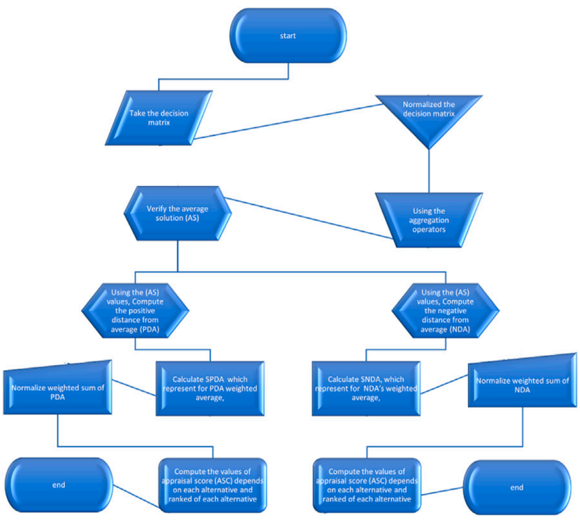
where W shows the score value.

Step 6: Calculate $SPDA$ and $SNDA$, which represent for PDA and $NDA's$ weighted average, respectively: $SPDA = \sum_{d=1}^{m} \text{Ⱡ}_d PDA_d$, $SNDA = \sum_{d=1}^{m} \text{Ⱡ}_d NDA_d$ $\text{Ⱡ}_d \in [0, 1]$ and $\Sigma_{d=1}^{m} \text{Ⱡ}_d = 1$.

Step 7: Normalize weighted sum of PDA and NDA is defined as repectively:

$$NSPDA = \frac{SPDA}{max\left(SPDA\right)}$$

$$NSPDA = \frac{SNDA}{max\left(SNDA\right)}$$



Step 8: Compute the values of appraisal score (ASC) depends on each alternative's as

$$ASC = \frac{1}{2}\left(\left(NSPDA + 1 - NSNDA\right)\right.$$

Step 9: Depending on the ASC calculations, alternatives are sorted in decreasing order, and the higher the ASC number, the better options will be chosen.

7.1 An illustrative example

Step 1: Consider the decision matrix as discussed in previous example.

$Gz = \quad (Gz_{jd})_{5\times 3}$

{(0.6, 0.4), (0.5, 0.3)}	{(0.7, 0.1), (0.3, 0.5)}	{(0.4, 0.1), (0.8, 0.2)}
{(0.3, 0.1), (0.4, 0.5)}	{(0.4, 0.3), (0.6, 0.1)}	{(0.2, 0.7), (0.1, 0.3)}
{(0.2, 0.8), (0.4, 0.1)}	{(0.6, 0.1), (0.6, 0.2)}	{(0.3, 0.5), (0.6, 0.2)}
{(0.5, 0.6), (0.7, 0.2)}	{(0.3, 0.6), (0.4, 0.1)}	{(0.6, 0.3), (0.1, 0.4)}
{(0.3, 0.4), (0.1, 0.5)}	{(0.6, 0.1), (0.7, 0.2)}	{(0.5, 0.4), (0.6, 0.1)}

Step 2: The normalized decision matrix is created using normalization as follows:

$$Gz_{jd} = \begin{array}{ll} \left(U_{Sjd}U_{Tjd} \quad V_{Sjd}V_{Tjd}\right), & \text{if C1} \\ \left(V_{Sjd}V_{Tjd} \quad U_{Sjd}U_{Tjd}\right), & \text{if c11} \end{array}$$

if R_d ($d = 1, 2, \ldots, m$) is a benefit criterion, the statement use CI, if R_d ($d = 1, 2, \ldots, m$) is a cost criterion, the statement CII use.Here the given system is already normalized

Step 3: Now we can apply PyFZNWGA, in MDM problem can be solved using the invented MDM approach using the PyFZNWG operator, which is illustrated by the following decision-making process.The overall collected PyFZN Gz_j (j = 1, 2, 3, 4, 5) are obtained as follow:

$$\begin{aligned} Gz_1 &= \{(0.552113, 0.159033), (0.268361, 0.141392)\}, \\ Gz_2 &= \{(0.288809, 0.276248), (0.174972, 0.20492)\}, \\ Gz_3 &= \{(0.330559, 0.342362), (0.224104, 0.041134)\}, \\ Gz_4 &= \{(0.4485, 0.476574), (0.315567, 0.193374)\}. \\ Gz_5 &= (0.4485, 0.252332), 0.055274, 0.132527` \end{aligned} \tag{7.1}$$

Step 4: The score values J (Gz_j) of $PyFZNWG$ for the corresponding Q_j = {1, 2, 3, 4, 5} are given below:

$$\begin{aligned} J\left(Gz_1\right) &= 0.52493, J\left(Gz_2\right) = 0.521964, \\ J\left(Gz_3\right) &= 0.551976, J\left(Gz_4\right) = 0.593963, \\ J\left(Gz_5\right) &= 0.551241. \end{aligned}$$

And verify the average solution ($A_V S$) as:

$$\begin{aligned} A_V\left(Gz_1\right) &= 0.413696, A_V\left(Gz_2\right) = 0.30131, \\ A_V\left(Gz_3\right) &= 0.207656, A_V\left(Gz_4\right) = 0.132527. \end{aligned}$$

Score funtion of average solution ($A_V S$) we have:

$$J\left[A_V\left(Gz\right)\right] = 0.548565$$

Step 5: Using the $A_V S$ values, the positive distance from average (PDA_v) and the negative distance from average (NDA_v) can be be calculated:

Positive distance from average

0.334586	0	0.292333	0.066889
0	0	0	0.546247
0	0.136246	0.079207	0
0.08413	0.581673	0.519664	0
0.08413	0	0	0.459126

Negative distance from average

0	0.472196	0	0
0.301882	0.083177	0.157394	0
0.200962	0	0	0.68962
0	0	0	0.382649
0	0.162551	0.733819	0

To compute the *PDA* and *NDA*, we can use the score function of *PyFZNs* mentioned in Definition 3 as follows:

Find *PDA* using average of *PyFZNWG*

0	0	0.00621865	0.082756939	0.004878476

Find *NDA* using average of *PyFZNWG*

0.043085	0.48492	0	0	0

Step 6: Calculate *SPDA* and *SNDA*, which represent for *PDA* and *NDA′s* weighted average, and attributes weighting vector L = = (0.333, 0.333, 0.333, 0.333, 0.333), we can obtain the results as: respectively:

Find *SPDA* using weight vector

0	0	0.00621865	0.082756939	0.004878476

Find *NPDA* using weight vector

0	0	0.00621865	0.082756939	0.004878476

Step 7: Normalize weighted sum of *PDA* and *NDA* is defined as repectively:

NSPDA

0	0	0.07514371	1.000002199	0.05894958

NSNDA

0.888491579	0.999996804	0	0	0

Step 8: Compute the values of appraisal score (*ASC*) depends on each alternative's as:

appraisal score (ASC)

0.05575421	0.0000015	0.537571856	1.000001099	0.529474792

Step 9: Ranking of *EDAS* method based on Pythagorean fuzzy Z-number weighted geometric aggregation operator

Ranking of EDAS method based on PyFZNWG

0.05575421	Q_1
0.0000015	Q_2
0.537571856	Q_3
1.000001099	Q_4
0.529474792	Q_5

$Q_4 \geq Q_3 \geq Q_5 \geq Q_1 \geq Q_2$.Hence the best supplier is Q_4.

An analysis has been performed using the suggested operators, and the results are summarised in the table below in order to study the trend of variation in score and ranking of all options with the change in the aggregation procedure. From this tabular number, we see that the best choice stays the same, indicating that the outcomes are unbiased and cannot be affected by decision-makers preferences on aggregating processes. So the ranking results are trustworthy. While the comparison graph of all methods which are used in this article is represented in Figure 1. Therefore, our suggested technique is more flexible since the decision-maker(s) may pick operators based on their preferences and actual scenarios.

Comparison of all method

PyFZNW	$Q_4 \geq Q_2 \geq Q_5 \geq Q_3 \geq Q_1$	best Q_4
PyFZNWG	$Q_4 \geq Q_5 \geq Q_3 \geq Q_2 \geq Q_1$	Q_4
PyFZNOW	$Q_4 \geq Q_5 \geq Q_2 \geq Q_3 \geq Q_1$	Q_4
PyFZNOWG	$Q_4 \geq Q_3 \geq Q_5 \geq Q_2 \geq Q_1$	Q_4
EDAS	$Q_4 \geq Q_3 \geq Q_5 \geq Q_1 \geq Q_2$	Q_4

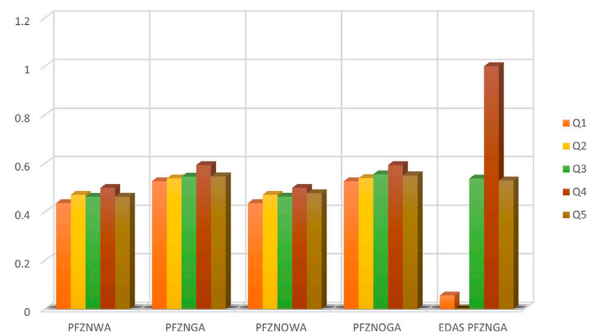


FIGURE 1
Ranking results.

TABLE 1 Comparison with existing studies.

Methods	$\mu(S)$	$\nu(S)$	Reliability	Range
Zadeh (1965)	yes	no	No	$0 \le \mu(S) \le 1$
Zadeh (2011)	yes	no	yes	$0 \le \mu(S, T) \le 1$
Atanassov and Atanassov (1999)	yes	yes	no	$0 \le \mu(S) + \nu(S) \le 1$
Yager (2013)	yes	yes	no	$0 \le \mu(S)^2 + \nu(S)^2 \le 1$
Proposed approach	yes	yes	yes	$0 \le \mu(S,T)^2 + \nu(S,T)^2 \le 1$

8 Comparasion analysis

Using the idea of constraints in combination with multiattribute and multiobjective decision making approaches, we devised a way to address challenging real-world problems. To demonstrate the usefulness of the proposed approach in contrast to the existing ones for multi-criteria decision-making, a comparative study was conducted using various structures developed by different researchers. Table 1 provides an analytical comparison of the CPHFNS technique to the existing approaches. When such data as CPHFNSS is supplied to a decision maker, none of the existing works can appropriately address it. While the suggested technique is capable of handling existing approaches data. Thus, our proposed methods are superior and more reliable than those now used.

9 Conclusion

In this study, we analyzed the limitation of the current Pythagorean sets and presented a new set, PyFZN, that may handle the issue of hybrid information representation that occurs when Pythagorean values and their related reliability measures are stated simultaneously. The novel score function, basic operations, and the PyFZNW and PyFZNWG operators of PyFZNs also introduced to aggregate information and MDM modeling in the PyFZN context. We also defined their properties and theorems with proofs. To deal with multicriteria decision making issues we offered an algorithm. We view the decision matrix Gz assessment measures of corresponding reliabilities as a special case of the definite. In this paper an example of supplier selection problem at large scale showed how well the created MDM technique worked in the PyFZN environment more effectively than the existing approaches. However, the created MDM strategy offers a fresh approach to resolving MDM issues with PyFZNs. We also presented the EDAS technique on the perposed concept and a comparative analysis with the existing studies to check the efficacy and supermecy of this study.

In the future to enhance the quality of the information provided we plan to apply various aggregation operators such as Einstein, Dombi, average hybrid, etc., with TOPSIS and VIKOR technique and justify their application with the help of medical diagnostics, network signaling, and artificial intelligence.

Data availability statement

The raw data supporting the conclusion of this article will be made available by the authors, without undue reservation.

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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