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# Duality, Matroids, Qubits, Twistors, and Surreal Numbers

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We show that *via* the Grassmann-Plücker relations, the various apparent unrelated concepts, such as duality, matroids, qubits, twistors, and surreal numbers are, in fact, deeply connected. Moreover, we conjecture the possibility that these concepts may be considered as underlying mathematical structures in quantum gravity.

**Keywords: duality, matroids, twistors, surreal numbers, quantum gravity**

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It is a fact that the duality concept is everywhere in both mathematics and physics. Of course, since the list of examples of this fact is very large and since we are concern with quantum gravity let us just briefly mention, as examples in which the duality concept plays a fundamental role, matroid theory [1, 2] [see also [3–9] and references therein] and surreal numbers [10–12] in mathematics and string theory [13] and loop quantum gravity [14] in physics. The origin of matroid theory can be traced back to graph theory were according to the Kuratowski theorem a graph has a dual if does not contain the complete graphs $K_5$ and $K_{3,3}$ (see [15]). A matroid is a generalization of the graph concept in which every matroid has a dual. One may understand why matroid theory is a generalization of graph theory by associating with every graph $G$ a matroid $M(G)$. So one must have $M(K_5)$ and $M(K_{3,3})$, but according to matroid theory one must have the corresponding duals $M^*(K_5)$ and $M^*(K_{3,3})$ which turns out to be non-graphic. A surreal number $x = \{X_L, X_R\}$ is written in terms of the dual sets $X_L$ left set and $X_R$ the right set which satisfies two main axioms (see below). Surprisingly these dual numbers contains the structure of real numbers among other numerical structures. On the other hand it is known that the origin of $M$-theory [16] was inspired by trying to make sense of a number of dualities between string theory and $p$-branes. For instance, in eleven dimensions the 1-brane is dual to the 5-brane (see [16]). Finally, it is known that loop quantum gravity emerges from the discovery of the Ashtekar variables which in turn arises by the requirement of the canonical formalism applied to the self-dual Ricci curvature tensor [see [14] and references therein].

Of course, the duality concepts refereed above may be at first sight quite different for each example. So the first step it is to introduce a formal definition of the concept of duality. It turns out that at least in matroid theory one finds such a formal definition [17]. Let $\mathcal{M}$ denote the family of all matroids $M$ which corresponding to the ground set $E$. The matroid duality is a map $^* : \mathcal{M} \to \mathcal{M}$ satisfying the two main axioms:

$$\text{(a)}\ ^{**}M = M \qquad (\forall M \in \mathcal{M}).$$
$$\text{(b)}\ E(^*M) = E(M) \qquad (\forall M \in \mathcal{M}).$$

Inspired by this definition of duality in oriented matroid theory let us propose a general tensor definition of duality structure. Consider a family $\mathcal{A}$ of all completely antisymmetric tensors $A$ ($p$-forms), which correspond to space of dimension $d$, together with an operation $+$ which can be any well-defined tensorial sum operation. The pair $(\mathcal{A}, +)$ determines a dual structure through the map $^* : \mathcal{A} \to \mathcal{A}$ if satisfies the following axioms:

(I) $^{**}A = A \quad (\forall A \in \mathcal{A})$.
(II) $d(^*A) = d(A) \quad (\forall A \in \mathcal{A})$.

Note that (II) plays the role of (b) in matroid theory.

Assuming the particular case that $\mathcal{A}$ corresponds to family of zero-rank tensors one may add two additional axioms, namely

(III) There exist in $\mathcal{A}$ a self dual element $^*0 = 0$ such that $A + 0 = 0 + A = A, \ (\forall A \in \mathcal{A})$.
(IV) For $\forall A \in \mathcal{A}$ one has $A + {}^*A = {}^*A + A = 0$.

One can prove that the element 0 in (III) is unique as follows: Assume that $(\mathcal{A}, +)$ is a dual structure with two self-dual elements 0 and $0'$. Then $0 = 0 + 0' = 0'$. Moreover, according to the axiom (IV) the element $^*A$ can be considered as the inverse of $A$. In order to show that the inverse $^*A$ is unique one takes recourse of the axiom (I) instead of the associativity axiom in group theory. In fact, assume that an arbitrary element $A$ in $\mathcal{A}$ has two inverses $^*A$ and $^*B$. Thus, one has (i) $A + {}^*A = 0$ and (ii) $A + {}^*B = 0$. Applying the axioms (I) and (III) to (ii) one obtains $^*A + B = 0$ and therefore according (i) one gets $^*A + B = {}^*A + A$ which means that $B = A$. The two axioms (III) and (IV) are similar to the definition of a field in number theory. For these reasons one it is straightforward to verify that the integer $Z$ and the real numbers $R$ are in fact dual structures.

The main goal of the present work is to comment about the possibility that the various concepts such as oriented matroids, qubits, twistors, and surreal numbers are linked by the duality symmetry. Moreover we shall argue that such a dual concept may be considered as an underlying mathematical tool in quantum gravity.

It turns out that the completely antisymmetric $\varepsilon$-symbol becomes the underlying mathematical object in all these connections. Specifically, the $\varepsilon$-symbol can be defined as

$$\varepsilon^{a_1...a_d} \in \{-1, 0, 1\}. \tag{1}$$

Here, the indices $a_1, ..., a_d$ run from 1 to $d$. This is a $d$-rank density tensor which values are $+1$ or $-1$ depending on even or odd permutations of $\varepsilon^{12...d}$, respectively. Moreover, $\varepsilon^{a_1...a_d}$ takes the value 0 unless the values of $a_1...a_d$ are all different. Lowering and rising the indices with a Kronecker delta $\delta_{ab}$ one finds that

$$\varepsilon^{a_1...a_d}\varepsilon_{b_1...b_d} = \delta^{a_1...a_d}_{b_1...b_d}, \tag{2}$$

where $\delta^{a_1...a_d}_{b_1...b_d}$ is a generalized Kronecker delta. A contraction in (2) of the last $n$-indices of the type $a_i$ with the last $n$-indices of the type $b_i$ leads to

$$\varepsilon^{a_1...a_{k-1}a_k...a_d}\varepsilon_{b_1...b_{k-1}a_k...a_d} = n!\delta^{a_1...a_{k-1}}_{b_1...b_{k-1}}, \tag{3}$$

with $n = d - k + 1$. In particular one has

$$\varepsilon^{a_1...a_d}\varepsilon_{a_1...a_d} = d!. \tag{4}$$

Let $v^i_a$ be any $d \times n$ matrix over some field $F$, where the index $i$ takes values in the ordinal set $E = \{1, ..., n\}$. Consider the object

$$\Sigma^{i_1...i_d} = \varepsilon^{a_1...a_d} v^{i_1}_{a_1}...v^{i_d}_{a_d}. \tag{5}$$

Using the $\varepsilon$-symbol property

$$\varepsilon^{a_1...[a_d}\varepsilon^{b_1...b_d]} = 0, \tag{6}$$

it is not difficult to prove that $\Sigma^{i_1...i_d}$ satisfies the Grassmann-Plücker relations [see [18] and references therein], namely

$$\Sigma^{i_1...[i_d}\Sigma^{j_1...j_d]} = 0. \tag{7}$$

Here, the brackets in the indices of (6) and (7) mean completely antisymmetric.

Through (5) one can define the object

$$\Sigma = \frac{1}{d!}\Sigma^{i_1...i_d} e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_d}, \tag{8}$$

where $e_{i_1}, e_{i_2}, ..., e_{i_d}$ are 1-form bases associated with the $\binom{n}{d}$-dimensional real vector space of alternating $d$-forms on $R^n$. It turns out that (8) can also be written as

$$\Sigma = \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge ... \wedge \mathbf{v}_d, \tag{9}$$

for some $\mathbf{v}_1, \mathbf{v}_2, ..., .\mathbf{v}_d \in R^n$. This means that $\Sigma^{i_1...i_d}$ corresponds to an alternating decomposable $d$-form [19].

A realizable chirotope $\chi$ is defined as

$$\chi^{i_1...i_d} = sign\Sigma^{i_1...i_d}. \tag{10}$$

In order to define non-realizable chirotopes it is convenient to write the expression (7) in the alternative form

$$\sum_{k=1}^{d+1} s_k = 0, \tag{11}$$

where

$$s_k = (-1)^k \Sigma^{i_1...i_{d-1}j_k}\Sigma^{j_1...\hat{j}_k...j_{d+1}}. \tag{12}$$

Here, $j_{d+1} = i_d$ and $\hat{j}_k$ establish the notation for omitting this index. Thus, for a general definition, one defines a $d$-rank chirotope $\chi : E^d \to \{-1, 0, 1\}$ if there exist $r_1, ..., r_{d+1} \in R^+$ such that

$$\sum_{k=1}^{d+1} r_k s_k = 0, \tag{13}$$

with

$$s_k = (-1)^k \chi^{i_1...i_{d-1}j_k}\chi^{j_1...\hat{j}_k...j_{d+1}}, \tag{14}$$

and $k = 1, ..., d + 1$. It is evident that (11) is a particular case of (13). Therefore, there are chirotopes that may be non-realizable. Moreover, this definition of a chirotope admits a straightforward generalization to the complex structure setting. In this case the complex chirotopes are called phirotopes [20–22].

Given a chirotope (or phirotope) $\chi^{i_1...i_d}$ its dual is defined as

$$^{*}\chi_{i_{d+1}...i_p}=\varepsilon_{i_1...i_d i_{d+1}...i_p}\chi^{i_1...i_d}. \tag{15}$$

Here $D = d + p$ is the total dimension of the ground state $E$. Observe that due to the relations (3) one gets

$$^{**}\chi = \chi, \tag{16}$$

which means that $\chi$ satisfies the axiom (I). It turns out that (16) is true for a general completely antisymmetric object $F$ ($d$-form) when its dual is defined in terms of the $\varepsilon$-symbol. In fact, when $D$ is even one can write $D = d + d = 2d$ and in this case one can define the self-dual (antiself-dual) tensor as

$$^{\pm}F = F \pm^{*} F \tag{17}$$

One observe that $^{\pm}F$ satisfies

$$^{*\pm}F = \pm^{\pm}F \tag{18}$$

Thus, one sees that for $D$ even the $^{\pm}F$ tensor is another self-dual (antiself-dual) notion other than the 0 element in the axioms (III) and (IV).

Let us now explain how the Grassmann-Plücker relation (7) is connected with qubit theory [see [23] and references therein]. For this purpose consider the general complex state $\mid \psi >\in C^{2^N}$

$$\mid \psi >= \sum_{A_1,A_2,...,A_N=0}^{1} Q_{A_1A_2...A_N} \mid A_1A_2...A_N >, \tag{19}$$

where the states $\mid A_1A_2...A_N >=\mid A_1 > \otimes \mid A_2 > ...\otimes \mid A_N >$ correspond to a standard basis of the $N$-qubit. For a 3-qubit (19) becomes

$$\mid \psi >= \sum_{A_1,A_2,A_3=0}^{1} Q_{A_1A_2A_3} \mid A_1A_2A_3 >, \tag{20}$$

while for 4-qubit one has

$$\mid \psi >= \sum_{A_1,A_2,A_3,A_4=0}^{1} Q_{A_1A_2A_3A_4} \mid A_1A_2A_3A_4 > . \tag{21}$$

It turns out that, in a particular subclass of $N$-qubit entanglement, the Hilbert space can be broken into the form $C^{2^N} = C^L \otimes C^l$, with $L = 2^{N-n}$ and $l = 2^n$. Such a partition allows a geometric interpretation in terms of the complex Grassmannian variety $Gr(L, l)$ of $l$-planes in $C^L$ via the Plücker embedding. It turns out that in this scenario the complex 3-qubit, 4-qubit admit a geometric interpretation in terms of the complex Grassmannian varieties $Gr(4, 2)$, $Gr(8, 2)$, respectively [see [23] for details]. The idea is to associate the first $N - n$ and the last $n$ indices of $Q_{A_1A_2...A_N}$ with a $L \times l$ matrix $\omega^{i_1}_{a_1}$. This can be interpreted as the coordinates of the Grassmannian $Gr(L, l)$ of $l$-planes in $C^L$. Using the matrix $\omega^{i_1}_{p_1}$ one can define the Plücker coordinates

$$\mathcal{Q}^{i_1...i_d} = \varepsilon^{a_1...a_d}\omega^{i_1}_{a_1}...\omega^{i_d}_{a_d}, \tag{22}$$

which one recognizes as the complex version of the decomposable tensor $\Sigma^{i_1...i_d}$ defined in (5). Moreover, one verifies that under the transformation $\omega \to S\omega$ with $S \in GL(l, C)$ the Plücker coordinates transform as $\mathcal{Q}^{i_1...i_d} \to Det(S)\mathcal{Q}^{i_1...i_d}$ and of course $\pm^{i_1...i_d}$ satisfies the Grassmann-Plücker relations

$$\mathcal{Q}^{i_1...[i_d}\mathcal{Q}^{j_1...j_d]} = 0. \tag{23}$$

Now, consider the quantity $\sigma_\mu = (\sigma_0, \sigma_{\hat{i}})$, where the $\sigma_{\hat{i}}$ denotes Pauli matrices and $\sigma_0$ is the identity matrix. By using $\sigma_\mu$ the linear momentum in 4-dimensions $p^\mu$ can be written as

$$p^{A\dot{B}} = \sigma^{A\dot{B}}_\mu p^\mu. \tag{24}$$

This is the spinorial representation of $p^\mu$. An interesting aspect emerges if one sets $Det(p^{A\dot{B}}) = 0$, corresponding to a null momentum $p^\mu p_\mu = 0$. This means that

$$\frac{1}{2!}\varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}p^{A\dot{B}}p^{C\dot{D}} = 0. \tag{25}$$

A solution to this equation is given by $p^{A\dot{B}} = \xi^A\eta^{\dot{B}}$. Since $p^\mu$ is real vector one verifies that $p^{A\dot{B}} = \bar{p}^{\dot{B}A}$ and therefore

$$\xi^A\eta^{\dot{B}} = \bar{\xi}^{\dot{B}}\bar{\eta}^A. \tag{26}$$

One finds that this last expression means that $\eta^{\dot{B}} = a\bar{\xi}^{\dot{B}}$, where due to (26) one sees that $a$ is real number which can be normalized in the form $a = \pm$. So one has found that, in the case of null momentum, one can write $p^{A\dot{B}}$ in terms of a more fundamental complex quantity $\xi^A$, namely

$$p^{A\dot{B}} = \pm\xi^A\bar{\xi}^{\dot{B}}. \tag{27}$$

Similar analysis applies to the total angular momentum $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + S^{\mu\nu}$, where $S^{\mu\nu}$ is the internal angular momentum satisfying the Tulczyjew second class constraint [24];

$$S^{\mu\nu}p_\nu = 0. \tag{28}$$

Observe that due to (28) and since $p^\mu$ is a null vector one has $M^{\mu\nu}p_\nu = -(x^\nu p_\nu)p^\mu$ This means that $\delta^{\tau\mu\nu}_{\alpha\beta\gamma}p^\alpha M^{\beta\gamma}p_\nu = 0$. In turn this leads to $\varepsilon_{\sigma\alpha\beta\gamma}\varepsilon^{\sigma\tau\mu\nu}p^\alpha M^{\beta\gamma}p_\nu = 0$ or

$\varepsilon_{\sigma\alpha\beta\gamma}\varepsilon^{\sigma\tau\mu\nu}p^{\alpha}S^{\beta\gamma}p_{\nu} = 0$. Therefore, if one defines the 4-vector $S_{\sigma} = \frac{1}{2}\varepsilon_{\sigma\alpha\beta\gamma}p^{\alpha}S^{\beta\gamma}$ one obtains $\varepsilon^{\sigma\tau\mu\nu}S_{\sigma}p_{\nu} = 0$ and consequently one discovers that

$$S_{\mu} = sp_{\mu}, \tag{29}$$

for some non-vanishing constant $s$ which is identified with the helicity of the system. This means that the spin $S_{\mu}$ is parallel or anti-parallel to $p_{\mu}$ depending of the sign of $s$. So, determining $p^{A\dot{B}}$ in terms of $\xi^{A}$ via (27) is equivalent to determine $S^{A\dot{B}}$ in the form $S^{A\dot{B}} = s\xi^{A}\bar{\xi}^{\dot{B}}$. Thus, considering (28) one sees that the left relevant part of $M^{\mu\nu}$ is

$$L^{\mu\nu} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu}, \tag{30}$$

but again since $p^{\mu}$ is a null vector one has $L^{\mu\nu}p_{\nu} = -(x^{\nu}p_{\nu})p^{\mu}$ which means that out of the six true degrees of freedom of $L^{\mu\nu} = -L^{\nu\mu}$ three are already determined by $p^{\mu}$. Therefore, the corresponding spinor representation of $L^{\mu\nu}$, namely $L^{A\dot{B}C\dot{D}} = \sigma_{\mu}^{A\dot{B}}\sigma_{\nu}^{C\dot{D}}L^{\mu\nu}$, can be written as

$$L^{A\dot{B}C\dot{D}} = \mu^{AC}\epsilon^{\dot{B}\dot{D}} + \epsilon^{AC}\mu^{\dot{B}\dot{D}}. \tag{31}$$

Here, $\mu^{AC} = \mu^{CA}$ is a symmetric matrix and therefore has only three independent complex degrees of freedom. In order to reduce $\mu^{AC}$ to only three real components which of course are related to the true three degrees of freedom of $L^{\mu\nu}$ one further writes $\mu^{AC}$ in the form $\mu^{AC} = \xi^{A}\pi^{C} + \xi^{C}\pi^{A}$. If to the coordinates $\xi_{\dot{A}}$ one adds the the spinor $\pi^{A}$ one is lead to the twistor structure $\mathcal{P}^{\alpha} = (\pi^{A}, \xi_{\dot{A}})$ [25] [see [26] and references therein] which can be identified with a point in $C^4$. This analysis revel that in the case of a null system the eight coordinates $(x^{\mu}, p^{\nu})$ in $R^8$ may in principle be associated with the coordinates $(\pi^{A}, \xi_{\dot{A}})$ in the twistor complex space $C^4$.

Consider the 2-index twistor

$$P^{\alpha\beta} = \mathcal{P}_1^{\alpha}\mathcal{P}_2^{\beta} - \mathcal{P}_2^{\alpha}\mathcal{P}_1^{\beta}, \tag{32}$$

which can also be written as

$$\mathcal{P}^{\alpha\beta} = \varepsilon^{ab}\mathcal{P}_a^{\alpha}\mathcal{P}_b^{\beta}. \tag{33}$$

If one defines $p_1^{\mu} = x^{\mu}$ and $p_2^{\mu} = p^{\mu}$ one sees that $L^{\mu\nu}$ can be written as

$$L^{\mu\nu} = \varepsilon^{ab}p_a^{\mu}p_b^{\nu} \tag{34}$$

and therefore one concludes that $\mathcal{P}^{\alpha\beta}$ can be understood as the complexification of $L^{\mu\nu}$. Of course, $\mathcal{P}^{\alpha\beta}$ satisfies the Grassmann-Plücker relations

$$\mathcal{P}^{\mu[\nu}\mathcal{P}^{\alpha\beta]} = 0. \tag{35}$$

It turns out that $\mathcal{P}^{\alpha\beta}$ can be used to associate points in $C^4$ with points in the complexified Minkowski space (see [25]). From the perspective of oriented complex matroids, $\mathcal{P}^{\alpha\beta}$ is just a representable phirotope. One is tempted to assume that a generalization of twistor theory may be also be associated with the phirotope theory.

Is it possible that twistors or qubits can be related to surreal number theory [10–12]? Consider the set

$$x = \{X_L \mid X_R\} \tag{36}$$

and call $X_L$ and $X_R$ the left and right sets of $x$, respectively. Conway develops the surreal numbers structure $\mathcal{S}$ from two axioms:

**Axiom 1**. Every surreal number corresponds to two sets $X_L$ and $X_R$ of previously created numbers, such that no member of the left set $x_L \in X_L$ is greater or equal to any member $x_R$ of the right set $X_R$.

Let us denote by the symbol $\ngeq$ the notion of no greater or equal to. So the axiom establishes that if $x$ is a surreal number then for each $x_L \in X_L$ and $x_R \in X_R$ one has $x_L \ngeq x_R$. This is denoted by $X_L \ngeq X_R$.

**Axiom 2**. One number $x = \{X_L \mid X_R\}$ is less than or equal to another number $y = \{Y_L \mid Y_R\}$ if and only the two conditions $X_L \ngeq y$ and $x \ngeq Y_R$ are satisfied.

This can be simplified by saying that $x \leq y$ if and only if $X_L \ngeq y$ and $x \ngeq Y_R$.

Observe that Conway definition relies in an inductive method; before a surreal number $x$ is introduced one needs to know the two sets $X_L$ and $X_R$ of surreal numbers. Using Conway algorithm one finds that at the $j$-day one obtains $2^{j+1} - 1$ numbers all of which are of form

$$x = \frac{m}{2^n}, \tag{37}$$

where $m$ is an integer and $n$ is a natural number, $n > 0$. Of course, the numbers (37) are dyadic rationals which are dense in the reals $R$.

The sum and product of surreal numbers are defined as

$$x + y = \{X_L + y, x + Y_L \mid X_R + y, x + Y_R\} \tag{38}$$

and

$$xy = \{X_L y + xY_L - X_L Y_L, X_R y + xY_R - X_R Y_R \mid X_L y + xY_R - X_L Y_R, X_R y + xY_L - X_R Y_L\}, \tag{39}$$

respectively. The importance of (38) and (39) is that allow us to prove that the surreal number structure is algebraically a closed field. Moreover, through (38) and (39) it is also possible to show that the real numbers $R$ are contained in the surreals $\mathcal{S}$ [see [10–12] for details]. Of course, in some sense the prove relies on the fact that the dyadic numbers (37) are dense in the reals $R$.

In 1986, Gonshor [12] introduced a different but equivalent definition of surreal numbers.

**Definition 1**. A surreal number is a function $f$ from initial segment of the ordinals into the set $\{+,-\}$.

For instance, if $f$ is the function so that $f(1)=+, f(2)=+, f(3)=-, f(4)=+$ then $f$ is the surreal number $(++-+)$. In the Gonshor approach one obtains the sequence: 1-day

$$-1=(-), \qquad (+)=+1, \tag{40}$$

in the 2-day

$$-2=(--), \quad -\frac{1}{2}=(-+), \quad (+-)=+\frac{1}{2}, \quad (++)=+2, \tag{41}$$

and 3-day

$$\begin{aligned} &-3=(---), \quad -\frac{3}{2}=(--+), \quad -\frac{3}{4}=(-+-),\\ &-\frac{1}{4}=(-++), (+--)=+\frac{1}{4}, \ (+-+)=+\frac{3}{4},\\ &(++-)=+\frac{3}{2}, \quad (+++)=+3, \end{aligned} \tag{42}$$

respectively. Moreover, in Gonshor approach one finds the different numbers through the formula

$$\mathcal{J}=l\mid\varepsilon_0\mid-\frac{\mid\varepsilon_1\mid}{2}+\sum_{i=2}^{s}\frac{\mid\varepsilon_i\mid}{2^i}, \tag{43}$$

where $\varepsilon_0,\varepsilon_1,\varepsilon_2,...,\varepsilon_q\in\{+,-\}$ and $\varepsilon_0\neq\varepsilon_1$. Furthermore, one has $\mid+\mid=+$ and $\mid-\mid=-$. As in the case of Conway definition, through (43) one gets the dyadic rationals. Just for clarity, let us consider the additional example:

$$(++-+-+)=2-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}=\frac{27}{16}. \tag{44}$$

By defining the order $x<y$ if $x(\alpha)<y(\alpha)$, where $\alpha$ is the first place where $x$ and $y$ differ and the convention $-<0<+$, it is possible to show that the Conway and Gonshor definitions of surreal numbers are equivalent [see [12] for details].

Suppose that instead of qubits we consider a rebit (real bits) which can be thought as $j$-tensor [4],

$$t_{A_1A_2...A_j}, \tag{45}$$

where the indices $A_1,A_2,...,A_j$ run from 0 to 1. Of course $j$ indicates the rank of $t_{A_1A_2...A_j}$. In tensorial analysis, (45) is a familiar object. One arrives to a link with surreal numbers by making the indices identification $0\rightarrow+$ and $1\rightarrow-$. For instance, the tensor $t_{0010}$ in the Gonshor notation becomes

$$t_{0010}\rightarrow t_{++-+}\rightarrow(++-+). \tag{46}$$

In terms of $t_{A_1A_2...A_j}$, the expressions (40), (41) and (42) read

$$-1=t_1, \qquad t_0=+1, \tag{47}$$

in the 2-day

$$-2=t_{11,} \quad -\frac{1}{2}=t_{10}, \quad t_{01}=\frac{1}{2}, \quad t_{00}=2, \tag{48}$$

and 3-day

$$\begin{aligned} &-3=t_{111}, \ -\tfrac{3}{2}=t_{110}, \ -\tfrac{3}{4}=t_{101}, \ -\tfrac{1}{4}=t_{100},\\ &t_{011}=+\tfrac{1}{4}, \ t_{010}=+\tfrac{3}{4}, \ t_{001}=+\tfrac{3}{2}, \ t_{000}=+3, \end{aligned} \tag{49}$$

respectively.

Note that there is a duality symmetry between positive and negative labels in surreal numbers. In fact, one can prove that this is general for any $j$-day. This could be anticipated because according to Conway definition a surreal number can be written in terms of the dual pair left and right sets $X_L$ and $X_R$. Further, the concept of duality it is even clearer in the Gonshor definition of surreal numbers since in such a case one has a functions $f$ with the image in the dual set $\{+,-\}$. In terms of the tensor $t_{A_1A_2...A_p}$ such a duality can be written in the form

$$t_{A_1A_2...A_p}+(-1)^p\varepsilon_{A_1B_1}\varepsilon_{A_2B_2}...\varepsilon_{A_pB_p}t^{B_1B_2...B_p}=0, \tag{50}$$

where

$$\varepsilon_{AB}=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}. \tag{51}$$

The identification of surreal numbers with rebits means that its complexification must be related to qubit theory and therefore with twistor theory. So one has discovered that the use of the completely antisymmetric object epsilon $\varepsilon^{a_1...a_d}$ allows to define the Plucker coordinates which must to satisfy the Grassmann-Plücker relation. In turn, we have proved that this relation is a common mathematical central notion in oriented matroids, qubit theoy, twistor theory and surreal number theory.

Moreover, it has been proved in Mosseri and Dandoloff [27], Mosseri [28], and Bernevig and Chen [29] that for normalized qubits the complex 1-qubit, 2-qubit, and 3-qubit are deeply related to division algebras via the Hopf maps, $S^3\xrightarrow{S^1}S^2$, $S^7\xrightarrow{S^3}S^4$, and $S^{15}\xrightarrow{S^7}S^8$, respectively. It seems that there does not exist a Hopf map for higher $N$-qubit states. So, from the perspective of Hopf maps, and therefore of division algebras, one arrives to the conclusion that 1-qubit, 2-qubit, and 3-qubit are more special than higher dimensional qubits [see [27–29] for details]. Again one wonders whether surreal numbers can contribute in this qubits theory framework.

The original idea of Penrose was to replace the continuity of the Minkowski space for new geometric framework which may allow for a discrete structure and in this way unify general relativity and quantum mechanics. In fact, one of the original motivation to introduce twistors was be able to have mathematical arena in which the discretization of the spacetime was possible. The hope was that the complex structure of twistors may be connected with quantum mechanics. In a sense the idea was to replace $R^4$ by $C^4$ and in this way, since the object in $C^4$ are complex, one may be able to connect with quantum mechanics which intrinsically is a complex structure. Ironically, according to the discussion in this work, it seems to us that the combinatorial structure searched by Penrose in connection with quantum gravity is not the twistors itself but the underlying oriented matroid theory. But ground set in oriented matroids can be constructed by strings of the set $\{+-\}$ which are the main tool in qubit theory and surreal numbers. All these comments suggested that the concepts such as chirotopes (phirotopes), qubits, twistors, and surreals must be considered mathematical tools underlying quantum gravity.

Let us analysis deeply the connection between surreal numbers and qubits. For this purpose we shall assume that one may be able to write a surreal complex numbers $\mathcal{Z}$ in the form

$$\mathcal{Z} = \mathcal{J}_1 + i\mathcal{J}_2, \tag{52}$$

where $\mathcal{J}_1$ and $\mathcal{J}_2$ are two surreal numbers according to the formula (43). This complexification of surreal numbers must establish a complete connection with the $N$-qubit structure if one assume the existence of a complex operator $\hat{\mathcal{Z}}_{A_1A_2...A_N}$ such that

$$\hat{\mathcal{Z}}_{A_1A_2...A_N} \mid A_1A_2...A_N > = \sum_{A_1,A_2,...,A_N=0}^{1} Q_{A_1A_2...A_N} \mid A_1A_2...A_N > = \mathcal{J} \mid A_1A_2...A_N > . \tag{53}$$

This is inspired in the observation that $\mathcal{J}$ in (43) can be associated with the eigenvalues of a $z$-component $\hat{J}_z$ of the total angular momentum $\hat{J}$ in quantum mechanics. Of course in such case one has $J_z = l \pm \frac{1}{2}$, with the identification of $\frac{1}{2}$-spin of the system. The surprise with surreal numbers is that predicts that besides $\frac{1}{2}$-spin system there must exist infinite number of $\mathcal{J}$-spins, according to the formula (43). Thus, for instance one must include particles with $\frac{1}{4}$-spin (see [30, 31]) and $\frac{1}{8}$-spin and in general particles with dyadic rational $\frac{m}{2^n}$-spin.

Traditionally, quantum mechanics enter in the above twistor formalism when one writes all possible gauge fields (and their associated field equations) in twistor language and proceed to quantize in the usual way. In the case of qubit theory things are different because, even from the begining, qubits refers to quantum states. Thus, concepts of quantum mechanics such as the density of states are constructed from the corresponding entanglement monotones [23]. Here, we would like to propose an alternative possible route to connect further our formalism with quantum mechanics. The central idea is to continue looking the surreal numbers as a quantities associated with different dyadic spins ($\frac{m}{2^n}$-spin). Let us explain in some detail this idea. As we mentioned $\mathcal{J}$ in (43) seems to play the analog of the eigenvalues of the $z$-component $\hat{J}_z$ of the angular momentum operator, namely $J_z = l \pm \frac{1}{2}$. Roughly speaking, from the point of view of number theory, the quantization of a physical system means to go from the real numbers (continuum) $R$ to natural numbers $N$ (discrete). In the case of surreal numbers things are different because one starts with the 0-day, 1-day, 2-day, and so on and in the $\omega$-day (this is the way mathematitians called) one obtains the real numbers $R$. In other words one starts with a discrete structure and finds the continuum scenario. Moreover, if in addition to (43) one uses the identity

$$2^{n+1} = 2 + 2 + 4 + 8 + ... + 2^n, \tag{54}$$

it is not difficult to show that $\mathcal{J}$ in (43) satisfies the expression

$$-l < \mathcal{J} < l. \tag{55}$$

Since $l < j$ one also has

$$-j < \mathcal{J} < j. \tag{56}$$

Here, one assumes that from (43) one has $j = l + s$. Of course, (56) is the analoguos inequality of the eigenvalue of the total angular momentum. Following this route of thoughts one first note that surreal numbers of the type $(+ + ... + +)$ (or the corresponding negative part) can be associated with higher integer-spins, $1, 2, 3, ...$, while surreal numbers of the type $(+ + ... + -)$ can be associated with half-inter spins, $1/2, 3/2, 5/2, ...$. This means that in principle bosons and fermions are part of the surreal structure and therefore supersymmetry must be present. Thus one must expect that a generalized supersymmetry can be obtained if one includes other surreal numbers such as $1/4, 3/4, 1/8, 3/8$, and so on. Since, as we mentioned, the dyadic rational $m/2^n$ are dense in the reals $R$ one should expect that eventually, in the $\omega$-day, the anyons may emerge. What about the graviton? This corresponds to the surreal number 2 or 2-spin. Thus, just as in string theory the graviton is just one resonance out of many or even infinity resonances, in our case the graviton is just a physical system with particular value 2-spin, but in principle one has all kind of dyadic-spin particles. Thus, according to these observations it seems that quantum gravity should not be seen as an isolated problem but as part of a much larger system in which all types of dyadic-spins are present.

Another source of interesting developments it may emerge from the analysis of singularities, both in balck-holes and cosmology. In fact, from the point of view of surreal numbers theory the black-hole singularity $2MG/c^2r \to \infty$, when $r \to 0$, and the Big-Bang singularity (of the radiation energy density) $\rho_r = \rho_0/a^4 \to \infty$, when $a \to 0$ are not a real problem because in such a mathematical theory all kind of infinite large and infinite small are present.

It is worth mentioning that in Atiyah [32], the twistor space and the Plücker coordinates are used to determine the geometry of the instantons solutions of Yang-Mills theory. It may interesting for further research to find the connection between instantons formalism and surreal number theory.

Finally, let us just mention that using fiber bundle concept in oriented matroid theory [33, 34] a connection with $p$-branes and phirotopes was established [6]. Thus according to the present development one may expect that eventually a link between $p$-branes and surreal numbers must be route to follow in the quest of quantum gravity.

## AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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