frontiers in Physics

ORIGINAL RESEARCH
published: 20 March 2020
doi: 10.3389/fphy.2020.00071

Performance of Three-Stage Sequential Estimation of the Normal Inverse Coefficient of Variation Under Type II Error Probability: A Monte Carlo Simulation Study

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This paper sheds light on the performance of the three-stage sequential estimation of the population inverse coefficient of variation of the normal distribution under a moderate sample size. We estimate the final sample size generated by the three-stage procedure, and the population mean, the population variance, the population inverse coefficient of variation, the asymptotic coverage probability, and the asymptotic regret incurred by estimating the population inverse coefficient of variation by its sample statistics under squared-error loss function plus linear sampling cost. Besides, we address the sensitivity of the constructed confidence interval to detect a potential shift that may occur in the population inverse coefficient of variation under uncontrolled and controlled optimal sample size against type II error probability. We do so by computing the characteristic operating function. Besides, we address the sensitivity of the three-stage procedure as the underlying distribution departs away from normality. We consider two classes of distributions: Student's *t* distribution and beta distribution. We use Monte Carlo simulations for this study. We write FORTRAN codes and use Microsoft developer studio software. The simulation results revealed that the controlled confidence intervals provide coverage probabilities that exceed the prescribed nominal value even for small optimal sample size contrary to the uncontrolled case that attains the nominal value only asymptotically. Moreover, under the controlled case, the sensitivity of the procedure to depict a potential shift in the parameter of concern becomes more sensitive than the uncontrolled case. Finally, the three-stage procedure is non-sensitive to departure from normality for normal likewise distributions.

Keywords: asymptotic consistency, asymptotic efficiency, inverse coefficient of variation, Monte Carlo simulation, normal distribution, squared-error loss function, three-stage procedure

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Specialty section:
This article was submitted to Mathematical Physics, a section of the journal Frontiers in Physics

Received: *06 January 2020*
Accepted: *02 March 2020*
Published: *20 March 2020*

Citation:
Yousef A (2020) Performance of Three-Stage Sequential Estimation of the Normal Inverse Coefficient of Variation Under Type II Error Probability: A Monte Carlo Simulation Study. Front. Phys. 8:71. doi: 10.3389/fphy.2020.00071

INTRODUCTION

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables from a normal distribution $N(\mu, \sigma^2)$ with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in \mathbb{R}^+$, both parameters are finite but unknown. Pearson [1] introduced the concept of coefficient of variation in the statistical literature. The population coefficient of variation is simply the ratio of the population standard deviation to the population mean, provided the mean is not zero. The higher the coefficient of variation, the greater the level of dispersion around the mean. It is a unit-free measure that allows for comparison between distributions of values whose scales of measurement are not comparable.

The measure has a wide range of applications across many fields of science; see Nairy and Rao [2] for a brief survey of recent applications in business, climatology, engineering, and other fields. Recently, Hima Bindu et al. [3] published a book, which provides necessary exposure of computational strategies, properties of the coefficient of variation, and extracting the metadata leading to efficient knowledge representation. It also compiles representational and classification strategies based on the measure through illustrative explanations. The disadvantage of the measure lies when the population mean equal zero, or when the mean approaches zero. For that reason, we recommend to work with the reciprocal of the measure, inverse coefficient of variation, that is $\eta = \frac{\mu}{\sigma}, \eta \in \mathbb{R}$.

Having observed a random sample $X_1, X_2, \ldots, X_n$ of size $(n \geq 2)$ from the normal distribution, we recommend using the customary measures $\overline{X}_n = n^{-1}\sum_1^n X_i$ and $S_n = (n-1)^{-1/2}\left\{\sum_1^n \left(X_i - \overline{X}_n\right)^2\right\}^{\frac{1}{2}}$ as initial point estimates for the population mean μ and the population standard deviation σ, respectively. Note $\left(\overline{X}_n,\ S_n^2\right)$ are complete sufficient statistics for $\left(\mu,\ \sigma^2\right)$. Consequently, the customary sample inverse coefficient of variation is $\hat{\eta}_n = \frac{\overline{X}_n}{S_n}$.

Lehman [4] obtained an exact form for the distribution function of the sample coefficient of variation, which depends on the non-central $t-$distribution, while Jayakumar and Sulthan [5] derived a density function for the sample coefficient of variation in terms of the confluent hypergeometric distribution. Moreover, they obtained the first two moments of the distribution and proved that the sample coefficient of variation is a biased estimator for the population coefficient of variation. Sharma and Krishna [6] found the asymptotic distribution for the sample inverse coefficient of variation without assuming normality. They derived an asymptotic confidence interval for the population inverse coefficient of variation mathematically and then invested the result in making inferences regarding Gamma and Weibull distributions. Albatineh et al. [7] examined the performance of the asymptotic confidence interval for a wide class of underlying distributions: normal, lognormal, χ^2 (Chi-squared-distribution), Gamma, and Weibull via Monte Carlo simulation. Gulha et al. [8] considered several confidence intervals for estimating the population coefficient of variation using parametric, non-parametric, and modified methods using Simulation. Their objective was to compare the performance of the existing and newly proposed methods. Banik and Kibria [9] also considered various confidence intervals for estimating the population coefficient of variation under several classes of distributions: symmetric and skewed distributions using simulation. They also include some bootstrap proposed interval estimators for estimating the coefficient of variation. Therefore, the inference for the coefficient of variation is limited to parametric methods or standard bootstrap. Wang et al. [10] used non-parametric methods based on empirical likelihood and modified jackknife empirical likelihood method for constructing confidence intervals for the coefficient of variation. They also propose bootstrap procedures for calibrating the test statistics.

In this paper, we propose sequential estimation for estimating the population inverse coefficient of variation of the normal distribution and prove that sequential estimation provides better results than the classical methods.

PROBLEM SETTING

Suppose we desire to construct a confidence interval for η such that

$$P\left(\left|\hat{\eta}_n - \eta\right| \leq d\right) \geq 1-\alpha, \text{for all } \mu \in \mathbb{R} \text{ and } \sigma > 0 \tag{1}$$

where $d\,(>0)$ and $0 < \alpha < 1$ are predetermined constants. That is the half-width of the interval is d, and the coverage probability is at least $100\,(1-\alpha)\,\%$.

It was shown from Yousef and Hamdy [11] Corollary 2 parts (i) and (ii), that as $n \to \infty$ $\sqrt{2n}\left(\hat{\eta}_n - \eta\right) N \xrightarrow{D} \left(0,\ 2+\eta^2\right)$, "D" denotes convergence in distribution. It follows that

$$P\left(\left|\frac{\sqrt{2n}\left(\hat{\eta}_n - \eta\right)}{\sqrt{2+\eta^2}}\right| \leq \frac{d\sqrt{2n}}{\sqrt{2+\eta^2}}\right) \geq 1-\alpha = 2\Phi(a) - 1$$
$$\Rightarrow 2\Phi\left(\frac{d\sqrt{2n}}{\sqrt{2+\eta^2}}\right) - 1 \geq 2\Phi(a) - 1, \tag{2}$$

where $\Phi(\cdot)$ is the cumulative distribution function of $\mathcal{N}(0,1)$ and $a = Z_{\frac{\alpha}{2}}$ the upper cut off point of $\mathcal{N}(0,1)$. Solving Equation (2) for n provides

$$n \geq n^* = \lambda\left(1+\eta^2/2\right),\ \lambda = a^2/d^2 \tag{3}$$

If η is known, then (3) is the optimal fixed sample size required to solve (1) uniformly for all $\mu \in \mathbb{R}$ and $\sigma > 0$. However, since η is unknown, then it has been shown by Dantzig [12] that no fixed sample size procedure could satisfy (1) except by using multistage sequential sampling procedures. In this paper, we use Hall's three-stage sequential sampling procedure.

Before we review Hall's three-stage procedure [13, 14], we summarize the customary asymptotic measures through which one judges the quality of inference, as presented in the literature. These asymptotic measures help in comparing different methods of multistage sampling.

Let N be the final random sample size generated by a multistage sampling procedure, and let n^* be as in (3). Then the multistage procedure is said to be (i) *first-order* asymptotically efficient if as $\lambda \to \infty$, $E\left(\frac{N}{n^*}\right) \to 1$ while it is (ii) *second-order* asymptotically efficient if as $\lambda \to \infty$, $E(N - n^*)$ is bounded by a finite number unrelated to n^*, in the sense of Ghosh and Mukhopadhyay [15].

Let I_N be the fixed-width confidence interval constructed via a multistage procedure. Then the procedure is (iii) *consistent* or exactly consistent if $P(\eta \in I_N) \geq 1-\alpha$, uniformly for all

$\forall$ μ and σ. while it is (iv) *asymptotically consistent* if as $\lambda \to \infty$, $P(\eta \in I_N) \to 1-\alpha$ in the sense of Stein [16], Mukhopadhyay [17], and Chow and Robbins [18].

Let R_N be the multistage risk encountered in estimating η by the corresponding sample measure $\hat{\eta}$ and R_{n^*} be the optimal fixed sample size risk had η been known. Then, the procedure is (v) *first-order* asymptotically risk efficient if as $\lambda \to \infty$, $\frac{R_N}{R_{n^*}} \to 1$ while it is (vi) asymptotically *second-order* regret if as $\lambda \to \infty$, $R_N - R_{n^*}$ is bounded by a finite number in the sense of Ghosh and Mukhopadhyay [15]. For more details about the procedures, see Mukhopadhyay and de Silva ([19], Ch. 6).

In addition to the above asymptotic measures, we address other factors for comparison: the practical implementations in real-life problems, the insensitivity to changes in the underline distribution, and the sensitivity to depict any changes in the parameter under consideration.

Three-Stage Sequential Procedure

Stein [16, 20] and Cox [21] introduced the two-stage procedure for solving (1) regarding the population normal mean. The two-stage procedure attains consistency, but unfortunately, it leads to oversampling, in other words, it is asymptotically inefficient. To overcome such deficiency, Anscombe [22], Ray [23], and Chow and Robbins [18] proposed the purely sequential procedure. The procedure attains efficiency and asymptotic consistency but lacks time consumption. As a compromising procedure, Hall [13] introduced the three-stage procedure to achieve two primary objectives, the operational savings made possible by sampling in batches and the asymptotic efficiency attained by the purely sequential sampling. The procedure based on three stages, as we describe later. The procedure combines the efficiency of Anscombe, and Chow and Robbins one-by-one purely sequential procedure and the operational saving made possible by sampling in bulks by applying Stein's group sampling techniques. It is a nice trade-off between purely sequential procedure and two-stage procedure ease of implementation. The procedure attains all properties except exact consistency.

Mukhopadhyay [24] made further developments to the three-stage sampling by focusing on higher-order moments of the stopping variable. Hamdy [25] extended Hall's results and proposed a three-stage sampling point estimation procedure to estimate the normal mean while Liu [26] extended Hall's results to tackle hypothesis-testing problems for the normal mean.

Yousef [27, 28] tackle the three-stage fixed-width confidence interval for the mean of a continuous distribution where $E|X_1|^6 < \infty$ but unknown under two cases; the first when the explicit form of the underlying function is known and the second when the underlying distribution can be approximated by Edgeworth series of order two. Heuristically, he showed that the kurtosis of the underlying distribution mainly influences the performance of the asymptotic coverage probability. He studied the asymptotic characteristics of each confidence interval and discussed the sensitivity of the three-stage procedure as the underlying distribution departs away from normality. Son et al. [29] proposed a triple sampling sequential procedure, which yields both a fixed-width confidence interval and a hypothesis testing for the normal mean while controlling Type II error probability. Yousef [28] extended their results to a wider class of underlying continuous distributions. Both Son et al. [29] and Yousef [28] provided second-order approximations to the characteristic operating function of the inference. See also Hamdy et al. [30].

For a complete list of three-stage estimation, see Ghosh et al. [31].

In this paper, we use the three-stage procedure to generate inference for the population inverse coefficient of variation η based on (3).

***The Pilot-Stage*:** take a pilot sample of size m from the normal distribution and calculate the sample mean, sample variance, and the sample inverse coefficient of variation.

The Main-Study Stage: let $[x]$ be the largest integer function and γ (design factor)$0 < \gamma < 1$. The stage depends on the stopping rule

$$T = max\left\{m, \left[\gamma\lambda\left(1+\frac{1}{2}\hat{\eta}_m^2\right)\right]+1\right\} \tag{4}$$

***The Fine-Tuning Stage*:** Apply the rule

$$N = max\{\, T, \left[\lambda\left(1+\frac{1}{2}\hat{\eta}_T^2\right)\right]+1\,\}. \tag{5}$$

Once the procedure terminates, we propose $\hat{\mu}_N = \bar{X}_N$, $\hat{\sigma}_N = S_N$ and $\hat{\eta}_N = \frac{\bar{X}_N}{S_N}$. The $100\,(1-\alpha)$ % fixed-width confidence interval of η is $I_N \in \left(\hat{\eta}_N - d, \hat{\eta}_N + d\right)$.

Review of Sequential Estimation of the Population Inverse Coefficient of Variation

Regarding sequential estimation of the population inverse coefficient of variation of the normal distribution, Chaturvedi and Rani [32] developed a purely sequential procedure to find a fixed-width confidence interval estimation for the inverse coefficient of variation of the normal distribution. They showed mathematically that the proposed procedure attains asymptotic efficiency and consistency in the sense of Chow and Robbins [18] without any numerical or simulation results.

Later, Yousef and Hamdy [33] tackle the same problem using Hall's three-stage sequential procedure. They found a unified optimal sample size in the form, $n^* = \lambda\left(\frac{\sigma^2}{2}\right)$ that tackle both point and interval estimation for the population normal mean. As an application, they found the asymptotic coverage probability of the population inverse coefficient of variation and the asymptotic regret under the squared-error loss function with linear sampling cost through Monte Carlo simulation. The results showed that the three-stage procedure attains asymptotic efficiency and consistency in the sense of Chow and Robbins [18]. Recently, Yousef and Hamdy [11] reconsidered the same problem but theoretically using an optimal sample size of the form, $n^* = \lambda\left(\frac{\eta^2}{2}\right)$, that is, the stopping rule directly depends on the population inverse coefficient of variation. They found a compact form for the asymptotic coverage probability for the population inverse coefficient of variation, as well as the asymptotic regret under a squared-error loss function plus linear sampling cost.

TABLE 1 | Three-stage sequential estimation of the population inverse coefficient of variation under (3) at $m = 15$, $\gamma = 0.5$, $\alpha = 5\%$, $\eta = 20$, 10, 5, 3, 2, 1.5, 1.0, and 0.5.

n^*	$\bar{N}$	$S_{\bar{N}}$	$\hat{\mu}$	$S_{\hat{\mu}}$	$\hat{\sigma}$	$S_{\hat{\sigma}}$	$\hat{\eta}$	$S_{\hat{\eta}}$	$\hat{\omega}$	$\hat{\nu}$	$1-\hat{\alpha}$
					$\mu = 10$, $\sigma = 0.5$, $\eta = 20$						
24	35.02	0.032	9.999	0.0005	0.5142	0.0003	19.811	0.0120	−0.99	0.982	0.9643
43	46.92	0.053	9.998	0.0004	0.5168	0.0003	19.671	0.0110	−17.96	0.792	0.9051
61	62.91	0.071	9.999	0.0003	0.5082	0.0002	19.874	0.0086	−28.69	0.766	0.9406
76	77.79	0.083	9.999	0.0003	0.5053	0.0002	19.930	0.0075	−36.22	0.762	0.9444
96	98.10	0.096	9.999	0.0002	0.5040	0.0002	19.949	0.0065	−45.89	0.761	0.9483
125	127.53	0.114	9.999	0.0002	0.5030	0.0001	19.961	0.0057	−59.96	0.760	0.9465
171	174.49	0.140	10.000	0.0002	0.5020	0.0001	19.977	0.0049	−81.95	0.759	0.9487
246	250.61	0.176	10.000	0.0001	0.5014	0.0001	19.983	0.0040	−118.35	0.760	0.9500
500	509.10	0.302	10.000	0.0001	0.5008	0.0001	19.988	0.0028	−240.94	0.759	0.9509
					$\mu = 10$, $\sigma = 1.0$, $\eta = 10$						
24	35.09	0.033	9.992	0.0010	1.0300	0.0007	9.889	0.0061	−0.95	0.976	0.9620
43	46.91	0.052	9.991	0.0008	1.0347	0.0006	9.820	0.0055	−18.01	0.791	0.9025
61	62.73	0.070	9.996	0.0006	1.0161	0.0005	9.936	0.0043	−28.87	0.764	0.9420
76	77.58	0.082	9.997	0.0005	1.0112	0.0004	9.958	0.0038	−36.47	0.761	0.9447
96	97.96	0.096	9.998	0.0005	1.0087	0.0003	9.967	0.0033	−46.10	0.760	0.9457
125	127.41	0.113	9.998	0.0004	1.0064	0.0003	9.976	0.0029	−60.13	0.760	0.9471
171	174.14	0.137	9.999	0.0003	1.0045	0.0002	9.984	0.0025	−82.38	0.760	0.9477
246	250.21	0.177	9.999	0.0003	1.0036	0.0002	9.984	0.0020	−118.93	0.758	0.9497
500	508.80	0.299	10.000	0.0002	1.0016	0.0001	9.994	0.0014	−241.25	0.759	0.9498
					$\mu = 10$, $\sigma = 2.0$, $\eta = 5$						
24	34.82	0.032	9.973	0.0021	2.0521	0.0013	4.951	0.0031	−1.18	0.977	0.9643
43	46.80	0.051	9.971	0.0015	2.0597	0.0012	4.922	0.0028	−18.00	0.789	0.9030
61	62.77	0.069	9.985	0.0012	2.0285	0.0009	4.970	0.0022	−28.79	0.764	0.9434
76	77.62	0.079	9.989	0.0010	2.0199	0.0008	4.981	0.0019	−36.39	0.760	0.9440
96	97.86	0.092	9.991	0.0009	2.0151	0.0007	4.985	0.0017	−46.14	0.759	0.9470
125	127.30	0.108	9.994	0.0008	2.0107	0.0006	4.991	0.0015	−60.16	0.760	0.9486
171	174.14	0.135	9.995	0.0007	2.0078	0.0005	4.993	0.0013	−82.32	0.759	0.9484
246	250.33	0.169	9.996	0.0006	2.0055	0.0004	4.995	0.0010	−118.66	0.759	0.9499
500	508.56	0.289	9.999	0.0004	2.0023	0.0003	4.999	0.0007	−241.30	0.758	0.9501
					$\mu = 10$, $\sigma = 10/3$,$\eta = 3$						
24	34.57	0.031	9.936	0.0034	3.4161	0.0022	2.965	0.0020	−1.44	0.970	0.9599
43	46.21	0.048	9.925	0.0026	3.4229	0.0020	2.948	0.0018	−18.57	0.786	0.8943
61	62.50	0.065	9.966	0.0020	3.3721	0.0015	2.983	0.0014	−29.02	0.762	0.9436
76	77.60	0.075	9.973	0.0018	3.3590	0.0013	2.990	0.0012	−36.36	0.761	0.9466
96	97.82	0.085	9.983	0.0015	3.3540	0.0011	2.992	0.0011	−46.13	0.760	0.9461
125	127.25	0.101	9.988	0.0013	3.3483	0.0010	2.995	0.0009	−60.16	0.759	0.9473
171	173.62	0.123	9.990	0.0011	3.3453	0.0008	2.995	0.0008	−82.86	0.758	0.9510
246	249.50	0.155	9.992	0.0010	3.3411	0.0007	2.997	0.0007	−119.46	0.758	0.9479
500	506.62	0.262	9.996	0.0007	3.3377	0.0005	2.998	0.0005	−243.41	0.757	0.9498
					$\mu = 10$, $\sigma = 5.0$, $\eta = 2$						
24	34.24	0.027	9.863	0.0052	5.0953	0.0035	1.974	0.0015	−1.74	0.963	0.9588
43	45.55	0.044	9.870	0.0039	5.0882	0.0028	1.970	0.0013	−19.06	0.778	0.8950
61	62.33	0.058	9.942	0.0030	5.0348	0.0021	1.992	0.0010	−29.08	0.760	0.9479
76	77.31	0.066	9.954	0.0026	5.0295	0.0019	1.992	0.0009	−36.63	0.760	0.9481
96	97.60	0.076	9.963	0.0023	5.0215	0.0016	1.995	0.0008	−46.32	0.760	0.9496
125	126.80	0.089	9.975	0.0020	5.0199	0.0014	1.995	0.0007	−60.66	0.758	0.9510
171	173.24	0.106	9.981	0.0017	5.0142	0.0012	1.996	0.0006	−83.22	0.757	0.9498
246	248.95	0.139	9.987	0.0014	5.0091	0.0010	1.998	0.0005	−119.99	0.756	0.9507

(Continued)

TABLE 1 | Continued

n^*	$\overline{N}$	$S_{\overline{N}}$	$\hat{\mu}$	$S_{\hat{\mu}}$	$\hat{\sigma}$	$S_{\hat{\sigma}}$	$\hat{\eta}$	$S_{\hat{\eta}}$	$\hat{\omega}$	$\hat{\nu}$	$1-\hat{\alpha}$
					$\mu = 10,\ \sigma = 5.0,\ \eta = 2.0$						
500	505.14	0.219	9.993	0.0010	5.0045	0.0007	1.999	0.0003	−244.80	0.755	0.9509
					$\mu = 10,\ \sigma = 20/3,\ \eta = 1.5$						
24	33.73	0.025	9.814	0.0068	6.7614	0.0048	1.481	0.0013	−2.19	0.954	0.9524
43	44.92	0.040	9.837	0.0052	6.7341	0.0035	1.482	0.0011	−19.55	0.772	0.9193
61	62.18	0.051	9.921	0.0039	6.6958	0.0027	1.494	0.0008	−29.20	0.761	0.9488
76	77.27	0.058	9.935	0.0035	6.6919	0.0024	1.494	0.0007	−36.63	0.759	0.9487
96	97.47	0.067	9.951	0.0031	6.6841	0.0022	1.496	0.0007	−46.40	0.758	0.9510
125	126.71	0.079	9.962	0.0027	6.6813	0.0019	1.497	0.0006	−60.67	0.757	0.9495
171	172.75	0.092	9.968	0.0023	6.6780	0.0016	1.497	0.0005	−83.67	0.755	0.9482
246	248.35	0.116	9.982	0.0019	6.6755	0.0013	1.498	0.0004	−120.55	0.755	0.9516
500	504.01	0.188	9.991	0.0013	6.6708	0.0009	1.499	0.0003	−245.87	0.754	0.9504
					$\mu = 10,\ \sigma = 10, \eta = 1.0$						
24	33.06	0.021	9.701	0.0101	10.0631	0.0075	0.983	0.0011	−2.82	0.942	0.9477
43	44.27	0.033	9.852	0.0070	10.0175	0.0049	0.995	0.0008	−20.05	0.766	0.9509
61	62.28	0.041	9.900	0.0057	10.0093	0.0041	0.997	0.0007	−29.04	0.762	0.9518
76	77.29	0.045	9.924	0.0052	10.0135	0.0036	0.998	0.0006	−36.53	0.760	0.9499
96	97.33	0.052	9.942	0.0046	10.0149	0.0032	0.998	0.0006	−46.48	0.758	0.9499
125	126.48	0.059	9.941	0.0040	10.0056	0.0028	0.997	0.0005	−60.88	0.756	0.9517
171	172.66	0.072	9.963	0.0034	10.0045	0.0024	0.999	0.0004	−83.67	0.754	0.9515
246	247.67	0.087	9.970	0.0029	10.0051	0.0020	0.998	0.0003	−121.21	0.754	0.9509
500	502.20	0.135	9.986	0.0020	10.0025	0.0014	0.999	0.0002	−247.66	0.752	0.9508
					$\mu = 10,\ \sigma = 20, \eta = 0.5$						
24	32.13	0.016	9.54	0.0207	19.910	0.0158	0.4893	0.0011	−3.62	0.925	0.9357
43	44.31	0.020	9.81	0.0133	19.941	0.0096	0.4972	0.0007	−19.97	0.768	0.9536
61	62.32	0.023	9.86	0.0113	19.957	0.0081	0.4980	0.0006	−28.96	0.763	0.9529
76	77.39	0.027	9.90	0.0101	19.958	0.0072	0.4989	0.0005	−36.38	0.761	0.9518
96	97.35	0.029	9.91	0.0091	19.966	0.0065	0.4990	0.0005	−46.42	0.758	0.9506
125	126.42	0.034	9.93	0.0080	19.976	0.0057	0.4991	0.0004	−60.86	0.757	0.9519
171	172.47	0.040	9.95	0.0068	19.987	0.0048	0.4993	0.0004	−83.81	0.755	0.9513
246	247.56	0.049	9.97	0.0057	19.995	0.0040	0.4996	0.0003	−121.22	0.754	0.9491
500	501.59	0.071	9.98	0.0040	19.992	0.0028	0.4998	0.0002	−248.19	0.752	0.9490

n^* indicates the optimal sample size.

Moreover, they found the characteristic operating function for a simple hypothesis against a shift that may occur in the population inverse coefficient of variation. They showed mathematically that the three-stage procedure attains asymptotic efficiency while under some proper choices of γ (the design factor) and α the procedure attains consistency. Collectively, the three-stage procedure attains the nominal value only asymptotically. In both cases, the asymptotic regret provides negative values.

Up to our knowledge, none of the existing papers in the literature of sequential estimation use Monte Carlo simulations to examine the performance of the three-stage procedure to tackle inference of the normal inverse coefficient of variation using the optimal size defined in (3).

In this paper, we continue the research of estimating the population inverse coefficient of variation of the normal distribution by examining the performance of the procedure under (3) and verify the theoretical results found by Yousef and Hamdy [11] under moderate sample sizes. We estimate all the parameters in concern; the final sample size N, the population, mean μ, the population variance σ^2, and the population inverse coefficient of variation η. We tackle two estimation problems; first, the point estimation problem under the squared-error loss function plus linear sampling cost, and second, the fixed-width confidence estimation problem under controlled optimal sample size against type II error probability. Besides, we discuss the sensitivity of the procedure to depict any potential shift in the population inverse coefficient of variation under both uncontrolled and controlled optimal sample size. Finally, we study the sensitivity of the procedure as the underlying distribution departs away from normality considering $t-$ *distribtuion* with different degrees of freedom (Leptokurtic) and *Beta distribution* with different parameters (Platykurtic). We use Monte Carlo simulations for this study using Microsoft Developer Studio software.

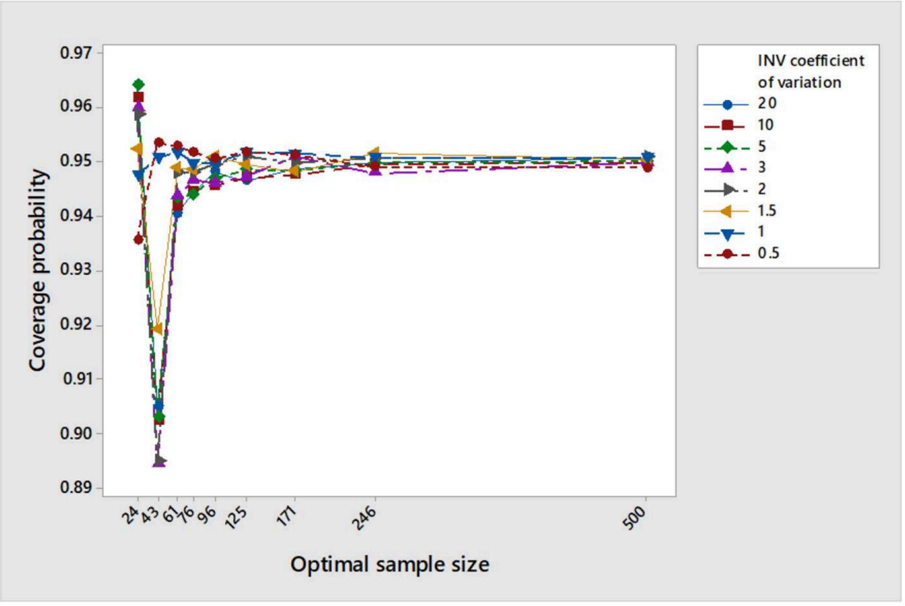

FIGURE 1 | Performance of the simulated coverage probability at $\mu = 10$, $\sigma = 0.5, 1, 2, \frac{10}{3}, 5, \frac{20}{3}, 10, 20$, $\gamma = 0.5$, $m = 15$, and $1 - \alpha = 0.95$.

SEQUENTIAL INFERENCE FOR THE POPULATION INVERSE COEFFICIENT OF VARIATION

Point Estimation Problem

Consider the loss incurred by estimation the population inverse coefficient of variation η by its customary estimate, the sample inverse coefficient of variation $\hat{\eta}_n$ given by

$$L_n(A) = A\left(\hat{\eta}_n - \eta\right)^2 + cn, \tag{6}$$

where A is a known constant, and c is the cost per unit sample. The risk associated with (6) is

$$R_n(A) = E\left(_n(A)\right) = \frac{A}{2n}\left(2 + \eta^2\right) + cn \tag{7}$$

By minimizing (7) concerning n yields

$$n^0 = \sqrt{A/2c}\sqrt{\left(2 + \eta^2\right)} \tag{8}$$

where n^0 is the optimal fixed sample size required for estimating η.

Now, if we set Equation (3) equal Equation (8), we find the optimal sample size needed to perform both point and confidence interval estimation for η with fixed-width $2d$ and coverage probability at least $100(1 - \alpha)$ %. That is, the constant A should be chosen such that

$$A = \left(a/d\right)^4\left(2 + \eta^2\right)c = \left(a/d\right)^2\left(cn^*\right) \tag{9}$$

As $d \to 0$, $A \to \infty$. For more details regarding A, see [33].

The optimal risk is $R_{n^*}\left(d\right) = 2cn^*$. While the three-stage sequential risk is

$$R_N\left(d\right) = AE\left(\hat{\eta}_N - \eta\right)^2 + cE\left(N\right) \tag{10}$$

The asymptotic regret, which is the difference between the risks of using the three-stage procedure minus the optimal risk, would be

$$\omega\left(d\right) = R_N\left(d\right) - R_{n^*}\left(d\right) \tag{11}$$

While the asymptotic relative risk (efficiency ratio) is the sequential risk relative to the optimal risk, that is

$$\nu\left(d\right) = \frac{R_N\left(d\right)}{R_{n^*}\left(d\right)} \tag{12}$$

Now, if $R_N\left(d\right) < R_{n^*}\left(d\right)$, then Equation (11) provides a negative regret see Martinsek [34] while Equation (12) yields $\nu\left(d\right) < 1$. This implies that the three-stage procedure provides a better estimation than the optimal had η been known.

The Asymptotic Coverage Probability of the Population Inverse Coefficient of Variation

Recall the three-stage sampling confidence interval $I_N = (\hat{\eta}_N - d,\ \hat{\eta}_N + d)$ of the inverse coefficient variation, the asymptotic coverage probability of η is

$$P\left(\eta\epsilon I_N\right) = \sum_{n=m}^{\infty}\left(P\left|\ \hat{\eta}_N - \eta\right| \le d,\ N = n\right)$$

$$= \sum_{n=m}^{\infty}\left(P\left|\hat{\eta}_N - \eta\right| \le d|N = n\right)P(\ N =\ n)$$

TABLE 2 | The impact of increasing the pilot sample m on the procedure at $\mu = 10$, $\sigma = 5$, $\eta = 2$.

n^*	$\bar{N}$	$S_{\bar{N}}$	$\hat{\mu}$	$S_{\hat{\mu}}$	$\hat{\sigma}$	$S_{\hat{\sigma}}$	$\hat{\eta}$	$S_{\hat{\eta}}$	$\hat{\omega}$	$1-\hat{\alpha}$
					m = 8					
24	27.61	0.049	9.77	0.0054	5.151	0.0039	1.9497	0.0018	−8.48	0.9044
43	46.25	0.076	9.92	0.0035	5.049	0.0025	1.9883	0.0012	−18.16	0.9494
61	65.27	0.095	9.94	0.0029	5.033	0.0021	1.9920	0.0010	−26.15	0.9484
76	81.24	0.124	9.96	0.0026	5.028	0.0018	1.9930	0.0009	−32.69	0.9504
96	102.47	0.145	9.97	0.0023	5.022	0.0016	1.9943	0.0008	−41.47	0.9521
125	133.46	0.189	9.97	0.0020	5.015	0.0014	1.9962	0.0007	−53.95	0.9519
171	182.36	0.252	9.98	0.0017	5.013	0.0012	1.9962	0.0006	−74.11	0.9521
246	261.55	0.345	9.99	0.0014	5.008	0.0010	1.9981	0.0005	−107.36	0.9520
500	531.03	0.696	9.99	0.0010	5.004	0.0007	1.9993	0.0003	−218.84	0.9516
					m = 10					
24	28.61	0.034	9.75	0.0053	5.182	0.0039	1.9335	0.0017	−7.61	0.9272
43	45.14	0.056	9.92	0.0035	5.046	0.0025	1.9895	0.0012	−19.26	0.9490
61	63.65	0.074	9.95	0.0029	5.032	0.0021	1.9927	0.0010	−27.75	0.9496
76	79.01	0.083	9.95	0.0026	5.027	0.0018	1.9925	0.0009	−34.92	0.9499
96	99.54	0.101	9.96	0.0023	5.023	0.0016	1.9933	0.0008	−44.43	0.9490
125	129.47	0.123	9.97	0.0020	5.019	0.0014	1.9952	0.0007	−57.98	0.9491
171	177.06	0.167	9.98	0.0017	5.012	0.0012	1.9975	0.0006	−79.34	0.9506
246	254.77	0.235	9.99	0.0014	5.009	0.0010	1.9979	0.0005	−114.15	0.9512
500	516.49	0.431	9.99	0.0010	5.004	0.0007	1.9993	0.0003	−233.39	0.9515
					m = 15					
24	34.24	0.030	9.87	0.0051	5.093	0.0035	1.9751	0.0015	−1.73	0.9566
43	45.45	0.044	9.87	0.0039	5.086	0.0028	1.9701	0.0013	−19.16	0.8942
61	62.32	0.058	9.94	0.0030	5.036	0.0021	1.9913	0.0010	−29.10	0.9491
76	77.33	0.066	9.96	0.0026	5.031	0.0019	1.9923	0.0009	−36.61	0.9490
96	97.53	0.076	9.96	0.0023	5.024	0.0016	1.9935	0.0008	−46.42	0.9479
125	126.87	0.088	9.97	0.0020	5.016	0.0014	1.9958	0.0007	−60.56	0.9503
171	173.10	0.105	9.98	0.0017	5.013	0.0012	1.9963	0.0006	−83.36	0.9498
246	249.06	0.137	9.99	0.0014	5.006	0.0010	1.9987	0.0005	−119.79	0.9488
500	504.94	0.223	9.99	0.0010	5.005	0.0007	1.9988	0.0003	−245.01	0.9519
					m = 20					
24	41.09	0.038	9.97	0.0048	4.977	0.0033	2.0443	0.0016	5.70	0.9511
43	49.65	0.037	9.83	0.0040	5.123	0.0028	1.9470	0.0013	−15.29	0.9252
61	62.66	0.054	9.92	0.0031	5.048	0.0022	1.9847	0.0011	−28.87	0.9353
76	77.11	0.063	9.95	0.0026	5.028	0.0019	1.9932	0.0009	−36.81	0.9494
96	97.20	0.071	9.97	0.0023	5.021	0.0016	1.9956	0.0008	−46.69	0.9481
125	126.31	0.082	9.97	0.0020	5.017	0.0014	1.9961	0.0007	−61.10	0.9490
171	172.18	0.097	9.98	0.0017	5.013	0.0012	1.9965	0.0006	−84.28	0.9519
246	247.38	0.118	9.99	0.0014	5.010	0.0010	1.9975	0.0005	−121.57	0.9493
500	502.43	0.178	9.99	0.0010	5.005	0.0007	1.9989	0.0003	−247.51	0.9481

n^* indicates the optimal sample size.

The results of Anscombe [35] provide that $\frac{\sqrt{2N}(\hat{\eta}_N-\eta)}{\sqrt{2+\eta^2}}\mathcal{N}(0,\ 1)$ as $\lambda \to \infty$ independent of the random variable $N = m,\ m+1,\ m+2,\ \ldots$. Thus,

$$P\left(\eta \epsilon I_N\right) = \sum_{n=m}^{\infty}\left(P\left|\frac{\sqrt{2n}\left(\hat{\eta}_N - \eta\right)}{\sqrt{2+\eta^2}}\right| \leq \frac{d\sqrt{2n}}{\sqrt{2+\eta^2}}\right)$$

$$P(\,N=n) = 2E\left\{\Phi\left(\frac{d\sqrt{2N}}{\sqrt{2+\eta^2}}\right)\right\} - 1 \qquad (13)$$

Constructing a Fixed-Width Confidence Interval With Controlled Type II Error Probability for the Population Inverse Coefficient of Variation

There is a close relationship between statistical testing hypotheses and confidence intervals in the sense that they can perform similar inference objectives. Confidence intervals, however, provide more information compared to the hypotheses testing counterpart see, Tukey [36]. They signify by their length, the

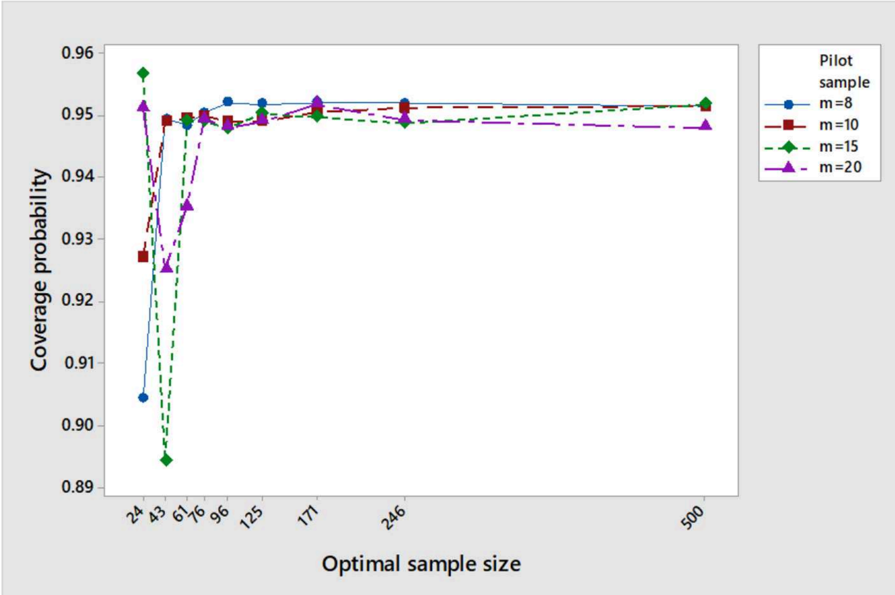

FIGURE 2 | The impact of increasing m on the coverage probability at $\mu = 10$, $\sigma = 5$, $\eta = 2$, $\gamma = 0.5$, and $1 - \alpha = 0.95$.

precision of estimation, and the direction of error. Moreover, confidence intervals show which parameter value should not be rejected if they were hypothesized as null values. Therefore, the sensitivity of confidence intervals to depict shifts in the real parameter value η_0 becomes a crucial issue to ensure the quality of inference.

Son et al. [29], Costanza et al. [37] were the first who brought up the idea for the normal mean, while Hamdy [38] considered the idea for estimating the location parameter of the exponential distribution.

From a practical standpoint, this issue is essential when constructing quality control charts to monitor the mean quality of service or production. We formulate the following hypotheses:

$$H_0: \ \eta = \eta_0, vs.$$
$$H_a: \eta = \eta_1, \eta_1 = \eta_0 \pm \left(k+1\right) d \notin I_N, \ \forall k \geq 0 \quad (14)$$

Both hypotheses make statements about the population value of the test statistic and are mutually exclusive. The null hypothesis H_0 asserts that no shift in the actual population inverse coefficient of variation occurred against the alternative hypothesis H_a which emphasized that the actual inverse coefficient of variation has shifted by a distance k Measured in unites of d.

The probability of not depicting a shift given that the shift has already occurred can be assessed by the type II error probability β_{kc}.

$$\begin{aligned}\beta_{kc} &= P\left(\eta_0 \in I_N | H_a\right) \\ &= P\left(\hat{\eta}_N - d \leq \eta \leq \hat{\eta}_N + d \middle| \eta_1 = \eta_0 \pm d\left(k+1\right)\right) \quad (15)\end{aligned}$$

Since the process has an equal probability of committing a type II error probability above the centerline or below the centerline, we, therefore, consider only the probability of committing a positive shift from the actual parameter value η_0.

Let τ be the probability of committing a type II error probability. Our objective is to control the probability of committing a type II error probability. We do so by finding the characteristic operating curve that gives the probability of acceptance of various possible values of η_1. The minimum sample size required to control both α *and* τ is

$$n^0 = \frac{\left(a+b\right)^2}{d^2}\left(1+\frac{\eta^2}{2}\right) \quad (16)$$

where $b = Z_{\frac{\tau}{2}}$ is the upper $\frac{\tau}{2}$ point of $N\left(0,\ 1\right)$. For more details, see Nelson [39, 40].

The second-order approximation of the controlled characteristic operating function under Equations (14) and (16) as $\lambda \to \infty$

$$\begin{aligned}\boldsymbol{\beta_{kc}} &= \boldsymbol{P\left(\eta \in I_N | H_a\right)} = \sum\nolimits_{n=m}^{\infty} \boldsymbol{P\left(\left|\hat{\eta}_N - \eta_1\right| \leq d | N = n\right) P\left(N = n\right)} \\ &= \sum_{n=m}^{\infty} \boldsymbol{P\left(-\left(2+k\right) d \leq \eta_N - \eta_0 \leq -kd\right) P\left(N = n\right)} \\ &= E_N\left(\Phi\left(-dk\frac{\sqrt{2N}}{\eta}\right)\right) - E_N\left(\Phi\left(-\left(2+k\right) d\frac{\sqrt{2N}}{\eta}\right)\right) \quad (17)\end{aligned}$$

The uncontrolled case occurs by setting $b = 0$ in (16) to give β_k.

MONTE CARLO SIMULATION

Since the sequential results are asymptotic, it is worth mentioning to estimate the above equations through Monte

TABLE 3 | The robustness of three–stage at **m = 15**, **γ = 0.5**, **α = 0.05**, **μ = 10**, **σ = 5**, **η = 2.0**.

				Uncontrolled optimal sample size defined in (3)					
n^*	24	43	61	76	96	125	171	246	500
$1-\hat{\alpha}$	0.9588	0.895	0.9479	0.9481	0.9496	0.951	0.9498	0.9507	0.9509
				Uncontrolled characteristic operating values $\hat{\beta}_k$					
k	24	43	61	76	96	125	171	246	500
0	0.5088	0.5146	0.5126	0.5114	0.5085	0.5050	0.5094	0.5055	0.5061
0.1	0.4355	0.4394	0.4367	0.4298	0.4324	0.4319	0.4281	0.4271	0.4241
0.2	0.3625	0.3633	0.3595	0.3595	0.3536	0.3538	0.3552	0.3503	0.3542
0.3	0.2949	0.2971	0.2881	0.2879	0.2912	0.2839	0.2847	0.2812	0.2782
0.4	0.2314	0.2403	0.2275	0.2265	0.2263	0.2230	0.2226	0.2216	0.2193
0.5	0.1808	0.1895	0.1696	0.1704	0.1677	0.1684	0.1684	0.1683	0.1672
0.6	0.1317	0.1517	0.1292	0.1267	0.1261	0.1251	0.1250	0.1226	0.1188
0.7	0.0929	0.1238	0.0912	0.0929	0.0917	0.0909	0.0876	0.0884	0.0877
0.8	0.0620	0.1034	0.0655	0.0658	0.0617	0.0621	0.0630	0.0597	0.0600
0.9	0.0412	0.0919	0.0449	0.0417	0.0419	0.0418	0.0415	0.0407	0.0401
1.0	0.0231	0.0852	0.0306	0.0306	0.0279	0.0285	0.0276	0.0277	0.0267
1.5	0.0006	0.0121	0.0027	0.0024	0.0022	0.0021	0.0020	0.0022	0.0018
2.0	0.0000	0.0007	0.0005	0.0001	0.0001	0.0000	0.0000	0.0001	0.0001
				Controlled optimal sample size defined in (15) τ = 0.05					
n^*	24	43	61	76	96	125	171	246	500
$1-\hat{\alpha}$	0.9999	0.9994	0.9993	0.9999	0.9999	0.9999	0.9999	0.9998	0.9999
				Controlled characteristic operating values $\hat{\beta}_{kc}$					
k	24	43	61	76	96	125	171	246	500
0	0.5102	0.5160	0.5096	0.5109	0.05059	0.5055	0.5076	0.5050	0.5017
0.1	0.3628	0.3622	0.3583	0.3587	0.3555	0.3552	0.3565	0.3562	0.3475
0.2	0.2313	0.2369	0.2243	0.2244	0.2250	0.2214	0.2213	0.2199	0.2166
0.3	0.2313	0.2369	0.2243	0.2244	0.2250	0.2214	0.2213	0.2199	0.2166
0.4	0.1343	0.1536	0.1301	0.1286	0.1250	0.1256	0.1254	0.1246	0.1230
0.5	0.0619	0.1045	0.0645	0.0644	0.0632	0.0612	0.0616	0.0621	0.0604
0.6	0.0242	0.0848	0.0291	0.0278	0.0285	0.0272	0.0260	0.0275	0.0265
0.7	0.0064	0.0419	0.0125	0.0125	0.0112	0.0101	0.0101	0.0102	0.0095
0.8	0.0013	0.0189	0.0044	0.0042	0.0045	0.0037	0.0038	0.0034	0.0033
0.9	0.0002	0.0067	0.0018	0.0015	0.0012	0.0011	0.0011	0.0009	0.0009
1.0	0.0000	0.0020	0.0008	0.0004	0.0004	0.0004	0.0003	0.0004	0.0003
1.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

n^* indicates the optimal sample size.

Carlo simulations. We do so by writing FORTRAN codes and run them using *Microsoft Developer Studio software*.

Simulation Methodology

The simulation process performs as follows: Fix the values of m, γ, α andñ*.

A. Generate an i-th sample of size $m \geq 8$ from the normal distribution, and compute X_m, S_m^2 and $\hat{\eta}_m$ as initial point estimates of μ, σ^2 and η, respectively.

B. Apply Equation (4), $T = max\left\{m, \left[\gamma\lambda\left(1+\hat{\eta}_m^2\right)\right]\right\}$. Furthermore, compute the numerical value of T.

- If $T \leq m$, then we have enough observations, and thus the experiment terminates. In this case $\hat{\mu}_N = \overline{X}_m$, $\hat{\sigma}_N^2 = S_m^2$ and $\hat{\eta}_N = \hat{\eta}_m$.
- If $T > m$ then sample extra observations of size $T - m$, say $X_{m+1}, X_{m+2}, X_{m+3}, \ldots, X_T$, then augment the new sample with the previous sample in (A) to have a sample of size T. Then compute the statistics $\overline{X}_T$, S_T^2 and $\hat{\eta}_T$ for the parameters μ, σ, and η, respectively.

C. Apply Equation (5), $N = max\left\{T, \left[\lambda\left(1+\hat{\eta}_T^2\right)\right]\right\}$ and compute N.

- If $N \leq T$, sampling is terminated with $\hat{\mu}_N = \overline{X}_T, \sigma_N^2 = S_T^2$ and $\hat{\eta}_N = \hat{\eta}_T$.
- If $N > T$, further observations needed. Sample the difference $N - T$ say $X_{T+1}, X_{T+2}, \ldots, \ldots, X_N$ Furthermore, augmented with the previous T observations. The updated sample is of size N, and the new estimates are $\hat{\mu}_N = \overline{X}_N$, $\sigma_N^2 = S_N^2$ and $\hat{\eta}_N = \hat{\eta}_N$.

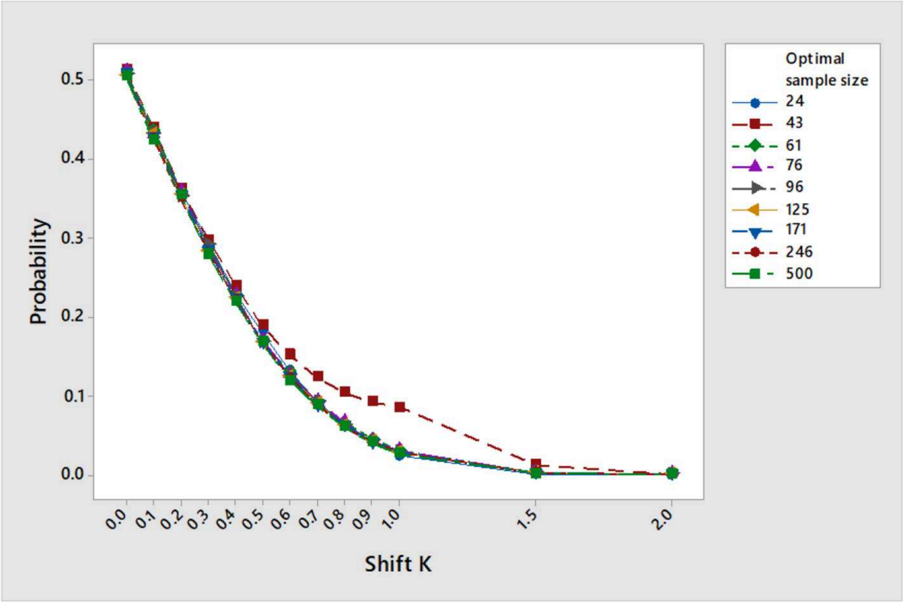

FIGURE 3 | Operating characteristic values under *uncontrolled* optimal sample size as the shift increases at $\mu = 10$, $\sigma = 5$, $\eta = 2$, $\gamma = 0.5$, and $\alpha = 0.05$.

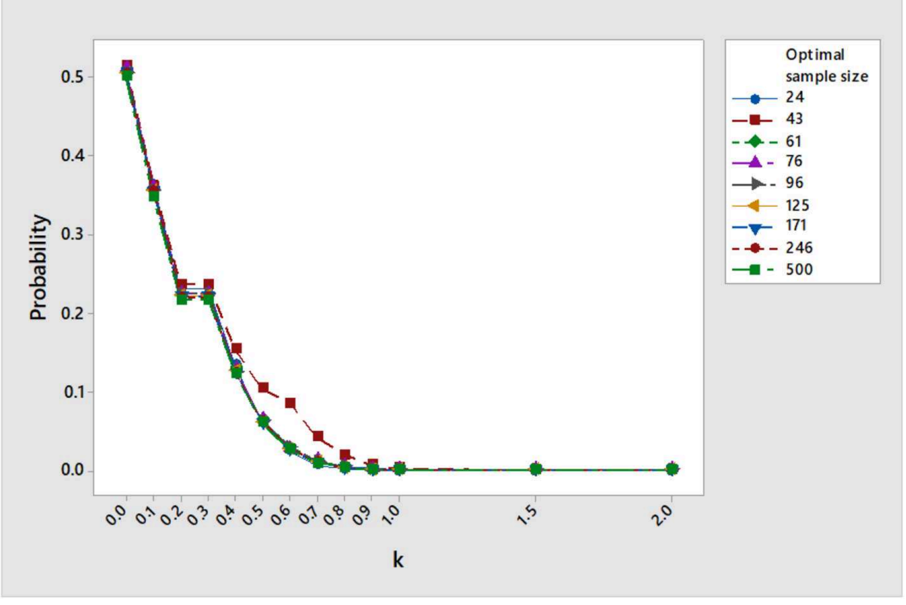

FIGURE 4 | Operating characteristic values under *controlled* optimal sample size as the shift increases at $\mu = 10$, $\sigma = 5$, $\eta = 2$, $\gamma = 0.5$, and $\alpha = 0.05$ and $\beta = 0.05$.

Upon termination, record the resultant sample size N_i^*, the simulated mean $\overline{X}_i$, the simulated standard deviation $\hat{\sigma}_i$ and the simulated inverse coefficient of variation $\hat{\eta}_i$ for $i = 1, 2, \ldots, L$.

D. As a result, record the observations $\left(N_1^*, N_2^*, \ldots, N_L^*\right), \left(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_L\right)$, $\left(\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_L\right)$, and $\left(\hat{\eta}_1, \hat{\eta}_2, \ldots, \hat{\eta}_L\right)$.

E. Calculate the estimated means for N, μ, σ and η respectively as follows

- $\overline{N} = L^{-1} \sum_1^L N_i^*$ is the estimated mean sample size,
- $\overline{X} = L^{-1} \sum_1^L \overline{X}_i$ is the estimated mean of the sample mean,
- $\overline{\sigma} = L^{-1} \sum_1^L \hat{\sigma}_i$ is the estimated mean of the sample variance and

TABLE 4 | Three–stage estimation of the inverse coefficient of variation for underlying $T(r)$, $r = 5,\ 10,\ 20$ $m = 15$, $\gamma = 0.5$, $\alpha = 5\%$.

n^*	$\overline{N}$	$S_{\overline{N}}$	$\hat{\mu}$	$S_{\hat{\mu}}$	$\hat{\sigma}$	$S_{\hat{\sigma}}$	$\hat{\eta}$	$S_{\hat{\eta}}$	$\hat{\omega}$	$\hat{\nu}$	$1-\hat{\alpha}$
						r = 5					
24	30.60	0.012	0.0003	0.0015	1.246	0.0016	−0.0001	0.0012	−4.99	0.894	0.8665
43	44.61	0.006	0.0000	0.0009	1.272	0.0010	−0.0002	0.0007	−19.64	0.771	0.9540
61	62.61	0.006	0.0004	0.0007	1.276	0.0009	0.0003	0.0006	−28.64	0.765	0.9538
76	77.62	0.006	−0.0007	0.0007	1.280	0.0008	−0.0005	0.0005	−36.13	0.762	0.9524
96	97.63	0.006	−0.0006	0.0006	1.282	0.0007	−0.0004	0.0005	−46.12	0.760	0.9515
125	126.62	0.006	−0.0008	0.0005	1.283	0.0006	−0.0006	0.0004	−60.63	0.758	0.9520
171	172.63	0.006	−0.0009	0.0004	1.285	0.0006	−0.0007	0.0003	−83.62	0.756	0.9511
246	247.63	0.006	0.0000	0.0004	1.287	0.0005	0.0000	0.0003	−121.12	0.754	0.9504
500	501.48	0.006	−0.0001	0.0003	1.288	0.0003	−0.0001	0.0002	−248.27	0.752	0.9494
						r = 10					
24	30.37	0.013	0.0005	0.0012	1.093	0.0011	0.0003	0.0012	−5.22	0.894	0.8656
43	44.61	0.006	−0.0009	0.0007	1.108	0.0006	−0.0006	0.0007	−19.64	0.771	0.9528
61	62.63	0.006	−0.0007	0.0006	1.110	0.0005	−0.0005	0.0006	−28.63	0.765	0.9518
76	77.62	0.006	−0.0008	0.0006	1.114	0.0005	−0.0007	0.0005	−36.14	0.762	0.9531
96	97.62	0.006	−0.0009	0.0005	1.115	0.0004	−0.0008	0.0005	−46.13	0.760	0.9518
125	126.62	0.006	−0.0004	0.0004	1.115	0.0004	−0.0004	0.0004	−60.64	0.757	0.9506
171	172.63	0.006	−0.0005	0.0004	1.116	0.0003	−0.0004	0.0003	−83.63	0.755	0.9503
246	247.63	0.006	−0.0003	0.0003	1.116	0.0003	−0.0002	0.0003	−121.13	0.754	0.9510
500	501.48	0.007	−0.0006	0.0002	1.117	0.0002	−0.0005	0.0002	−248.27	0.752	0.9502
						r = 20					
24	30.60	0.011	−0.0005	0.0012	1.035	0.0009	−0.0004	0.0012	−4.99	0.896	0.8667
43	44.61	0.006	−0.0004	0.0007	1.047	0.0005	−0.0004	0.0007	−19.65	0.772	0.9543
61	62.62	0.006	0.0001	0.0006	1.049	0.0005	0.0002	0.0006	−28.64	0.765	0.9522
76	77.61	0.006	0.0000	0.0005	1.051	0.0004	0.0000	0.0005	−36.14	0.762	0.9535
96	97.62	0.006	−0.0001	0.0005	1.051	0.0004	−0.0001	0.0005	−46.13	0.760	0.9526
125	126.63	0.006	−0.0011	0.0004	1.052	0.0003	−0.0010	0.0004	−60.63	0.757	0.9518
171	172.62	0.006	−0.0004	0.0004	1.052	0.0003	−0.0005	0.0003	−83.63	0.755	0.9511
246	247.62	0.006	0.0003	0.0003	1.053	0.0002	0.0003	0.0003	−121.13	0.754	0.9508
500	501.49	0.006	−0.0002	0.0002	1.054	0.0002	−0.0002	0.0002	−248.26	0.752	0.9503

n^* indicates the optimal sample size.

- $\overline{\eta} = L^{-1}\sum_1^L \hat{\eta}_i$ is the estimated mean of the sample inverse coefficient of variation across replicates.

F. The simulated standard errors are

- $S_{\overline{N}} = \left(L^2 - L\right)^{\frac{-1}{2}} \left\{\sum_1^L \left(N_i^* - \overline{N}\right)^2\right\}^{\frac{-1}{2}}$, $S_{\hat{\mu}} = \left(L^2 - L\right)^{\frac{-1}{2}} \left\{\sum_1^L \left(\overline{X}_i - \overline{X}\right)^2\right\}^{\frac{-1}{2}}$,
- $S_{\hat{\sigma}} = \left(L^2 - L\right)^{\frac{-1}{2}} \left\{\sum_1^L \left(\hat{\sigma}_i - \overline{\sigma}\right)^2\right\}^{\frac{-1}{2}}$ and $S_{\overline{\eta}} = \left(L^2 - L\right)^{\frac{-1}{2}} \left\{\sum_1^L \left(\hat{\eta}_i - \overline{\eta}\right)^2\right\}^{\frac{-1}{2}}$.

G. The simulated regret is $\hat{\omega}\left(d\right) = AL^{-1}\left\{\sum_1^L \left(\hat{\eta}_i - \overline{\eta}\right)^2\right\} + c\overline{N} - R\left(n^*\right)$.

H. The simulated relative risk $\hat{\nu}\left(d\right)$

I. The simulated coverage probability is

- $(1 \hat{-} \alpha) = \frac{\#(\hat{\eta}_i - d < \eta < \hat{\eta}_i + d)}{L}$, $i = 1, \ldots, L$

J. The simulated controlled operating characteristic function

$\hat{\beta}_{kc} = \frac{\#(\hat{\eta}_i + kd < \eta < \hat{\eta}_i + (2+k)d)}{L}$, $i = 1, \ldots, L$; $k = 0\ (0.1)\ 1, 1.5$ and 2

The study covers two points; the performance of the procedure at fixed m and the performance of the procedure as m changes from $m = 8,\ 10,\ 15$, and 20.

Simulation Experiment and Results

To conduct the simulation study, a series of $L = 50,000$ replications were generated from $N(\mu, \sigma)$, with $\mu = 10$, $\sigma = 0.5,\ 1.0,\ 2.0,\ \frac{10}{3}, 5,\ 20/3$ and 10 provided $\eta = 20,\ 10,\ 5,\ 3,\ 2,\ 1.5,\ 1$ and 0.5.

The optimal sample sizes are chosen to represent small, medium to large performance, that is

$n^* = 24,\ 43,\ 61,\ 76,\ 96,\ 125,\ 171,\ 246$, and 500.

While the design factor is chosen to be $\gamma = 0.5$, and the pilot samples are taken $m = 8,\ 10, 15$, and 20. As small to moderate pilot samples. For brevity, we consider $\alpha = 5\%$, which gives $a = 1.96$.

TABLE 5 | Three–stage estimation of the inverse coefficient of variation for underlying *Beta* (0.5, 0.5) and (1, 1) $m = 15$, $\gamma = 0.5$, $\alpha = 5\%$.

						Beta (0.5, 0.5)					
n^*	$\bar{N}$	$S_{\bar{N}}$	$\hat{\mu}$	$S_{\hat{\mu}}$	$\hat{\sigma}$	$S_{\hat{\sigma}}$	$\hat{\eta}$	$S_{\hat{\eta}}$	$\hat{\omega}$	$\hat{\nu}$	$1-\hat{\alpha}$
24	33.04	0.022	0.4891	0.0004	0.356	0.0001	1.381	0.0010	−3.08	0.936	0.9886
43	43.51	0.033	0.4934	0.0003	0.354	0.0001	1.399	0.0008	−21.05	0.755	0.9788
61	61.42	0.040	0.4960	0.0002	0.354	0.0001	1.405	0.0006	−30.12	0.752	0.9866
76	76.44	0.045	0.4966	0.0002	0.354	0.0001	1.406	0.0006	−37.64	0.752	0.9869
96	96.42	0.051	0.4975	0.0002	0.354	0.0001	1.408	0.0005	−47.64	0.752	0.9871
125	125.49	0.064	0.4978	0.0001	0.354	0.0001	1.409	0.0004	−62.09	0.752	0.9867
171	171.59	0.072	0.4985	0.0001	0.354	0.0000	1.410	0.0004	−84.99	0.752	0.9875
246	246.76	0.088	0.4989	0.0001	0.354	0.0000	1.411	0.0003	−122.34	0.752	0.9872
500	501.53	0.136	0.4995	0.0001	0.354	0.0000	1.413	0.0002	−248.48	0.751	0.9874
						Beta (1, 1)					
24	33.38	0.022	0.4913	0.0003	0.291	0.0001	1.701	0.0012	−2.72	0.940	0.9873
43	43.88	0.035	0.4939	0.0002	0.290	0.0001	1.713	0.0009	−20.72	0.759	0.9624
61	61.50	0.044	0.4969	0.0002	0.289	0.0001	1.723	0.0007	−30.02	0.754	0.9855
76	76.59	0.050	0.4975	0.0001	0.289	0.0001	1.725	0.0007	−37.43	0.754	0.9852
96	96.56	0.057	0.4980	0.0001	0.289	0.0001	1.726	0.0006	−47.48	0.753	0.9856
125	125.71	0.066	0.4984	0.0001	0.289	0.0001	1.727	0.0005	−61.83	0.753	0.9856
171	171.80	0.077	0.4988	0.0001	0.289	0.0000	1.728	0.0004	−84.75	0.752	0.9853
246	247.07	0.097	0.4992	0.0001	0.289	0.0000	1.730	0.0004	−121.96	0.752	0.9859
500	501.75	0.154	0.4996	0.0001	0.289	0.0000	1.731	0.0003	−248.32	0.752	0.9854

n^* indicates the optimal sample size.

Table 1 demonstrates the simulation results at $m = 15$ as the optimal sample size increases. We noticed the following

1. Regarding the final sample size N; $\overline{N} > n^*$ for all values of n^* and the absolute difference between $\overline{N}$ and n^* reduces as the optimal sample size increases. The simulated standard errors increase as n^* increases.
2. Regarding the population mean, the simulated mean converges asymptotically to the population mean. That is $\hat{\mu}$ is asymptotically unbiased estimator to μ. The standard errors decrease as n^* increases.
3. Regarding the population standard variance, the estimates converge to the population standard deviation asymptotically. $\hat{\sigma}$ is asymptotically unbiased to σ. The standard errors decrease as n^* increases.
4. The simulated regret has negative values, which indicates that the three-stage procedure provides estimates for the population inverse coefficient of variation better than the optimal had n^* been known.
5. Regarding the relative risk, the simulation results reveal that
   - For fixed n^* as η decreases the estimated values $\hat{\nu}(d)$ decreases slightly.
   - For fixed η as n^* increases the simulated values $\hat{\nu}(d)$ decreases.

a. At $n^* = 500$, $\hat{\nu}(d)$ converges asymptotically to 0.759 at $\eta = 20$ and 10 and approaches 0.752 at $\eta = 1$. Even for $\eta < 1$, $\hat{\nu}(d)$ will converge asymptotically to 0.752.

6. Regarding the relative risk, the simulated $\hat{\nu}(d)$ converges asymptotically to nearly 0.75. This implies that the sequential risk is 25% less than the optimal risk.
7. Regarding the simulated coverage probability, the three-stage procedure attains the desired nominal value asymptotically (asymptotic consistency). The coverage improves as n^* increases. **Figure 1** shows the performance of the coverage probability as the optimal sample size increases at $\mu = 10$ while $\sigma = 0.5, 1, 2, \frac{10}{3}, 5, \frac{20}{3}, 10$ and 20. In other words, as η decreases.

Now let us record the impact of increasing m on the performance of the procedure. To do so, we present **Table 2**. We noticed the following

1. At $n^* = 24$ and 43: As the pilot sample increases the absolute difference between the optimal sample size and the simulated final sample size increases, while at $n^* = 61$, 76, 96, and 125, the absolute difference decreases slightly. At $n^* = 171$, 246, and 500, the absolute difference decreases significantly to approach the desired optimal sample size. The corresponding standard deviations decreases.
2. For fixed n^*, as the pilot sample increases, the simulated mean is nearly approaching the population mean. While the estimations become better as the optimal sample size increases with decreased standard deviation.
3. The simulated standard deviation approaches the population standard deviation as the optimal sample size increases. The biased decreases as the optimal sample size increases.
4. In general, the simulated regret decreases as the pilot sample increases except at n^* =24.
5. **Figure 2** demonstrates the performance of the simulated coverage probability as m increases.

Table 3 shows the asymptotic coverage probability and the asymptotic characteristic operating function under the uncontrolled optimal sample size given in (3). The procedure attains the desired nominal value, only asymptotically. As k increases the simulated $\hat{\beta}_k$ decreases and approaches zero at $k = 2$. The other part of **Table 3** shows the results under the controlled optimal sample size given in (16) against type II error probability. The procedure exceeds the desired nominal value, even for small optimal sample sizes. The simulated $\hat{\beta}_{kC}$ decreases significantly to zero and $\hat{\beta}_{kc} < \hat{\beta}_k$. This indicates that by controlling the confidence interval against type II error probability, the procedure becomes more sensitive toward a remarkable shift. **Figures 3**, **4** reflect the previous comments.

THE SENSITIVITY OF THE NORMAL-BASED THREE-STAGE PROCEDURE FOR UNDERLYING DISTRIBUTION

Assume we need to estimate the population inverse coefficient of variation for a class of non-normal distributions using the normal-based optimal sample size in (3). How sensitive is the three-stage procedure toward estimation? To examine this and without loss of generality, we consider two families of underlying distributions; the student's $\mathbf{t}\,(\mathbf{r})$, $\mathbf{r}$=**5, 10** and 20, $\mathbf{r}$ indicates the degrees of freedom and the family of beta distribution; Beta(**0.5, 0.5**) and Beta(**1, 1**). **Table 4** shows the asymptotic results for the **t**-distribution with the selected degrees of freedom. We obtain better estimation for all parameters, and the procedure satisfies all asymptotic measures except consistency. The simulated relative risk $\hat{\boldsymbol{\nu}}\,(\mathbf{d})$ converges asymptotically to 0.752. **Table 5** shows the results for the beta distribution. Here the three-stage procedure provides coverage probabilities that exceed the prescribed nominal value. This may resort to the structural behavior of the beta distribution since it belongs to uniform power series functions. Again $\hat{\boldsymbol{\nu}}\,(\mathbf{d})$ approaches nearly to 0.752.

CONCLUSION

We examine the performance of the three-stage procedure for estimating the population inverse coefficient of variation of the normal distribution. We estimated all parameters in concern and found that the three-stage procedure attains efficiency and asymptotic consistency as the width of the interval approaches zero. By controlling the confidence intervals against type II error probabilities, the procedure provides coverage probabilities that exceed the prescribed nominal value and becomes more sensitive toward any potential shift that may occur in the population inverse coefficient of variation. Regarding the sensitivity of the procedure as the underlying distribution departs away from normality, we found that the three-stage procedure is robust for likewise normal distributions.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/supplementary material.

AUTHOR CONTRIBUTIONS

AY did all the study, wrote Fortran programs and ran them using Microsoft developer studio software, and wrote the draft and the final form of the paper.

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Conflict of Interest: The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.