



# Application of New Iterative Method to Time Fractional Whitham–Broer–Kaup Equations

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### Specialty section:

This article was submitted to  
Mathematical Physics,  
a section of the journal  
Frontiers in Physics

**Received:** 21 December 2019

**Accepted:** 20 March 2020

**Published:** 13 May 2020

### Citation:

Nawaz R, Kumam P, Farid S,  
Shutaywi M, Shah Z and Deebani W  
(2020) Application of New Iterative  
Method to Time Fractional  
Whitham–Broer–Kaup Equations.  
Front. Phys. 8:104.  
doi: 10.3389/fphy.2020.00104

This article presents the fractional Laplace transform with the help of new iterative method (NIM) is extended for an estimated solution of coupled system of fractional order PDEs. The time fractional Whitham–Broer–Kaup system is taken as a test example where derivatives are given in the Caputo sense. Numerical results found by the proposed method are compared with that of ADM, VIM, and OHAM. Numerical consequences display that the proposed method is reliable and operative for solution of fractional order coupled system of PDEs. The proposed method shows better accuracy in even two iterations compared to the methods given above.

**Keywords:** fractional Whitham–Broer–Kaup equations, coupled system of time fractional PDEs, new iterative method, fractional calculus (FC), Whitham–Broer–Kaup system, Caputo sense, ADM, VIM and OHAM

## INTRODUCTION

As we know that many technical and engineering issues that arises in day-by-day existence are modeled via mathematical tools form fractional calculus (FC), i.e., fractional calculus can be used to simulate various real phenomena involving long memory, e.g., using fractional derivative, one can model HIV/AIDS model based on the effect of screening of unaware infectives [1]. Maximum problems that arise are non-linear, and it is not usually probable to locate systematic results of such problems since some researchers introduced new approaches for finding the exact solution of FPDEs [2]. However, these methods also have some drawbacks, and we cannot use it for any type of problems. To fulfill these need, researchers introduced many semi analytical techniques such as HPM [3], HPTM [4], HAM [5], FDM [6], RPSM [7], etc.

NIM was introduced by Daftardar-Gejji and Jafari in 2006 and is also known as the DJ method for the solution of non-linear equations. This method is the modification of ADM in which the complex Adomian polynomials are replaced by Jafari polynomials. Therefore, we have no need to compute tedious Adomian's polynomial in each iteration.

In this presentation, we have extended the applications of the DJ method to a solution of coupled WBK equations of fractional order using the fractional Laplace Transform. Using the Laplace

transform for fractional PDEs is effortless compared to the Riemann Liouville integral operator for fractional PDEs as well as a system of fractional PDEs.

The fractional-order WBK equations describe the propagation of shallow water waves [8] with different dispersion relations. The WBK equations are of the form:

$$D_t^\alpha u + uu_x + v_x + bu_{xx} = 0$$

$$D_t^\alpha v + (uv)_x + au_{xxx} - bv_{xx} = 0,$$

where  $u(x, t)$  denotes the horizontal velocity,  $v(x, t)$  is the height that deviates from the equilibrium position,  $a, b$  are real constants that are represented in different diffusion powers, and  $D_t^\alpha 0 < \alpha \leq 1$  is the Caputo derivative operator. For  $\alpha = 1$ , we get the usual WBK equations. It is also essential to show that when  $a = 1$  and  $b = 0$ , we have fractional order modified Boussinesq (MB) equation, and when  $a = 0, b = 1/2$ , we get the fractional order approximate long wave (ALW) equation. These equations took the attention of many researchers in recent decades [9–11].

The present paper is divided into five sections. The Fundamental Theory of Proposed Method section is devoted to the analysis of the DJ method as well as the implementation of the Laplace transform for fractional PDEs are given. In the Application of Laplace Transform with DJ method to Fractional Whitham-Broer-Kaup Equations section, the application of Laplace transform to FPDEs are given. In the Results and Discussion section, the results of the proposed method are compared with VIM, ADM, and OHAM solutions for time-fractional WBK, time fractional MB, and time-fractional ALW equations, while in the Conclusion section, the conclusion of the work is given.

## FUNDAMENTAL THEORY OF PROPOSED METHOD

### New Iterative Method [12–16]

Daftardar-Gejji and Jafari consider the following equation [12]:

Consider the equations of the form:

$$v_i = f_i + \varsigma_i(v_1, v_2) + \xi_i(v_1, v_2), \quad i = 1, 2. \tag{1}$$

where  $f_i$  are known functions,  $\varsigma_i, \xi_i$  are linear and non-linear functions of  $v_i$ . Assuming that equation (1) have a solution of the series form:

$$v_i = \sum_{j=0}^{\infty} v_{i,j}, \quad i = 1, 2. \tag{2}$$

Since  $\varsigma_i$  is linear, so we write it as:

$$\varsigma_i \left( \sum_{j=0}^{\infty} (v_{1,j}, v_{2,j}) \right) = \sum_{j=0}^{\infty} \varsigma_i (v_{1,j}, v_{2,j}), \tag{3}$$

Decomposition of non-linear operators is as follows:

$$\xi_i \left( \sum_{j=0}^{\infty} v_{i,j} \right) = \xi_i (v_{1,0}, v_{2,0})$$

$$+ \sum_{j=1}^{\infty} \left\{ \xi_i \left( \sum_{k=0}^j v_{1,k}, \sum_{k=0}^j v_{2,k} \right) - \xi_i \left( \sum_{k=0}^{j-1} v_{1,k}, \sum_{k=0}^{j-1} v_{2,k} \right) \right\},$$

$$= \sum_{j=0}^{\infty} G_{i,j}. \tag{4}$$

where  $G_{i,0} = \xi_i (v_{1,0}, v_{2,0})$  and  $G_{i,j} = \xi_i \left( \sum_{k=0}^j v_{1,k}, \sum_{k=0}^j v_{2,k} \right) - \xi_i \left( \sum_{k=0}^{j-1} v_{1,k}, \sum_{k=0}^{j-1} v_{2,k} \right), j \geq 1. \quad i = 1, 2.$

Hence, equation (1) is equivalent to:

$$\sum_{j=0}^{\infty} v_{i,j} = f_i + \sum_{j=0}^{\infty} \varsigma_i (v_{1,j}, v_{2,j}) + \sum_{j=0}^{\infty} G_{i,j}. \tag{5}$$

Further, the recurrence relation is defined as follows:

$$v_{i,0} = f_i,$$

$$v_{i,1} = \varsigma_i (v_{1,0}, v_{2,0}) + G_{i,0},$$

$$v_{i,2} = \varsigma_i (v_{1,1}, v_{2,1}) + G_{i,1},$$

$$\dots$$

$$v_{i,m+1} = \varsigma_i (v_{1,m}, v_{2,m}) + G_{i,m}, \quad m = 1, 2, \dots \tag{6}$$

The  $k$ th-order approximation is given by:

$$v_i = \sum_{j=0}^{k-1} v_{i,j}.$$

For convergence analysis, we refer to Daftardar-Gejji and Jafari [13] where explanatory example is solved.

### Laplace Transform and Fractional Partial Differential Equations [4]

Consider the following equations:

$$D_t^\alpha v_i(x, t) + \varsigma v_i(x, t) + \xi v_i(x, t) = 0, \tag{7}$$

$0 < \alpha \leq 1,$   
with ICs.

$$v_i(x, 0) = f_i(x). \tag{8}$$

where  $\varsigma$  is the linear operator,  $\xi$  is the non-linear operator, and  $D_t^\alpha v_i(x, t)$  is the Caputo fractional derivative of a function  $v_i(x, t)$ , which is defined as:

$$D_t^\alpha v_i(x, t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \left( \frac{v_i^n(x, \gamma)}{(t - \gamma)^{\alpha+1-n}} \right) d\gamma, \quad (9)$$

$$L[D_t^\alpha v_i(x, t)] + L[\zeta v_i(x, t)] + L[\xi v_i(x, t)] = 0. \quad (11)$$

$(n - 1 < \alpha \leq n, n \in \mathbb{N})$ .

Using the property of Laplace transform for Caputo fractional derivatives is:

Using equation (10), we have:

$$L[D_t^\alpha v_i] = s^\alpha L[v_i(x, t)] - \sum_{k=0}^{n-1} v_i^k(x, 0^+) s^{\alpha-1-k}. \quad (10)$$

$$L[v_i(x, t)] = \frac{1}{s} v_i(x, 0) - \frac{1}{s^\alpha} L[\zeta v_i(x, t)] - \frac{1}{s^\alpha} L[\xi v_i(x, t)]. \quad (12)$$

Taking the Laplace transform on both sides of equation (10) we get:

Taking the inverse Laplace transform on both sides of equation (12), we get:

**TABLE 1** | Second-order DJ solution for  $u(x, t)$  in comparison with ADM, VIM, and OHAM solutions at  $\alpha = 1$  for WBK equation.

$(x, t)$	Absolute error of ADM [17]	Absolute error of VIM [18]	Absolute error of OHAM [19]	Absolute error of 2nd-order NIM
(0.1,0.1)	$1.04892 \times 10^{-4}$	$1.23033 \times 10^{-4}$	$1.07078 \times 10^{-4}$	$1.67111 \times 10^{-12}$
(0.1,0.3)	$9.64474 \times 10^{-5}$	$3.69597 \times 10^{-4}$	$3.04565 \times 10^{-4}$	$4.51196 \times 10^{-11}$
(0.1,0.5)	$8.88312 \times 10^{-5}$	$6.16873 \times 10^{-4}$	$4.81303 \times 10^{-4}$	$2.08888 \times 10^{-10}$
(0.2,0.1)	$4.25408 \times 10^{-4}$	$1.19869 \times 10^{-4}$	$1.04388 \times 10^{-4}$	$1.57879 \times 10^{-12}$
(0.2,0.3)	$3.91098 \times 10^{-4}$	$3.60098 \times 10^{-4}$	$2.97260 \times 10^{-4}$	$4.26227 \times 10^{-11}$
(0.2,0.5)	$3.60161 \times 10^{-4}$	$6.01006 \times 10^{-4}$	$4.70138 \times 10^{-4}$	$1.97328 \times 10^{-10}$
(0.3,0.1)	$9.71922 \times 10^{-4}$	$1.16789 \times 10^{-4}$	$1.01776 \times 10^{-4}$	$1.49181 \times 10^{-12}$
(0.3,0.3)	$8.93309 \times 10^{-4}$	$3.50866 \times 10^{-4}$	$2.90150 \times 10^{-4}$	$4.02799 \times 10^{-11}$
(0.3,0.5)	$8.22452 \times 10^{-4}$	$5.85610 \times 10^{-4}$	$4.59590 \times 10^{-4}$	$1.86481 \times 10^{-10}$
(0.4,0.1)	$1.75596 \times 10^{-3}$	$1.13829 \times 10^{-4}$	$9.92418 \times 10^{-5}$	$1.41043 \times 10^{-12}$
(0.4,0.3)	$1.61430 \times 10^{-3}$	$3.41948 \times 10^{-4}$	$2.83229 \times 10^{-4}$	$3.80803 \times 10^{-11}$
(0.4,0.5)	$1.48578 \times 10^{-3}$	$5.70710 \times 10^{-4}$	$4.49118 \times 10^{-4}$	$1.76298 \times 10^{-10}$
(0.5,0.1)	$2.79519 \times 10^{-3}$	$1.10936 \times 10^{-4}$	$9.67808 \times 10^{-4}$	$1.33388 \times 10^{-12}$
(0.5,0.3)	$2.56714 \times 10^{-3}$	$3.33274 \times 10^{-4}$	$2.76492 \times 10^{-4}$	$3.60145 \times 10^{-11}$
(0.5,0.5)	$2.36184 \times 10^{-3}$	$5.56235 \times 10^{-4}$	$4.38895 \times 10^{-4}$	$1.66734 \times 10^{-10}$

**TABLE 2** | Second-order DJ solution for  $u(x, t)$  in comparison with ADM, VIM, and OHAM solutions at  $\alpha = 1$  for MB equation.

$(x, t)$	Absolute error of ADM [17]	Absolute error of VIM [18]	Absolute error of OHAM [19]	Absolute error of 2nd-order NIM
(0.1,0.1)	$8.16297 \times 10^{-7}$	$6.35269 \times 10^{-5}$	$6.35267 \times 10^{-5}$	$4.57301 \times 10^{-13}$
(0.1,0.3)	$7.64245 \times 10^{-7}$	$1.90854 \times 10^{-4}$	$1.90854 \times 10^{-4}$	$1.23478 \times 10^{-11}$
(0.1,0.5)	$7.16083 \times 10^{-7}$	$3.18549 \times 10^{-4}$	$3.18548 \times 10^{-4}$	$5.71662 \times 10^{-11}$
(0.2,0.1)	$3.26243 \times 10^{-6}$	$6.18930 \times 10^{-5}$	$6.18931 \times 10^{-5}$	$4.32265 \times 10^{-13}$
(0.2,0.3)	$3.05458 \times 10^{-6}$	$1.85945 \times 10^{-4}$	$1.85945 \times 10^{-4}$	$1.16698 \times 10^{-11}$
(0.2,0.5)	$2.86226 \times 10^{-6}$	$3.10352 \times 10^{-4}$	$3.10352 \times 10^{-4}$	$5.40272 \times 10^{-11}$
(0.3,0.1)	$7.33445 \times 10^{-6}$	$6.03095 \times 10^{-5}$	$6.03098 \times 10^{-5}$	$4.08618 \times 10^{-13}$
(0.3,0.3)	$6.86758 \times 10^{-6}$	$1.81187 \times 10^{-4}$	$1.81187 \times 10^{-4}$	$1.10335 \times 10^{-11}$
(0.3,0.5)	$6.43557 \times 10^{-6}$	$3.02408 \times 10^{-4}$	$3.02408 \times 10^{-4}$	$5.10809 \times 10^{-11}$
(0.4,0.1)	$1.30286 \times 10^{-5}$	$5.87746 \times 10^{-5}$	$5.87749 \times 10^{-5}$	$3.86524 \times 10^{-13}$
(0.4,0.3)	$1.22000 \times 10^{-5}$	$1.76574 \times 10^{-4}$	$1.76574 \times 10^{-4}$	$1.04358 \times 10^{-11}$
(0.4,0.5)	$1.14333 \times 10^{-5}$	$2.94707 \times 10^{-4}$	$2.94708 \times 10^{-4}$	$4.83143 \times 10^{-11}$
(0.5,0.1)	$2.03415 \times 10^{-5}$	$5.72867 \times 10^{-5}$	$5.72865 \times 10^{-4}$	$3.65707 \times 10^{-13}$
(0.5,0.3)	$1.90489 \times 10^{-5}$	$1.72102 \times 10^{-4}$	$1.72102 \times 10^{-4}$	$9.87438 \times 10^{-12}$
(0.5,0.5)	$1.78528 \times 10^{-5}$	$2.87241 \times 10^{-4}$	$2.87240 \times 10^{-4}$	$4.5715 \times 10^{-11}$

$$v_i(x, t) = v_i(x, 0) - L^{-1}\left[\frac{1}{s^\alpha}L[\zeta v_i(x, t)]\right] - L^{-1}\left[\frac{1}{s^\alpha}L[\xi v_i(x, t)]\right]. \tag{13}$$

$$\begin{aligned} D_t^\alpha u + uu_x + v_x + bu_{xx} &= 0, \\ D_t^\alpha v + (uv)_x + au_{xxx} - bv_{xx} &= 0. \end{aligned} \tag{14}$$

Subject to ICs

Now, we apply a new iterative technique that was derived in the New Iterative Method section.

### APPLICATION OF LAPLACE TRANSFORM WITH DJ METHOD TO FRACTIONAL WHITHAM-BROER-KAUP EQUATIONS

$$\begin{aligned} u(x, 0) &= \lambda - 2Bk \coth(k\xi), \\ v(x, 0) &= -2B(B + b)k^2 \operatorname{csch}^2(k\xi), \end{aligned} \tag{15}$$

where  $\beta = \sqrt{a + b^2}$ ,  $\xi = x + \text{cand } \lambda, c, k$ , are any constants. For  $\alpha = 1$ , the exact solution of the system is as follows:

#### Problem 3.1: Time Fractional WBK Equation

**TABLE 3 |** Second-order DJ solution for  $u(x, t)$  in comparison with ADM, VIM, and OHAM solutions at  $\alpha = 1$  for ALW equation.

$(x, t)$	Absolute error of ADM [17]	Absolute error of VIM [18]	Absolute error of OHAM [19]	Absolute error of 2nd-order NIM
(0.1,0.1)	$8.02989 \times 10^{-6}$	$3.17634 \times 10^{-5}$	$3.17634 \times 10^{-5}$	$1.20348 \times 10^{-13}$
(0.1,0.3)	$7.38281 \times 10^{-6}$	$9.54273 \times 10^{-5}$	$9.54269 \times 10^{-5}$	$3.25026 \times 10^{-12}$
(0.1,0.5)	$6.79923 \times 10^{-6}$	$1.59274 \times 10^{-4}$	$1.59274 \times 10^{-4}$	$1.50478 \times 10^{-11}$
(0.2,0.1)	$3.23228 \times 10^{-5}$	$3.09466 \times 10^{-5}$	$3.09465 \times 10^{-5}$	$1.13895 \times 10^{-13}$
(0.2,0.3)	$2.97172 \times 10^{-5}$	$9.29725 \times 10^{-5}$	$9.29723 \times 10^{-5}$	$3.07447 \times 10^{-12}$
(0.2,0.5)	$2.73673 \times 10^{-5}$	$1.55176 \times 10^{-4}$	$1.55176 \times 10^{-4}$	$1.42339 \times 10^{-11}$
(0.3,0.1)	$7.32051 \times 10^{-5}$	$3.01549 \times 10^{-5}$	$3.01549 \times 10^{-5}$	$1.07747 \times 10^{-13}$
(0.3,0.3)	$6.73006 \times 10^{-5}$	$9.05935 \times 10^{-5}$	$9.05932 \times 10^{-5}$	$2.90939 \times 10^{-12}$
(0.3,0.5)	$6.19760 \times 10^{-5}$	$1.51204 \times 10^{-4}$	$1.51204 \times 10^{-4}$	$1.34695 \times 10^{-11}$
(0.4,0.1)	$1.31032 \times 10^{-4}$	$2.93874 \times 10^{-5}$	$2.93874 \times 10^{-5}$	$1.02029 \times 10^{-13}$
(0.4,0.3)	$1.20455 \times 10^{-4}$	$8.82871 \times 10^{-5}$	$8.82870 \times 10^{-5}$	$2.75424 \times 10^{-12}$
(0.4,0.5)	$1.10919 \times 10^{-4}$	$1.47354 \times 10^{-4}$	$1.47354 \times 10^{-4}$	$1.27514 \times 10^{-11}$
(0.5,0.1)	$2.06186 \times 10^{-4}$	$2.86433 \times 10^{-5}$	$2.86432 \times 10^{-5}$	$9.66033 \times 10^{-14}$
(0.5,0.3)	$1.89528 \times 10^{-4}$	$8.60509 \times 10^{-5}$	$8.60506 \times 10^{-5}$	$2.60846 \times 10^{-12}$
(0.5,0.5)	$1.74510 \times 10^{-4}$	$1.43620 \times 10^{-4}$	$1.43620 \times 10^{-4}$	$1.20763 \times 10^{-11}$

**TABLE 4 |** Second-order DJ solution for  $v(x, t)$  in comparison with ADM, VIM, and OHAM solutions at  $\alpha = 1$  for WBK equation.

$(x, t)$	Absolute error of ADM [17]	Absolute error of VIM [18]	Absolute error of OHAM [19]	Absolute error of 2nd-order NIM
(0.1,0.1)	$6.41419 \times 10^{-3}$	$1.10430 \times 10^{-4}$	$5.86860 \times 10^{-5}$	$3.28081 \times 10^{-12}$
(0.1,0.3)	$5.99783 \times 10^{-3}$	$3.31865 \times 10^{-4}$	$3.04565 \times 10^{-4}$	$8.85812 \times 10^{-11}$
(0.1,0.5)	$5.61507 \times 10^{-3}$	$5.54071 \times 10^{-4}$	$3.08812 \times 10^{-4}$	$4.10099 \times 10^{-10}$
(0.2,0.1)	$1.33181 \times 10^{-2}$	$1.07016 \times 10^{-4}$	$5.56884 \times 10^{-5}$	$3.07768 \times 10^{-12}$
(0.2,0.3)	$1.24441 \times 10^{-2}$	$3.21601 \times 10^{-4}$	$2.97260 \times 10^{-4}$	$8.30963 \times 10^{-11}$
(0.2,0.5)	$1.16416 \times 10^{-2}$	$5.36927 \times 10^{-4}$	$2.92626 \times 10^{-4}$	$3.84706 \times 10^{-10}$
(0.3,0.1)	$2.07641 \times 10^{-2}$	$1.03737 \times 10^{-4}$	$5.28609 \times 10^{-5}$	$2.88849 \times 10^{-12}$
(0.3,0.3)	$1.93852 \times 10^{-2}$	$3.11737 \times 10^{-4}$	$2.90150 \times 10^{-4}$	$7.79908 \times 10^{-11}$
(0.3,0.5)	$1.81209 \times 10^{-2}$	$5.20447 \times 10^{-4}$	$2.77382 \times 10^{-4}$	$3.6107 \times 10^{-10}$
(0.4,0.1)	$2.88100 \times 10^{-2}$	$1.00579 \times 10^{-4}$	$5.01929 \times 10^{-5}$	$2.71246 \times 10^{-12}$
(0.4,0.3)	$2.68724 \times 10^{-2}$	$3.02245 \times 10^{-4}$	$2.83229 \times 10^{-4}$	$7.32356 \times 10^{-11}$
(0.4,0.5)	$2.50985 \times 10^{-2}$	$5.04593 \times 10^{-4}$	$2.63019 \times 10^{-4}$	$3.39055 \times 10^{-10}$
(0.5,0.1)	$3.75193 \times 10^{-2}$	$9.75385 \times 10^{-5}$	$4.76741 \times 10^{-5}$	$2.54828 \times 10^{-12}$
(0.5,0.3)	$3.49617 \times 10^{-2}$	$2.93107 \times 10^{-4}$	$2.76492 \times 10^{-4}$	$6.88039 \times 10^{-11}$
(0.5,0.5)	$3.26239 \times 10^{-2}$	$4.89335 \times 10^{-4}$	$2.49480 \times 10^{-4}$	$3.18537 \times 10^{-10}$

Now, using the basic idea of the DJ method discussed in the Fundamental Theory of Proposed Method section, we have:

$$\begin{aligned}
 u(x, t) &= \lambda - 2Bk \coth(k(\xi - \lambda t)), \\
 v(x, t) &= -2B(B + b)k^2 \operatorname{csch}^2(k(\xi - \lambda t)).
 \end{aligned}
 \tag{16}$$

Applying Laplace transform and inverse Laplace transform to equation (3.1), we have:

$$\begin{aligned}
 u(x, t) &= u(x, 0) \\
 &+ L^{-1} \left[ \frac{1}{s^\alpha} L[-(u(x, t)u_x(x, t) + v_x(x, t) + bu_{xx}(x, t))] \right], \\
 v(x, t) &= v(x, 0) + L^{-1} \left[ \frac{1}{s^\alpha} L[-((u(x, t)v(x, t))_x \right. \\
 &\left. + au_{xxx}(x, t) - bv_{xx}(x, t))] \right].
 \end{aligned}
 \tag{17}$$

$$\begin{aligned}
 u_0 &= \lambda - 2Bk \coth(k\xi), \\
 v_0 &= -2B(B + b)k^2 \operatorname{csch}^2(k\xi),
 \end{aligned}
 \tag{18}$$

$$u_1 = \frac{-2Bk^2 t^\alpha \lambda \operatorname{csc} h^2(k(x + c))}{\Gamma(1 + \alpha)},
 \tag{19}$$

$$\begin{aligned}
 v_1 &= \frac{1}{\Gamma(1 + \alpha)} 4Bk^3 t^\alpha \operatorname{csc} h^2(k(x + c))(- (b + B)\lambda \coth(k(x + c)) \\
 &- (a + b^2 - B^2)k(2 + 3 \operatorname{csc} h^2(k(c + x))),
 \end{aligned}
 \tag{20}$$

**TABLE 5** | Second-order DJ solution for  $v(x, t)$  in comparison with ADM, VIM, and OHAM solutions at  $\alpha = 1$  for MB equation.

$(x, t)$	Absolute error of ADM [17]	Absolute error of VIM [18]	Absolute error of OHAM [19]	Absolute error of 2nd-order NIM
(0.1,0.1)	$5.88676 \times 10^{-5}$	$1.65942 \times 10^{-5}$	$1.65942 \times 10^{-5}$	$2.59213 \times 10^{-13}$
(0.1,0.3)	$5.56914 \times 10^{-5}$	$4.98691 \times 10^{-5}$	$4.98691 \times 10^{-5}$	$6.99872 \times 10^{-12}$
(0.1,0.5)	$5.27169 \times 10^{-5}$	$8.32598 \times 10^{-5}$	$8.26491 \times 10^{-4}$	$3.24016 \times 10^{-11}$
(0.2,0.1)	$1.18213 \times 10^{-4}$	$1.60813 \times 10^{-5}$	$1.60812 \times 10^{-5}$	$2.43233 \times 10^{-13}$
(0.2,0.3)	$1.11833 \times 10^{-4}$	$4.83269 \times 10^{-5}$	$4.83269 \times 10^{-5}$	$6.56712 \times 10^{-12}$
(0.2,0.5)	$1.05858 \times 10^{-4}$	$8.06837 \times 10^{-5}$	$7.94290 \times 10^{-4}$	$3.04035 \times 10^{-11}$
(0.3,0.1)	$1.78041 \times 10^{-4}$	$1.55880 \times 10^{-5}$	$1.55880 \times 10^{-5}$	$2.28336 \times 10^{-13}$
(0.3,0.3)	$1.68429 \times 10^{-4}$	$4.68440 \times 10^{-5}$	$4.68439 \times 10^{-5}$	$6.16531 \times 10^{-12}$
(0.3,0.5)	$1.59428 \times 10^{-4}$	$7.82068 \times 10^{-5}$	$7.63646 \times 10^{-4}$	$2.85432 \times 10^{-11}$
(0.4,0.1)	$2.38356 \times 10^{-4}$	$1.51135 \times 10^{-5}$	$1.51135 \times 10^{-5}$	$2.14485 \times 10^{-13}$
(0.4,0.3)	$2.25483 \times 10^{-4}$	$4.54174 \times 10^{-5}$	$4.54174 \times 10^{-5}$	$5.79099 \times 10^{-12}$
(0.4,0.5)	$2.13430 \times 10^{-4}$	$7.58243 \times 10^{-5}$	$7.34471 \times 10^{-4}$	$2.68103 \times 10^{-11}$
(0.5,0.1)	$2.99162 \times 10^{-4}$	$1.46569 \times 10^{-5}$	$1.46569 \times 10^{-5}$	$2.01559 \times 10^{-13}$
(0.5,0.3)	$2.83001 \times 10^{-4}$	$4.40448 \times 10^{-5}$	$4.40448 \times 10^{-5}$	$5.44208 \times 10^{-12}$
(0.5,0.5)	$2.67868 \times 10^{-4}$	$7.35317 \times 10^{-5}$	$7.06678 \times 10^{-4}$	$2.51949 \times 10^{-11}$

**TABLE 6** | Second-order DJ solution for  $v(x, t)$  in comparison with second-order ADM, VIM, and OHAM solutions at  $\alpha = 1$  for ALW equation.

$(x, t)$	Absolute error of ADM [17]	Absolute error of VIM [18]	Absolute error of OHAM [19]	Absolute error of 2nd-order NIM
(0.1,0.1)	$4.81902 \times 10^{-4}$	$8.29712 \times 10^{-6}$	$8.29711 \times 10^{-6}$	$6.71962 \times 10^{-14}$
(0.1,0.3)	$4.50818 \times 10^{-4}$	$2.49346 \times 10^{-5}$	$2.49345 \times 10^{-5}$	$1.81427 \times 10^{-12}$
(0.1,0.5)	$4.22221 \times 10^{-4}$	$4.16299 \times 10^{-5}$	$4.16298 \times 10^{-5}$	$8.39947 \times 10^{-12}$
(0.2,0.1)	$9.76644 \times 10^{-4}$	$8.04063 \times 10^{-6}$	$8.04063 \times 10^{-6}$	$6.30876 \times 10^{-14}$
(0.2,0.3)	$9.13502 \times 10^{-4}$	$2.41634 \times 10^{-5}$	$2.41634 \times 10^{-5}$	$1.70328 \times 10^{-12}$
(0.2,0.5)	$8.55426 \times 10^{-4}$	$4.03419 \times 10^{-5}$	$4.03418 \times 10^{-5}$	$7.88563 \times 10^{-12}$
(0.3,0.1)	$1.48482 \times 10^{-3}$	$7.79401 \times 10^{-6}$	$7.79400 \times 10^{-6}$	$5.92521 \times 10^{-14}$
(0.3,0.3)	$1.38858 \times 10^{-3}$	$2.34220 \times 10^{-5}$	$2.34219 \times 10^{-5}$	$1.59992 \times 10^{-12}$
(0.3,0.5)	$1.30009 \times 10^{-3}$	$3.91034 \times 10^{-5}$	$3.91034 \times 10^{-5}$	$7.40708 \times 10^{-12}$
(0.4,0.1)	$2.00705 \times 10^{-3}$	$7.55675 \times 10^{-6}$	$7.55675 \times 10^{-6}$	$5.56907 \times 10^{-14}$
(0.4,0.3)	$1.87661 \times 10^{-3}$	$2.27087 \times 10^{-5}$	$2.27087 \times 10^{-5}$	$1.50359 \times 10^{-12}$
(0.4,0.5)	$1.75670 \times 10^{-3}$	$3.79121 \times 10^{-5}$	$3.79121 \times 10^{-5}$	$6.96112 \times 10^{-12}$
(0.5,0.1)	$2.54396 \times 10^{-3}$	$7.32847 \times 10^{-6}$	$7.32846 \times 10^{-6}$	$5.23618 \times 10^{-14}$
(0.5,0.3)	$2.37815 \times 10^{-3}$	$2.20224 \times 10^{-5}$	$2.20224 \times 10^{-5}$	$1.41377 \times 10^{-12}$
(0.5,0.5)	$2.22578 \times 10^{-3}$	$3.67658 \times 10^{-5}$	$3.67658 \times 10^{-5}$	$6.54527 \times 10^{-12}$

$$u_2 = \frac{1}{(\Gamma(1 + \alpha))^2 \Gamma(1 + 2\alpha) \Gamma(1 + 3\alpha)} (2Bk^3 t^{2\alpha} \coth(k(x + c)) \csc h^4(k(x + c)) (4Bk^2 t^\alpha \lambda^2 (\Gamma(1 + 2\alpha))^2 + (-20(a + b^2 - B^2)k^2 + \lambda^2 - (4(a + b^2 - B^2)k^2 + \lambda^2) \cos h(2k(x + c)))) (\Gamma(1 + \alpha))^2 \Gamma(1 + 3\alpha)), \tag{21}$$

$$v_2 = Bk^4 t^{2\alpha} \csc h^6(k(x + c)) \left[ \frac{1}{\sqrt{\pi} \Gamma(1 + \alpha) \Gamma(1 + 3\alpha)} 2^{3+2\alpha} Bk^2 t^\alpha \lambda \csc h(k(x + c)) \right]$$

$$\begin{aligned} & \Gamma\left(\frac{1}{2} + \alpha\right) (16(a + b^2 - B^2)k \cosh(k(x + c)) \\ & + 2(a + b^2 - B^2)k \cosh(3k(x + c)) \\ & + (b + B)\lambda(2 \sinh(k(c + x)) + \sinh(3k(x + c)))) \\ & - \frac{1}{\Gamma(1 + 2\alpha)} (12(11b - 5B)(a + b^2 - B^2)k^2 \\ & - 3(b + B)\lambda^2 + 2(4(13b - 7B)(a + b^2 - B^2)k^2 \\ & + (b + B)\lambda^2) \cosh(2k(x + c)) + (4(b - B)(a + b^2 - B^2)k^2 \\ & + (b + B)\lambda^2) \cosh(4k(x + c)) \\ & + 4(a + b^2 - B^2)k\lambda(10 \sinh(2k(x + c)) + \sinh(4k(x + c))))]. \end{aligned} \tag{22}$$

Three terms approximate the solution for equation (14):

$$\begin{aligned} u &= u_0 + u_1 + u_2, \\ v &= v_0 + v_1 + v_2. \end{aligned} \tag{23}$$

We take  $k = 0.1, \lambda = 0.005, a = b = 1.5$  and  $c = 10$  in the above problem.

**Problem 3.2:** Time Fractional MB Equation

$$\begin{aligned} D_t^\alpha u + uu_x + v_x &= 0, \\ D_t^\alpha v + (uv)_x + u_{xxx} &= 0, \end{aligned} \tag{24}$$

Subject to ICs

$$\begin{aligned} u(x, 0) &= \lambda - 2k \coth(k\xi), \\ v(x, 0) &= -2k^2 \csc h^2(k\xi). \end{aligned} \tag{25}$$

where  $\xi = x + c$  and  $k, \lambda, c$  are arbitrary constants.

For  $\alpha = 1$ , the exact solution of the system is as follows:

$$\begin{aligned} u(x, t) &= \lambda - 2k \coth(k(\xi - \lambda t)), \\ v(x, t) &= -2k^2 \csc h^2(k(\xi - \lambda t)). \end{aligned} \tag{26}$$

According to the DJ method described in the Fundamental Theory of Proposed Method section, we have:

$$\begin{aligned} u(x, t) &= u(x, 0) + L^{-1} \left[ \frac{1}{S^\alpha} L[-(u(x, t)u_x(x, t) + v_x(x, t))] \right], \\ v(x, t) &= v(x, 0) + L^{-1} \left[ \frac{1}{S^\alpha} L[-((u(x, t)v(x, t))_x + u_{xxx}(x, t))] \right], \end{aligned} \tag{27}$$

so that

$$\begin{aligned} u_0 &= \lambda - 2k \coth(k(x + c)), \\ v_0 &= -2k^2 \csc h^2(k(x + c)), \end{aligned} \tag{28}$$

$$u_1 = -\frac{2k^2 t^\alpha \lambda \csc h^2(k(x + c))}{\Gamma(1 + \alpha)}, \tag{29}$$

$$v_1 = -\frac{4k^3 t^\alpha \lambda \coth(k(x + c)) \csc h^2(k(x + c))}{\Gamma(1 + \alpha)}, \tag{30}$$

$$\begin{aligned} u_2 &= \frac{2k^3 t^{2\alpha} \lambda^2 \csc h^4(k(x + c))}{\Gamma(1 + 2\alpha)} \left\{ \frac{4k^2 t^\alpha \coth(k(x + c)) (\Gamma(1 + 2\alpha))^2}{(\Gamma(1 + \alpha))^2 \Gamma(1 + 3\alpha)} \right. \\ & \left. - \sinh(2k(x + c)) \right\}, \end{aligned} \tag{31}$$

$$\begin{aligned} v_2 &= \frac{4k^4 t^{2\alpha} \lambda^2 \csc h^4(k(x + c))}{\Gamma(1 + 2\alpha)} \{-2 - \cosh(2k(x + c)) + \\ & \frac{2k^2 t^\alpha (3 + 2 \cosh(2k(x + c))) \csc h^2(k(x + c)) (\Gamma(1 + 2\alpha))^2}{(\Gamma(1 + \alpha))^2 \Gamma(1 + 3\alpha)}\}. \end{aligned} \tag{32}$$

Three terms approximate the solution for equation (25):

$$\begin{aligned} u &= u_0 + u_1 + u_2, \\ v &= v_0 + v_1 + v_2. \end{aligned} \tag{33}$$

**Problem 3.3:** Time Fractional ALW Equation

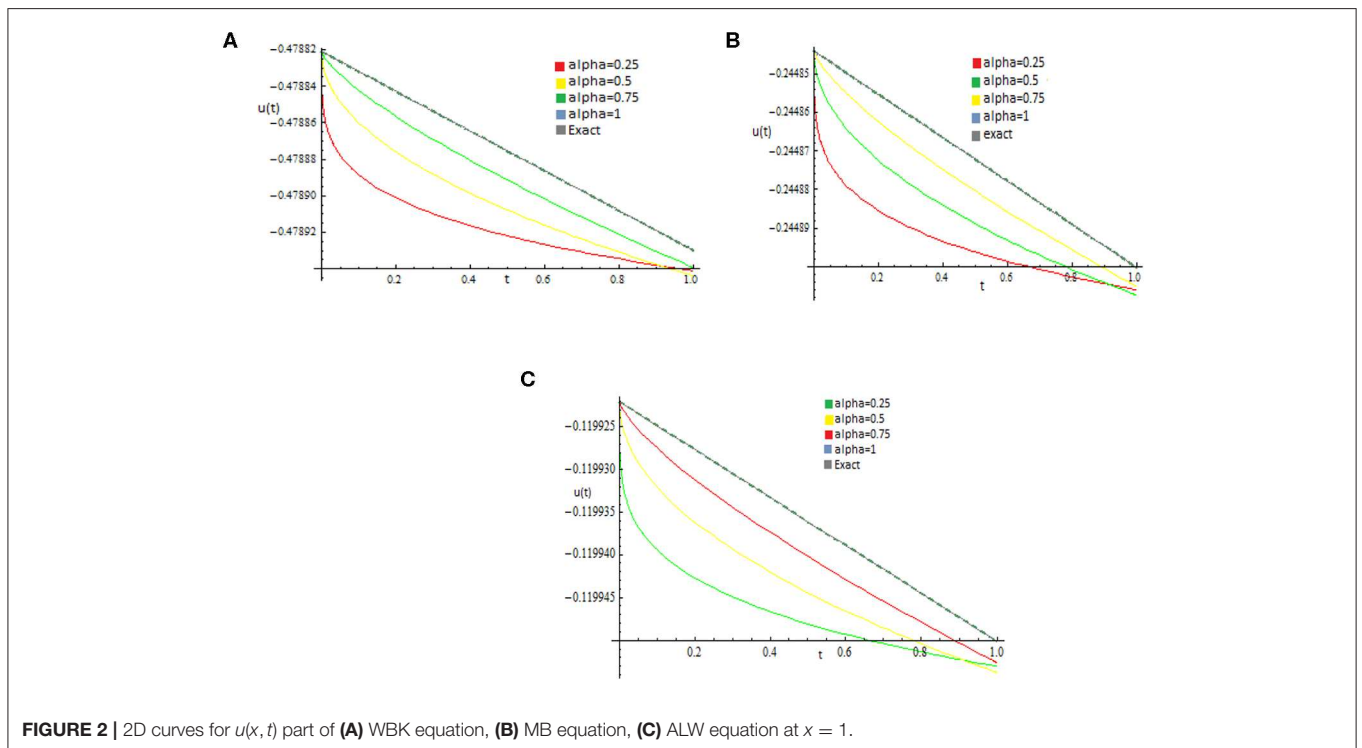
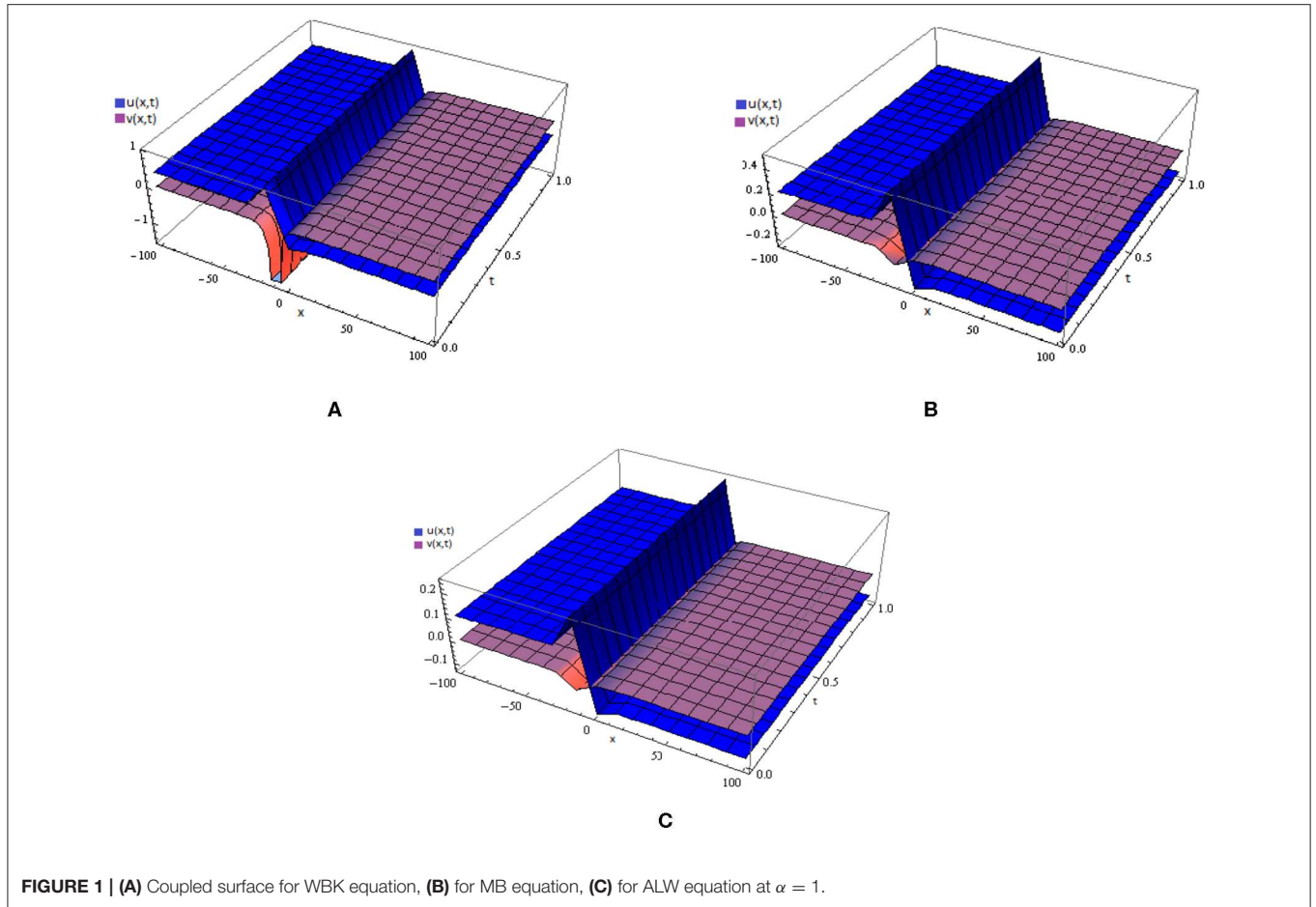
$$\begin{aligned} D_t^\alpha u + uu_x + \frac{1}{2}u_{xx} + v_x &= 0, \\ D_t^\alpha v + (uv)_x - \frac{1}{2}v_{xx} &= 0, \end{aligned} \tag{34}$$

subject to Ics

$$\begin{aligned} u(x, 0) &= \lambda - k \coth(k\xi), \\ v(x, 0) &= -k^2 \csc h^2(k\xi). \end{aligned} \tag{35}$$

where  $\xi = x + c$  and  $\lambda, c, k$  are arbitrary constants.

For  $\alpha = 1$ , the exact solution of the system is as follows:



$$\begin{aligned} u(x, t) &= \lambda - k \coth(k(\xi - \lambda t)), \\ v(x, t) &= -k^2 \csc h^2(k(\xi - \lambda t)). \end{aligned} \tag{36}$$

According to the DJ method described in the Fundamental Theory of Proposed Method section, we have:

$$\begin{aligned} u(x, t) &= u(x, 0) \\ &+ L^{-1} \left[ \frac{1}{S^\alpha} L[-(u(x, t)u_x(x, t) + v_x(x, t) + \frac{1}{2}u_{xx}(x, t))] \right], \\ v(x, t) &= v(x, 0) + L^{-1} \left[ \frac{1}{S^\alpha} L[-((u(x, t)v(x, t))_x - \frac{1}{2}v_{xx}(x, t))] \right]. \end{aligned} \tag{37}$$

So that

$$\begin{aligned} u_0 &= \lambda - k \coth(k(x + c)), \\ v_0 &= -k^2 \csc h^2(k(x + c)), \end{aligned} \tag{38}$$

$$u_1 = -\frac{k^2 t^\alpha \lambda \csc h^2(k(x + c))}{\Gamma(1 + \alpha)}, \tag{39}$$

$$v_1 = -\frac{2k^3 t^\alpha \lambda \coth(k(x + c)) \csc h^2(k(x + c))}{\Gamma(1 + \alpha)}, \tag{40}$$

$$\begin{aligned} u_2 &= \frac{1}{\Gamma(1 + 2\alpha)} k^3 t^{2\alpha} \lambda^2 \csc h^4(k(x + c)) \\ &\left\{ \frac{2k^2 t^\alpha \coth(k(x + c))(\Gamma(1 + 2\alpha))^2}{(\Gamma(1 + \alpha))^2 \Gamma(1 + 3\alpha)} - \sinh(2k(c + x)) \right\}, \end{aligned} \tag{41}$$

$$\begin{aligned} v_2 &= \frac{2k^4 t^{2\alpha} \lambda^2 \csc h^4(k(x + c))}{\Gamma(1 + 2\alpha)} \{-2 - \cosh(2k(x + c))\} \\ &+ \frac{1}{(\Gamma(1 + \alpha))^2 \Gamma(1 + 3\alpha)} (k^2 t^\alpha \\ &(3 + 2 \cosh(2k(x + c))) \csc h^2(k(x + c))(\Gamma(1 + 2\alpha))^2). \end{aligned} \tag{42}$$

Three terms approximate the solution for equation (26):

$$\begin{aligned} u &= u_0 + u_1 + u_2, \\ v &= v_0 + v_1 + v_2. \end{aligned} \tag{43}$$

Values of the parameters are taken to be same as problem 3.1.

## RESULTS AND DISCUSSION

The DJ method is experienced upon the fractional WBK, MB, and ALW equations. Mathematical 7 have been used for most computations.

Tables 1–3 show the estimation of absolute errors of the second-order DJ solution with ADM, VIM, and second-order OHAM solutions for  $u(x, t)$  of fractional WBK, MB, and ALW equations at  $\alpha = 1$ , respectively. Tables 4–6 shows the estimation of absolute errors of second-order DJ solution with ADM, VIM, and second-order OHAM solutions for  $v(x, t)$  of fractional WBK, MB, and ALW equations at  $\alpha = 1$ , respectively. The tabulated results show that the second-order approximate solutions by the DJ method are

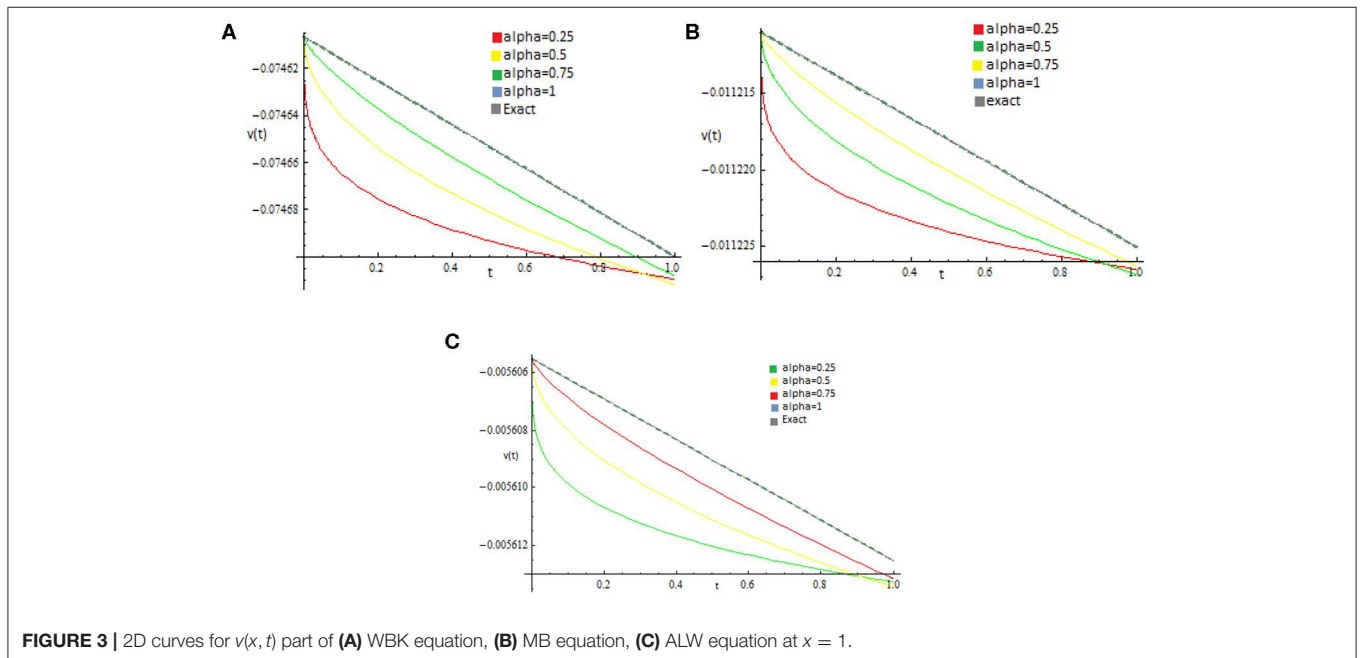
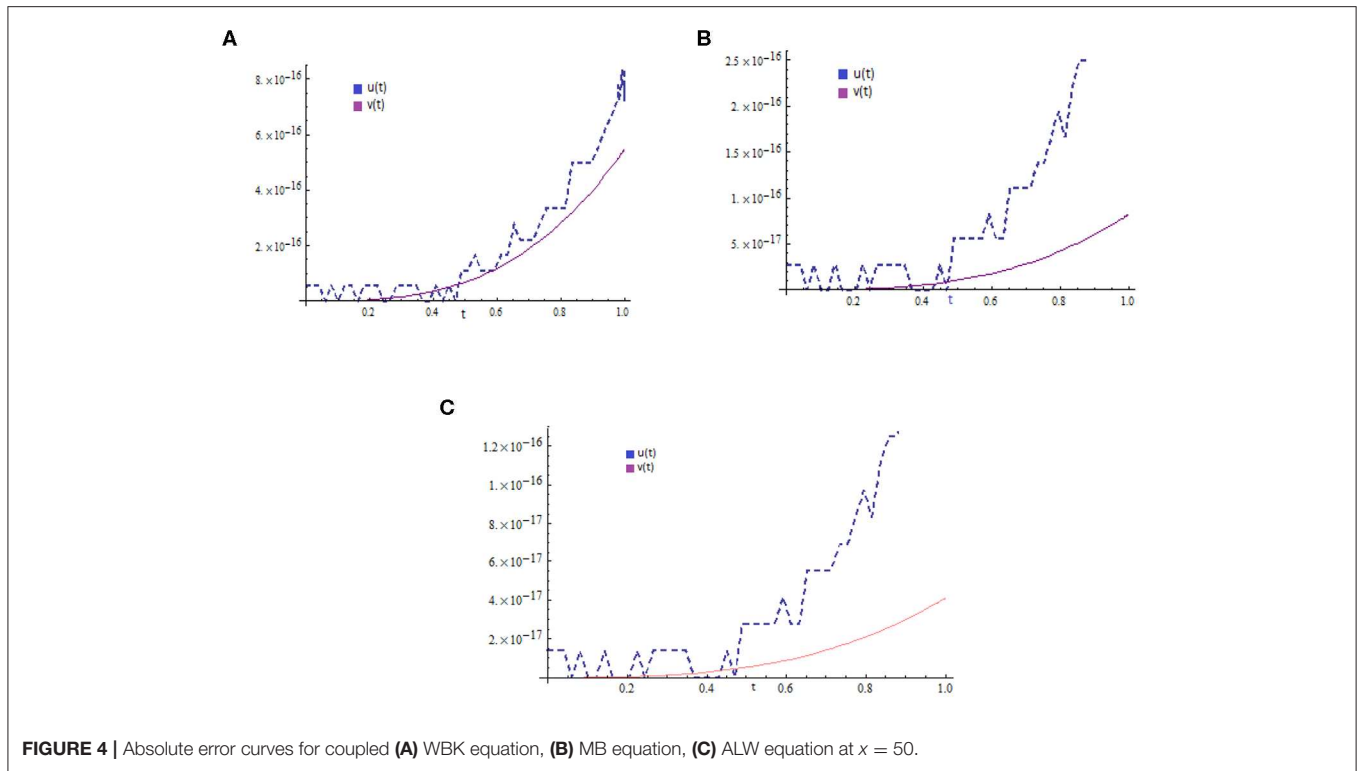


FIGURE 3 | 2D curves for  $v(x, t)$  part of (A) WBK equation, (B) MB equation, (C) ALW equation at  $x = 1$ .





closer to exact solutions than those of ADM, VIM, and OHAM solutions.

**Figures 1A–C** show the coupled surface of the second-order approximate solution by NIM for  $u(x, t)$  and  $v(x, t)$ , part of WBK, MB, and ALW equations at  $\alpha = 1$ , respectively. **Figures 2, 3** show the 2D plots of the second-order approximate solution by NIM for  $u(x, t)$  and  $v(x, t)$  of WBK, MB, and ALW equations at  $x = 1$  and different values of  $\alpha$ , respectively. **Figures 4A–C** show the absolute error graph for the coupled WBK, MB, and ALW equation at  $x = 50$ .

It is clear from 2D figures that as the value of  $\alpha$  increases to 1, the approximate solutions tend closer to the exact solution.

## CONCLUSION

The DJ method converges rapidly to the exact solution at lower order of approximations for the WBK system. The results obtained by the proposed method are very encouraging in assessment with ADM, VIM, and OHAM. As a result, it would be more appealing for researchers to apply this method for solving systems of non-linear PDEs in different fields of science especially in fluid dynamics and physics. The accurateness of the technique can more be improved by taking higher-order estimation of the proposed method.

## DATA AVAILABILITY STATEMENT

All data and related metadata underlying the findings is reported in a submitted article.

## AUTHOR CONTRIBUTIONS

RN and PK developed the numerical method and led the manuscript preparation. ZS and SF contributed to the code development and to the article preparation. MS and WD contributed to the analysis and discussion of the results and help in revision. All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

## FUNDING

This research was funded by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT.

## ACKNOWLEDGMENTS

This project was supported by the Theoretical and Computational Science (TaCS) Center under Computational and Applied Science for Smart Innovation Research Cluster (CLASSIC), Faculty of Science, KMUTT.

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**Conflict of Interest:** The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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