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Stability of Hybrid SDEs Driven by fBm

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In this paper, the exponential stability of stochastic differential equations driven by multiplicative fractional Brownian motion (fBm) with Markovian switching is investigated. The quasi-linear cases with the Hurst parameter $H \in (1/2, 1)$ and linear cases with $H \in (0, 1/2)$ and $H \in (1/2, 1)$ are all studied in this work. An example is presented as a demonstration.

Keywords: stochastic differential equation (SDEs), stability, fractional brownian motion, markovian switching, hybrid system

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1 INTRODUCTION

In the natural world, it is a common phenomena that many practical systems may face random abrupt changes in their structures and parameters, such as environmental variance, changing of subsystem interconnections and so on. To deal with these abrupt changes, Markovian switching systems, a particular class of hybrid systems, are investigated and widely used [1, 2]. Especially in signal processing, financial engineering, queueing networks, wireless communications and so on (see, e.g. [1, 3]).

In recent years, much attention has been paid to the stability of stochastic hybrid systems. For example, Mao [4] considers the exponential stability of general nonlinear stochastic hybrid systems. In [5], the criteria of moment exponential stability are obtained for stochastic hybrid delayed systems with Lévy noise in mean square. Zhou [6] investigates the pth moment exponential stability of the same systems. Some sufficient conditions for asymptotic stability in distribution of SDEs with Markovian switching are reported in [7]. See also [8, 9] for more results about Markovian switching.

On the other hand, it is generally known that if $H \in (0, 1/2)$ and $H \in (1/2, 1)$, $\{B_t^H\}_{t\geq 0}$ has a long range dependence, which means if we put

$$r(n) = \mathrm{cov}(B_1^H, (B_{n+1}^H - B_n^H)),$$

then $\sum_{n=1}^{\infty} r(n) = \infty$. Besides, the process $\{B_t^H\}_{t\geq 0}$ is also self-similar for any $H \in (0, 1)$. Since the pioneering work of Hurst [10, 11] and Mandelbrot [12], the fractional Brownian motion has been suggested as a useful tool in many fields such as mathematical finance [13, 14] and weather derivatives [15]. Even though fractional Brownian motion is not a semimartingale, more and more financial models have been extended to fBm (see, e.g. [16, 17]). Therefore, in this paper, the risk assets are described by hybrid stochastic systems driven by multiplicative fBm. Then it is a natural and interesting question that under what conditions, this stochastic systems have some exponential stability. For the sake of clarity, we only consider the one dimensional cases. For more details about fractional noise, we refer the reader to [18–21].

The main purpose of this paper is to discuss the exponential stability of a risky asset, with price dynamics:

$$\begin{cases} \mathrm{d}X_t = f(X_t, t, r_t)\mathrm{d}t + g(X_t, t, r_t)\mathrm{d}B_t^H, \\ X_0 = x_0 > 0, \end{cases} \tag{1}$$

where $g(X_t, t, r_t) = \sigma(t, r_t)X_t$, $\{r_t\}_{t\geq 0}$ is a Markov chain taking values in $\mathbb{S} = \{1, 2, \dots, N\}$, $\{B_t^H\}_{t\geq 0}$ is a standard fractional Brownian motion. Moreover, $f(x, t, r_t)$: $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}$ and $\sigma(t, r_t)$: $\mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}$.

In this paper, the initial value x_0 is assumed to be deterministic, otherwise more calculations about Wick product are required.

Equation 1 can be regarded as the result of the following N fractional stochastic differential equations:

$$\begin{cases} \mathrm{d}X_t = f(X_t, t, i)\mathrm{d}t + g(X_t, t, i)\mathrm{d}B_t^H, & 1 \le i \le N, \\ X_0 = x_0 > 0, \end{cases}$$

switching from one to another according to the movement of $\{r_t\}_{t\ge 0}$.

Throughout this paper, unless otherwise specified, we let C denote a general constant and p denote a non-negative constant. Let $C^{2,1}(\mathbb{R}\times\mathbb{R}_+\times\mathbb{S};\mathbb{R})$ denote the family of all real value functions on $\mathbb{R}\times\mathbb{R}_+\times\mathbb{S}$ which are continuously twice differentiable with respect to the first variables and once differentiable with respect to the second variables.

This paper is organized as follows. For the convenience of the reader, we briefly recall some of the basic results in **Section 2**. In **Section 3**, we investigate the solution and an extended Itô's Formula for the general hybrid fractional stochastic differential **Equation 1**. **Section 3** is devoted to the linear cases. In this section the moment exponential stability and almost sure exponential stability are discussed respectively. In **Section 4**, some useful criteria for the exponential stability with respect to quasi-linear cases are presented. Finally, a numerical example and graphical illustration are presented in **Section 6**.

2 PRELIMINARIES

2.1 Markov Chain

Let $\{r_t\}_{t\ge 0}$ be a right-continuous Markov chain taking values in a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$. The generator $Q = (q_{ij})_{N\times N}$ is given by

$$\mathbb{P}\{r_{t+\triangle} = j \mid r_t = i\} = \begin{cases} q_{ij}\triangle + o(\triangle), & \text{if } i \ne j, \\ 1 + q_{ij}\triangle + o(\triangle), & \text{if } i = j, \end{cases}$$

where $\triangle > 0$.

Here q_{ij} is the transition rate from i to j if $i \ne j$. According to [22, 23], a continuous-time Markov chain $\{r_t\}_{t\ge 0}$ with generator $Q = (q_{ij})_{N\times N}$ can be represented as a stochastic integral with respect to a Poisson random measure. Then we have

$$\mathrm{d}r_t = \int_{\mathbb{R}} h(r_{t-}, y)\nu(\mathrm{d}t \times \mathrm{d}y),$$

with initial condition $r_0 = i_0$, where $\nu(dt \times dy)$ is a Poisson random measure with intensity $\mathrm{d}t \times m(\mathrm{d}y)$. Here $m(\cdot)$ is the Lebesgue measure on $\mathbb{R}$.

Throughout this paper, unless otherwise specified, the Markov chain $\{r_t\}_{t\ge 0}$ has the invariant probability measure $\mu = (\mu_i)_{i\in\mathbb{S}}$ and is assumed to be independent of $\{B_t^H\}_{t\ge 0}$. Almost every sample path of the Markov chain $\{r_t\}_{t\ge 0}$ is assumed to be a right-continuous step function with a finite number of simple jumps in any finite time interval $[0, T]$. The generator $Q = (q_{ij})_{N\times N}$ is assumed to be irreducible and conservative, i.e., $q_i := -q_{ii} = \sum_{i\ne j} q_{ij} < \infty$. For more details about Markovian switching we further refer the reader to [24–26].

2.2 Fractional Brownian Motion and Wick Product

We recall some of the basic results of fBm briefly, which will be needed throughout this paper. For more details about fBm we refer the reader to [16, 17, 27, 28]. If $H \in (0, 1/2) \cup (1/2, 1)$, then the (standard) fractional Brownian motion with Hurst parameter H is a continuous centered Gaussian process $\{B_t^H\}_{t\ge 0}$ with $\mathbb{E}(B_t^H) = 0$ and covariance function:

$$\mathbb{R}^H(s,t) = \mathbb{E}(B_s^H B_t^H) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s-t|^{2H}), \quad s, t \ge 0.$$

To simplify the representation, it is always assumed that $B_0^H = 0$.

Besides, $\{B_t^H\}_{t\ge 0}$ has the following Wiener integral representation:

$$B_t^H = \int_0^t K^H(t,s)\mathrm{d}W_s,$$

where $\{W_t\}_{t\ge 0}$ is a Wiener process and $K^H(t, s)$ is the kernel function defined by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H}\int_0^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\mathrm{d}u,$$

in which $c_H = \left(\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$, where $\mathrm{B}(\cdot, \cdot)$ is the Beta function, and $s < t$. In this paper, $\{B_t^H\}_{t\ge 0}$ generates a filtration $\{\mathcal{F}_t, t \ge 0\}$ with $\mathcal{F}_t = \sigma\{B_s^H, t \ge 0\}$. Denote $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ the complete probability space, with the filtration described above.

Let $\mathcal{I}$ be the set of all finite multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ for some $n \ge 1$ of non-negative integers. Denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\alpha! = \alpha_1!\cdots\alpha_n!$.

Define the Hermite polynomials:

$$h_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}), \quad n \ge 0,$$

and Hermite functions:

$$\tilde{h}_n(x) = \pi^{-\frac{1}{4}}(n!)^{-\frac{1}{2}}h_n(x)e^{-\frac{x^2}{4}}, \quad n \ge 0.$$

Let $S(\mathbb{R})$ denote the Schwartz space of rapidly decreasing infinitely differentiable $\mathbb{R}$-valued functions. Denote the dual space of $S(\mathbb{R})$ by $S'(\mathbb{R})$. Define

$$\mathcal{H}_\alpha(\omega) = \prod_{i=1}^n h_{\alpha_i}(\langle\tilde{h}_i(x), \omega\rangle),$$

the product of Hermite polynomials. Consider a square integrable random variable

$$F = F(\omega) \in L^2(S'(\mathbb{R}), \mathcal{F}, P).$$

According to [17, 29], every $F(\omega)$ has a unique representation:

$$F(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha \mathcal{H}_\alpha(\omega),$$

besides,

$$\|F\|^2_{L^2(\omega)} = \sum_{\alpha\in\mathcal{I}} \alpha! c_\alpha^2 < \infty.$$

Definition 2.1. (Wick Product) For $F, G \in L^2(S'(\mathbb{R}), \mathcal{F}, P)$, set $F(\omega) = \sum_{\alpha\in\mathcal{I}} c_\alpha \mathcal{H}_\alpha(\omega)$ and $G(\omega) = \sum_{\beta\in\mathcal{I}} d_\beta \mathcal{H}_\beta(\omega)$. Their Wick product is defined by

$$\begin{aligned} F \diamond G(\omega) &= \sum_{\alpha,\beta\in\mathcal{I}} a_\alpha b_\beta \mathcal{H}_{\alpha+\beta}(\omega) \\ &= \sum_{\gamma\in\mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) \mathcal{H}_\gamma(\omega). \end{aligned}$$

2.3 Malliavin Derivative

Let $L^p := L^p(\Omega, \mathcal{F}, P)$ be the space of all random variables $\Omega \to \mathbb{R}$, such that

$$\|F\|_p = \mathbb{E}(|F|^p)^{1/p} < \infty,$$

and let

$$L_\phi^2(\mathbb{R}_+) = \left\{ f \,|\, f: \mathbb{R}_+ \to \mathbb{R}, |f|_\phi^2 := \int_0^\infty \int_0^\infty f(s) f(t) \phi(s,t) \mathrm{d}s \mathrm{d}t < \infty \right\},$$

where $\phi(s, t) = H(2H - 1)|s - t|^{2H-2}$.

Definition 2.2. The ϕ-derivative of $F \in L^p$ in the direction of Φ_g is defined by

$$D_{\Phi_g} F(\omega) = \lim_{\delta\to 0} \frac{1}{\delta} \left\{ F\left(\omega + \delta \int_0^{\cdot} (\Phi_g)(u) \mathrm{d}u \right) - F(\omega) \right\},$$

if the limit exists in L^p. Moreover if there exists a process $(D_s^\phi F_s, s \geq 0)$ such that

$$D_{\Phi_g} F = \int_0^\infty D_s^\phi F_s g_s \mathrm{d}s \quad a.s.,$$

for all $g \in L_\phi^2$, then F is said to be ϕ-differentiable.

According to [16, 30], let $\mathcal{A}(0, T)$ be the family of stochastic process on [0, T] such that $F \in \mathcal{A}(0, T)$ if $\mathbb{E}|F|_\phi^2 < \infty$ and F is ϕ-differentiable, the trace of $(D_s^\phi F_t, 0 \leq s \leq T, 0 \leq t \leq T)$ exists and $\mathbb{E}\int_0^T (D_s^\phi F_s)^2 ds < \infty$, and for each sequence of partitions $\pi_n, n \in \mathbb{N}$ such that $|\pi_n| \to 0$, as $n \to \infty$. Moreover

$$\sum_{i=0}^{n-1} \mathbb{E}\left\{ \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} |D_s^\phi F_{t_i^{(n)}}^\pi - D_s^\phi F_s| \mathrm{d}s \right\}^2 \to 0,$$

and

$$\mathbb{E}|F^\pi - F|_\phi^2 \to 0,$$

as $n \to \infty$. Here π_n: $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = T$, and $|\pi_n| = \max_{i\in\{0,1,\ldots,n-1\}}\{t_{i+1}^{(n)} - t_i^{(n)}\}$.

Now we define the B_t^H-integral considered in [16].

Definition 2.3. Let $\{F_t\}_{t\geq 0}$ be a stochastic process such that $F \in \mathcal{A}(0, T)$. Define $\int_0^T F_s \mathrm{d}B_s^H$ by

$$\int_0^T F_s \mathrm{d}B_s^H = \lim_{|\pi|\to 0} \sum_{i=0}^{n-1} F_{t_i}^\pi \diamond (B_{t_{i+1}}^H - B_{t_i}^H),$$

where $|\pi| = \max_{i\in\{0,1,\ldots,n-1\}}\{t_{i+1} - t_i\}$.

Remark 2.1. : According to Theorem 3.6.1 in [16], if $F_s \in \mathcal{A}(0, T)$, then the stochastic integral satisfies $\mathbb{E}\int_0^T F_s \mathrm{d}B_s^H = 0$, and

$$\mathbb{E}\left| \int_0^T F_s \mathrm{d}B_s^H \right|^2 = \mathbb{E}\left[\left(\int_0^T D_s^\phi F_s \mathrm{d}s \right)^2 + |1_{[0,T]} F|_\phi^2 \right]$$

What's more, according to Definition 3.4.1 in [16], the stochastic integral can be extended by

$$\int_{\mathbb{R}} F_t \mathrm{d}B_t^H := \int_{\mathbb{R}} F_t \diamond W^H(t) dt,$$

where F: $\mathbb{R} \to (\mathcal{S})_H^*$ is a given function such that $F_t \diamond W^H(t)$ is dt – integrable in $(\mathcal{S})_H^*$. Here $(\mathcal{S})_H^*$ is the fractional Hida distribution space defined by Definition 3.1.11 in [16]. In particular, the integral on [0, T] can be defined by

$$\int_0^T F_t \mathrm{d}B_t^H = \int_{\mathbb{R}} F_t I_{[0,T]}(t) \mathrm{d}B_t^H.$$

3 HYBRID FRACTIONAL SYSTEMS

In this section, firstly, we consider the existence and uniqueness of solution for **Eq. 1**. Then, an extended Itô's Formula is presented.

3.1 Existence and Uniqueness

To ensure the existence and uniqueness, we impose the following assumptions.

Assumption 3.1. Let $f = f(x, t, i)$: $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}$ satisfy the hypothesises:

1) For each fixed $i \in \mathbb{S}$, $f(x, t, i)$ is measurable in all the arguments.
2) For each fixed $i \in \mathbb{S}$, there exists a constant $C > 0$, such that
$|f(x, t, i) - f(y, t, i)| \leq C|x - y|, \quad \forall x, y \in \mathbb{R}, \quad \forall t \in \mathbb{R}_+$.
3) For each fixed $i \in \mathbb{S}$, there exists a constant $C > 0$, such that

$$|f(x, t, i)| \leq C(1 + |x|), \quad \forall (x, t) \in \mathbb{R} \times \mathbb{R}_+.$$

Assumption 3.2. Let $\sigma = \sigma(t, i)$: $\mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}$ satisfy the hypothesises:

1) For each fixed $i \in \mathbb{S}$, $\sigma(t, i)$ is nonrandom;
2) For each fixed $i \in \mathbb{S}$, $\sigma(t, i) \in L^{\frac{1}{H}}(\mathbb{R}_+)$.

Lemma 3.1. : Let **Assumptions 3.1, 3.2** hold. Then **Eq. 1** has a unique solution.

Proof: The existence and uniqueness can be proved similar to that for Theorem 2.6 in [31], so we omit it here.

3.2 The Itô Formula

Next, we first review the results in [16, 30] on the Itô formula with respect to fBm. Then we extend it to SDEs driven by fBm with Markovian switching.

Lemma 3.2. [16] (The Itô Formula) Let $(F_u, 0 \leq u \leq T)$ be a stochastic process in $\mathcal{A}(0, T)$. Assume that there exists an $\alpha > 1 - H$ and $C > 0$ such that

$$\mathbb{E}|F_u - F_v|^2 \leq C|u - v|^{2\alpha},$$

where $|u - v| \leq \delta$ for some $\delta > 0$ and

$$\lim_{0\leq u,v\leq t,|u-v|\to 0} \mathbb{E}|D_u^\phi (F_u - F_v)|^2 = 0.$$

Let $\sup_{0\leq s\leq T}|G_s| < \infty$ and $\tilde{g} = \tilde{g}(x,t) \in C^{2,1}(\mathbb{R}\times\mathbb{R}_+;\mathbb{R})$ with bounded derivatives. Moreover, for $\eta_t = \int_0^t F_u \mathrm{d}B_u^H$, it is assumed that $\mathbb{E}\int_0^T |F_s D_s^\phi \eta_s|\mathrm{d}s < \infty$ and $(\frac{\partial \tilde{g}}{\partial x}(s,\eta_s)F_s, s\in[0,T])$ is in $\mathcal{A}(0,T)$. Denote $x_t = x_0 + \int_0^t G_u \mathrm{d}u + \int_0^t F_u \mathrm{d}B_u^H$, $x_0 \in \mathbb{R}$ for $t \in [0,T]$. Let $(\frac{\partial \tilde{g}}{\partial x}(x_s,s)F_s, s\in[0,T]) \in \mathcal{A}(0,T)$, $\mathbb{E}[\sup_{0\leq s\leq t}|G_s|] < \infty$. Then for $t \in [0,T]$,

$$\begin{aligned}\tilde{g}(x_t,t) &= \tilde{g}(x_0,0) + \int_0^t \frac{\partial \tilde{g}}{\partial s}(x_s,s)\mathrm{d}s + \int_0^t \frac{\partial \tilde{g}}{\partial x}(x_s,s)G_s\mathrm{d}s \\ &\quad + \int_0^t \frac{\partial \tilde{g}}{\partial x}(x_s,s)F_s\mathrm{d}B_s^H + \int_0^t \frac{\partial^2 \tilde{g}}{\partial x^2}(x_s,s)F_s D_s^\phi x_s \mathrm{d}s.\end{aligned}$$

Here $D_s^\phi x_s$ is the Malliavin derivative defined in **Definition 2.2**.

In particular, for the process $X_t^{(i)} = X_0^{(i)} + \int_0^t f(X_s^{(i)},s,i)\mathrm{d}s + \int_0^t g(X_s^{(i)},s,i)\mathrm{d}B_s^H$, with each fixed $i\in\mathbb{S}$, we have that

$$\begin{aligned}F(X_t^{(i)},t,i) &= F(X_0^{(i)},0,i) + \int_0^t \frac{\partial F}{\partial s}(X_s^{(i)},s,i)\mathrm{d}s \\ &\quad + \int_0^t \frac{\partial F}{\partial x}(X_s^{(i)},s,i)f(X_s^{(i)},s,i)\mathrm{d}s + \int_0^t \frac{\partial F}{\partial x}(X_s^{(i)},s,i)g(X_s^{(i)},s,i)\mathrm{d}B_s^H \\ &\quad + \int_0^t \frac{\partial^2 F}{\partial x^2}(X_s^{(i)},s,i)g(X_s^{(i)},s,i)D_s^\phi X_s^{(i)}\mathrm{d}s,\end{aligned} \tag{2}$$

Formally,

$$\begin{aligned}\mathrm{d}F(X_t^{(i)},t,i) &= F_t(X_t^{(i)},t,i)\mathrm{d}t + F_{xx}(X_t^{(i)},t,i)g(X_t^{(i)},t,i)D_s^\phi X_s^{(i)}\mathrm{d}t \\ &\quad + F_x(X_t^{(i)},t,i)f(X_t^{(i)},t,i)\mathrm{d}t + F_x(X_t^{(i)},t,i)g(X_t^{(i)},t,i)\mathrm{d}B_t^H,\end{aligned}$$

Let

$$\begin{aligned}\mathcal{L}^{(i)}F(X_t^{(i)},t,i) &= F_t(X_t^{(i)},t,i) + F_x(X_t^{(i)},t,i)f(X_t^{(i)},t,i) \\ &\quad + F_{xx}(X_t^{(i)},t,i)g(X_t^{(i)},t,i)D_s^\phi X_s.\end{aligned} \tag{3}$$

Substituting **Eq. 3** into **Eq. 2**, we get

$$\begin{aligned}F(X_t^{(i)},t,i) &= F(X_0^{(i)},0,i) + \int_0^t \mathcal{L}^{(i)}F(X_s^{(i)},s,i)\mathrm{d}s \\ &\quad + \int_0^t F_x(X_s^{(i)},s,i)g(X_s^{(i)},s,i)\mathrm{d}B_s^H.\end{aligned} \tag{4}$$

In the sequel of this paper, unless otherwise specified, we let the coefficients of **Eq. 1** satisfy the conditions in **Lemma 3.2**, for each fixed $i\in\mathbb{S}$. Set $V(X_t,t,r_t)\in C^{2,1}(\mathbb{R}\times\mathbb{R}_+\times\mathbb{S};\mathbb{R}_+)$. Next we consider the Itô formula which reveals how V maps (X_t, t, r_t) into a new process $V(X_t, t, r_t)$, where $\{X_t\}_{t\geq 0}$ is a stochastic process with the stochastic differential **Eq. 1**.

Lemma 3.3. If $V(X_t,t,r_t)\in C^{2,1}(\mathbb{R}\times\mathbb{R}_+\times\mathbb{S};\mathbb{R}_+)$, then for any $0 \leq s < t$,

$$\begin{aligned}\mathbb{E}V(X_t,t,r_t) &= \mathbb{E}V(X_s,s,r_s) + \mathbb{E}\int_s^t \mathcal{A}V(X_u,u,r_u)\mathrm{d}u \\ &\quad + \mathbb{E}\int_s^t V_x(X_u,u,r_u)g(X_u,u,r_u)\mathrm{d}B_u^H\end{aligned} \tag{5}$$

where $\mathcal{A}V$ is defined by

$$\mathcal{A}V(x,t,i) = \mathcal{L}^{(i)}V(x,t,i) + \sum_{j=1}^N \gamma_{ij}V(x,t,j).$$

Proof: This result can be obtained similarly to that in [31] and we therefore omit it. For further details we also refer to [2, 23].

4 LINEAR HYBRID FRACTIONAL SYSTEMS

There are many models for financial markets with fBm (see, e.g. [16]). The simplest nontrivial type of market is the fBm version of the classical Black Scholes market, in which linear fractional SDEs is used. Thus, we would like to give some new criteria for switching linear fractional SDEs with $H \in (0,\frac{1}{2})$ or $H\in(\frac{1}{2},1)$. At first, we present a definition and a useful lemma.

Definition 4.1. Let $H \in (0, 1)$. The operator M is defined on functions $f \in S(\mathbb{R})$ by

$$Mf(x) = -\frac{d}{dx}\frac{C_H}{(H-1/2)}\int_{\mathbb{R}}(t-s)|t-x|^{H-\frac{3}{2}}f(t)\mathrm{d}t \tag{6}$$

where

$$C_H = \left\{2\Gamma\left(H-\frac{1}{2}\right)\cos\left[\frac{\pi}{2}\left(H-\frac{1}{2}\right)\right]\right\}^{-1}$$

$$[\Gamma(2H+1)\sin(\pi H)]^{\frac{1}{2}}.$$

Here $\Gamma(\cdot)$ denotes the classical Gamma function.

According to [16], **Eq. 6** can be restated as follows.

For $H \in (0, 1/2)$, we have

$$Mf(x) = C_H\int_{\mathbb{R}}\frac{f(x-t)-f(x)}{|t|^{3/2-H}}\mathrm{d}t.$$

For $H = 1/2$, we have

$$Mf(x) = f(x).$$

For $H \in (1/2, 1)$, we have

$$Mf(x) = C_H\int_{\mathbb{R}}\frac{f(t)}{|t-x|^{3/2-H}}\mathrm{d}t.$$

Lemma 4.1. Let $\{r_t\}_{t\geq 0}$ be a right-continuous Markov chain which takes values in a finite state space $\mathbb{S} = \{1,2,\ldots,N\}$. Assume that it is irreducible and positive recurrent with invarient measure μ. If $\alpha(\cdot)\colon \mathbb{S}\to\mathbb{R}$ is a function verifying

$$\alpha := \sum_{i\in\mathbb{S}}\mu(i)\alpha(i) > 0.$$

Then there exists constants C, $c > 0$ such that:

$$ce^{-\alpha t} \leq \mathbb{E}\left[e^{-\int_0^t \alpha(r_s)\mathrm{d}s}\right] \leq Ce^{-\alpha t},$$

for any initial condition r_0 and every $t \geq 0$.

Proof: It is a consequence of Perron-Frobenius theorem and the study of eigenvalues. See Proposition 4.1 in [25], Proposition 4.2 in [25], and Lemma 2.7 in [26], for further details.

In **Eq. 1**, let us consider the case $g(x, t, r_t) = \sigma(t, r_t)x = t^h b(r_t)x$, $f(x, t, r_t) = \alpha(r_t)x$, where $\alpha(i)$ and $b(i)$ are constants for each $i \in \mathbb{S}$. This means that we are considering the following linear equation:

$$\begin{cases} \mathrm{d}X_t = \alpha(r_t)X_t\mathrm{d}t + \sigma(t, r_t)X_t\mathrm{d}B_t^H, \\ X_0 = x_0. \end{cases} \tag{7}$$

Set $\bar{b} = \max\{|b(i)|, i \in \mathbb{S}\}$ and $\underline{b} = \min\{|b(i)|, i \in \mathbb{S}\}$. x_0 is the deterministic initial value. For the sake of clarity, we firstly set $h = 1/2 - H$.

4.1 *p*th Moment Exponential Stability

Theorem 4.1. Let $\{X_t\}_{t\geq 0}$ be the solution of **Eq. 7** with $H \in (1/2, 1)$, $h = 1/2 - H$.

1) If $\sum_{i\in S}\mu_i\alpha(i) - \frac{(1-p)\underline{b}^2}{2} < 0$, then $\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|X_t|^p) < 0$.
2) If $\sum_{i\in \mathbb{S}}\mu_i\alpha(i) - \frac{(1-p)\bar{b}^2}{2} > 0$, then $\lim_{t\to\infty} \mathbb{E}|X_t|^p = \infty$.

Proof. According to [16], without too many calculations, we obtain that $\{X_t\}_{t\geq 0}$ has the following form:

$$X_t = x_0 \exp\left[\int_0^t \sigma(r_s)\mathrm{d}B_s^H + \int_0^t \alpha(r_s)\mathrm{d}s - \frac{1}{2}\int_{\mathbb{R}} (M_s(\sigma(t, r_s)I_{[0,t]}(s)))^2\mathrm{d}s\right], \tag{8}$$

where M_s is the operator M acting on the variable s. Let $x_0 \neq 0$. It follows from **Eq. 8** that

$$\mathbb{E}|X_t|^p = \mathbb{E}\left(|x_0|\exp\left[\int_0^t \sigma(t, r_s)\mathrm{d}B_s^H + \int_0^t \alpha(r_s)\mathrm{d}s - \frac{1}{2}\int_{\mathbb{R}} (M_s(\sigma(t, r_s)I_{[0,t]}(s)))^2\mathrm{d}s\right]\right)^p \tag{9}$$

We then see from **Eq. 9** that

$$\mathbb{E}|X_t|^p = \mathbb{E}\left(\exp\left(p\left[\int_0^t \alpha(r_s)\mathrm{d}s - \frac{1-p}{2}\int_{\mathbb{R}} (M_s(\sigma(t, r_s)I_{[0,t]}(s)))^2\mathrm{d}s\right]\right)\zeta_t\right), \tag{10}$$

where

$$\zeta_t = |x_0|^p \exp\int_0^t p\sigma(s, r_s)\mathrm{d}B_s^H - \frac{p^2}{2} \int_{\mathbb{R}} (M_s(\sigma(t, r_s)I_{[0,t]}(s)))^2\mathrm{d}s.$$

Noting that ζ_t is the solution to the equation

$$\mathrm{d}\zeta_t = p\sigma(t, r_t)\zeta_t\mathrm{d}B_t^H,$$

with initial value $\zeta_0 = |x_0|^p$. Thus

$$\zeta_t = |x_0|^p + \int_0^t p\sigma(t, r_s)\mathrm{d}B_s^H,$$

which yields

$$\mathbb{E}\zeta_t = \mathbb{E}\left[|x_0|^p + \int_0^t p\sigma(t, r_s)\mathrm{d}B_s^H\right] = |x_0|^p. \tag{11}$$

Substituting **Eq. 11** into **Eq. 10** gives

$$\mathbb{E}|X_t|^p = \mathbb{E}\exp\left(p\left[\int_0^t \alpha(r_s)\mathrm{d}s - \frac{1-p}{2}\int_{\mathbb{R}} (M_s(\sigma(r_s)I_{[0,t]}(s)))^2\mathrm{d}s\right]\right)|x_0|^p. \tag{12}$$

Note that

$$\int_{\mathbb{R}} (M_s(\underline{b}s^h I_{[0,t]}(s)))^2\mathrm{d}s \leq \int_{\mathbb{R}} (M_s(\sigma(t, r_s)I_{[0,t]}(s)))^2 \mathrm{d}s \leq \int_{\mathbb{R}} (M_s(\bar{b}s^h I_{[0,t]}(s)))^2\mathrm{d}s.$$

Consequently, by **Definition 4.1** and [16], one has

$$\underline{b}^2 t \leq \int_{\mathbb{R}} (M_s(\sigma(r_s)I_{[0,t]}(s)))^2\mathrm{d}s \leq \bar{b}^2 t. \tag{13}$$

Making use of **Eqs 12**, **13**, we obtain that

$$\mathbb{E}\exp\left(p\left[\int_0^t \alpha(r_s)\mathrm{d}s - \frac{1-p}{2}\bar{b}^2 t\right)\right]|x_0|^p \leq \mathbb{E}|X_t|^p$$
$$\leq \mathbb{E}\exp\left(p\left[\int_0^t \alpha(r_s)\mathrm{d}s - \frac{1-p}{2}\underline{b}^2 t\right)\right]|x_0|^p.$$

Therefore, by **Lemma 4.1** and **Eq. 12**, the required assertions follow. The proof is complete.

Theorem 4.2. Let $\{X_t\}_{t\geq 0}$ be the solution of **Eq. 7** with $H \in (0, 1/2)$, $h = 1/2 - H$.

1) If $\sum_{i\in S}\mu_i\alpha(i) < \frac{(1-p)\underline{b}^2}{2}$, then $\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|X_t|^p) < 0$.
2) If $\sum_{i\in \mathbb{S}}\mu_i\alpha(i) > \frac{(1-p)\bar{b}^2}{2}$, then $\lim_{t\to\infty} \mathbb{E}|X_t|^p = \infty$.

Proof: Similar to **Theorem 4.1**, we write the solution as follows.

$$\mathbb{E}|X_t|^p = \mathbb{E}\exp\left(p\left[\int_0^t \alpha(r_s)\mathrm{d}s + \frac{p-1}{2}\int_{\mathbb{R}} (M_s(\sigma(r_s)I_{[0,t]}(s)))^2\mathrm{d}s\right]\right)|x_0|^p. \tag{14}$$

Note that M_s is the operator M acting on the variable s, where

$$Mf(x) = C_H\int_{\mathbb{R}} \frac{f(x-t) - f(x)}{|t|^{3/2-H}}\mathrm{d}t.$$

According to [16], we also have that

$$\underline{b}^2 t \leq \int_{\mathbb{R}} (M_s(\sigma(t, r_s)I_{[0,t]}(s)))^2\mathrm{d}s \leq \bar{b}^2 t. \tag{15}$$

Consequently, by **Lemma 4.1**, the result follows. The proof is complete.

Remark 4.1. In the above **Theorems 4.1, 4.2**, the parameter h is supposed to be $H - 1/2$. Noting that by **Eqs 13**, **15** and together with the **Definition 4.1**, the stability of solution for

Eq. 7 with $h < 1/2 - H$ or $h > 1/2 - H$ can be deduced respectively without too many difficulties.

Remark 4.2. Take $H = 1/2$. It's easy to show that if $\sum_{i\in\mathbb{S}}\mu_i\alpha(i) = \alpha < \frac{(1-p)\underline{\sigma}^2}{2}$, then $\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|X_t|^p) < 0$, and if $\sum_{i\in\mathbb{S}}\mu_i\alpha(i) = \alpha > \frac{(1-p)\bar{\sigma}^2}{2}$, then $\lim_{t\to\infty}\mathbb{E}|X_t|^p = \infty$, which coincide with the results of SDEs driven by Brownian motion in [4, 32].

4.2 Almost Sure Exponential Stability

To proceed, we need to introduce the definition of almost sure stability and a useful lemma.

Definition 4.2. The equilibrium point $x = 0$ is said to be almost surely exponential stable if

$$\limsup_{t\to\infty}\frac{1}{t}\log|X_t| < 0 \quad a.s.$$

for any $x_0 \in \mathbb{R}$.

Lemma 4.2. (Law of the iterated logarithm) For a standard fBm $\{B_t^H\}_{t\geq 0}$, we have that

$$\limsup_{t\to\infty}\frac{B_t^H}{t^H\sqrt{\log\log t}} = C_H, \tag{16}$$

where $C_H > 0$ is a suitable constant.

Proof: By [33], we have

$$\limsup_{t\to 0^+}\frac{B_t^H}{t^H\sqrt{\log\log t^{-1}}} = c_H,$$

where c_H is a suitable constant. Then the thesis follows by the self-similarity of fBm and a change of variable $t \to 1/t$.

For the sake of clarity, we firstly set $h = 0$. Namely, let us consider

$$\begin{cases} dX_t = \alpha(r_t)X_t\mathrm{d}t + b(r_t)X_t\mathrm{d}B_t^H, \\ X_0 = x_0. \end{cases} \tag{17}$$

Noting that **Eq. 17** is exactly the geometry fBm with Markovian Switching. We proceed to discuss the almost sure exponential stability about it.

Theorem 4.3. 1) If $0 < H < 1/2$, the equilibrium point x = 0 of the system **Eq. 17** is almost surely exponential stable when $\sum_{i\in\mathbb{S}}\mu_i\alpha(i) < 0$, but unstable when $\sum_{i\in\mathbb{S}}\mu_i\alpha(i) > 0$; 2) If $H = 1/2$, the equilibrium point $x = 0$ of the system **Eq. 17** is almost surely exponential stable when $\sum_{i\in\mathbb{S}}\mu_i\alpha(i) < \frac{1}{2}\underline{b}^2$, but unstable when $\sum_{i\in\mathbb{S}}\mu_i\alpha(i) > \frac{1}{2}\bar{b}^2$; 3) If $1/2 < H < 1$, the equilibrium point $x = 0$ of the system **Eq. 17** is almost surely exponential stable for all parameters $\alpha(i)$ and $\sigma(i)$, $i \in \mathbb{S}$.

Proof: Define

$$\lambda = \limsup_{t\to\infty}\frac{1}{t}\log|X_t|.$$

From **Eqs 8**, **16**, we have

$$\begin{aligned}\lambda &= \limsup_{t\to\infty}\frac{1}{t}\log|X_t| \\ &= \limsup_{t\to\infty}\frac{1}{t}\log\left|x_0\exp\left[\int_0^t\sigma(r_s)\mathrm{d}B_s^H + \int_0^t\alpha(r_s)\mathrm{d}s - \frac{1}{2}\int_{\mathbb{R}}(M_s(\sigma(r_s)I_{[0,t]}(s)))^2\mathrm{d}s\right]\right| \\ &= \lim_{t\to\infty}\left(\sum_{i\in\mathbb{S}}\mu_i\alpha(i) - \frac{1}{2t}\int_{\mathbb{R}}(M_s(\sigma(r_s)I_{[0,t]}(s)))^2\mathrm{d}s\right).\end{aligned}$$

By **Definition 4.1** and [16], one has

$$\underline{b}^2t^{2H} \leq \int_{\mathbb{R}}(M_s(b(r_s)I_{[0,t]}(s)))^2\mathrm{d}s \leq \bar{b}^2t^{2H}. \tag{18}$$

Making use of **Eq. 18**, we get

$$\lambda = \begin{cases}\sum_{i\in\mathbb{S}}\mu_i\alpha(i), & 0 < H < 1/2; \\ -\infty, & 1/2 < H < 1.\end{cases}$$

Especially, when $H = 1/2$, we have that

$$\sum_{i\in\mathbb{S}}\mu_i\alpha(i) - \frac{1}{2}\bar{b}^2 \leq \lambda \leq \sum_{i\in\mathbb{S}}\mu_i\alpha(i) - \frac{1}{2}\underline{b}^2.$$

Therefore, the required results follows. The proof is complete.

Remark 4.3. Making use of **Eq. 18**, one can discuss the almost sure exponential stability for **Eq. 7** with $h \neq 0$. The proofs are similar to **Theorem 4.3** and are omitted.

5 QUASI-LINEAR HYBRID FRACTIONAL SYSTEMS

We now apply the extended Itô Formula in **Section 3** to discuss the stability for quasi-linear fractional SDEs with Markovian switching.

Theorem 5.1. : Let **Assumptions 3.1, 3.2** hold. If there exists a function $V \in C^{2,1}(\mathbb{R}\times\mathbb{R}_+\times\mathbb{S};\mathbb{R}_+)$ and positive constants a_1, a_2, b and $p \geq 1$, such that

$$a_1|X_t|^p \leq |V(X_t,t,i)| \leq a_2|X_t|^p, \tag{19}$$

$$\mathcal{L}^{(i)}V(X_t,t,i) \leq -b|X_t|^p, \tag{20}$$

for all $X_t \in \mathbb{R}$, $t \geq t_0$, $i \in \mathbb{S}$.

Then the solution of **Eq. 1** is pth moment exponential stable. More precisely,

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|X_t|^p) < 0.$$

Proof: According to **Lemma 3.1**, **Eq. 1** has a unique solution. Denote it $\{X_t\}_{t\geq 0}$. Set

$$U(X_t,t,i) = e^{\lambda t}V(X_t,t,i),$$

where $\lambda \in (\eta, \frac{b}{a_2})$, $\eta > 0$. Making use of **Definition 2.3** and **Lemma 3.2**, one has $\mathcal{A}U = e^{\lambda t}(\lambda V + \mathcal{A}V)$ and $(U_xg, s \in [0,T]) \in \mathcal{A}(0,T)$.

Applying the conditions **Eq. 19**, **20**, together with the generalized Itô**Eq. 5** and **Remark 2.1**, we obtain that for any $t \in [0, T]$

$$a_1 e^{\eta t}\mathbb{E}|X_t|^p \le \mathbb{E}U(X_t,t,i) = \mathbb{E}V(X_0,0,r_0) + \mathbb{E}\int_0^t \mathcal{A}U\mathrm{d}s + \mathbb{E}\int_0^t U_x g \mathrm{B}_s^H = \mathbb{E}V(X_0,0,r_0) + \mathbb{E}\int_0^t \mathcal{L}^{(r_s)}U\mathrm{d}s$$
$$= \mathbb{E}V(X_0,0,r_0) + \mathbb{E}\int_0^t e^{\lambda s}(\lambda V + \mathcal{A}V)\mathrm{d}s \le \mathbb{E}V(X_0,0,r_0) + \mathbb{E}\int_0^t e^{\lambda s}(\lambda a_2 - b)|X_t|^p\mathrm{d}s.$$

Thus we obtain that

$$a_1 e^{\eta t}\mathbb{E}|X_t|^p \le \mathbb{E}V(X_0,0,r_0) + \mathbb{E}\int_0^t e^{\lambda s}(\lambda a_2 - b)|X_t|^p\mathrm{d}s. \quad (21)$$

Dividing both sides of **Eq. 21** by $a_1 e^{\eta t}$, noting that $\lambda a_2 - b < 0$, we get

$$\mathbb{E}|X_t|^p \le \frac{e^{-\eta t}}{a_1}\mathbb{E}V(X_0,0,r_0) + \frac{e^{-\eta t}}{a_1}\mathbb{E}\int_0^t e^{\lambda s}(\lambda a_2 - b)|X_t|^p\mathrm{d}s \le \frac{e^{-\eta t}}{a_1}\mathbb{E}V(X_0,0,r_0).$$

Consequently,

$$\sup_{t\in[0,T]} a_1 e^{\eta t}\mathbb{E}|X_t|^p \le \mathbb{E}V(X_0,0,r_0).$$

Letting $T \to \infty$ gives

$$\sup_{t\ge 0}\mathbb{E}|X_t|^p \le \frac{e^{-\eta t}}{a_1}\mathbb{E}V(X_0,0,r_0),$$

and the required assertion follows. The proof is complete.

In the sequel of this section, we give another useful criterion and prove it briefly.

Theorem 5.2. Assume that **Eq. 1** has a unique solution and there exist a function $V \in C^{2,1}(\mathbb{R}\times\mathbb{R}_+\times\mathbb{S};\mathbb{R}_+)$ and positive constants b_1, b_2, $p \ge 1$ and $\beta_i \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $t \ge t_0$, $i \in \mathbb{S}$,

$$b_1|x|^p \le |V(x,t,i)| \le b_2|x|^p,$$
$$\mathcal{L}^{(i)}V(x,t,i) \le \beta_i V(x,t,i),$$

and

$$\sum_{i\in\mathbb{S}}\mu_i\beta_i < 0$$

Then **Eq. 1** is pth moment exponential stable.

Proof: Set $\bar{\beta}_i = \frac{1}{\theta}\beta_i$, where $\theta \in (0,1)$. Let $\delta = -\sum_{i\in\mathbb{S}}\mu_i\bar{\beta}_i = -\mu\bar{\beta}$. Let 1 denote the vector which all elements are 1. Then,

$$\mu(\bar{\beta} + \delta 1) = \mu\bar{\beta} + \delta = -\delta + \delta = 0. \quad (22)$$

By [1], **Eq. 22** implies the Poisson equation:

$$Qc = \bar{\beta} + \delta 1. \quad (23)$$

Note that **Eq. 23** has the solution $c = (c_1,\ldots,c_N)^T$. Hence,

$$-\delta = \bar{\beta}_i - \sum_{j=1}^N q_{ij}c_j, \quad i \in \mathbb{S}. \quad (24)$$

For each $i \in \mathbb{S}$, set $U(x,t,i) = (1-\theta c_i)V(x,t,i)$, where $\theta \in (0,1)$ is already defined and sufficiently small satisfying $1 - \theta c_i > 0$.

Then, for any $t \in [0, T]$ we get

$$\begin{aligned}\mathcal{A}U(x,t,i) &= (1-\theta c_i)\mathcal{L}^{(i)}V(x,t,i) + \sum_{i\ne j} q_{ij}(U(x,t,j) - U(x,t,i)) \\ &= (1-\theta c_i)\mathcal{L}^{(i)}V(x,t,i) - \theta V(x,t,i)\sum_{i\ne j}q_{ij}(c_j - c_i) \\ &\le (1-\theta c_i)\theta V(x,t,i)\left[\bar{\beta}_i - \sum_{i\ne j}q_{ij}\frac{c_j - c_i}{(1-\theta c_i)}\right].\end{aligned} \quad (25)$$

According to [1, 31], one has

$$\begin{aligned}\sum_{i\ne j}q_{ij}\frac{c_j - c_i}{(1-\theta c_i)} &= \sum_{i\ne j}q_{ij}c_j + \sum_{i\ne j}q_{ij}\frac{\theta c_i c_j - c_i}{1-\theta c_i} \\ &= \sum_{j=1}^N q_{ij}c_j + \sum_{i\ne j}q_{ij}\frac{c_i(c_j - c_i)}{1-\theta c_i}\theta = \sum_{j=1}^N q_{ij}c_j + o(\theta).\end{aligned} \quad (26)$$

Making use of **Eqs 25**, **26**, we obtain that

$$\mathcal{A}U(x,t,i) \le (1-\theta c_i)\theta V(x,t,i)\left[\bar{\beta}_i - \sum_{j=1}^N q_{ij}c_j + o(\theta)\right]. \quad (27)$$

Substituting **Eq. 24** into **Eq. 27**, we get

$$\mathcal{A}U(x,t,i) \le (1-\theta c_i)\theta V(x,t,i)[o(\theta) - \delta] = \kappa U(x,t,i),$$

where $\kappa < 0$. Making use of **Theorem 5.1**, the desired criterion follows.

On the other hand, we can prove it in another way. Set $\eta > 0$ and $\lambda \in (\eta, -\kappa)$. Define

$$\bar{U}(X_t,t,i) = \frac{e^{\lambda t}}{1-\theta c_i}U(X_t,t,i).$$

Compute

$$\begin{aligned}b_1 e^{\eta t}\mathbb{E}|X_t|^p &\le \mathbb{E}\bar{U}(X_t,t,i) = \mathbb{E}U(X_0,0,i_0) + \mathbb{E}\int_0^t \mathcal{A}\bar{U}\mathrm{d}s + \mathbb{E}\int_0^t \bar{U}_x g\mathrm{d}B_s^H \\ &= \mathbb{E}U(X_0,0,i_0) + \mathbb{E}\int_0^t e^{\lambda s}(\lambda U + \mathcal{A}U)\mathrm{d}s \le \mathbb{E}U(X_0,0,i_0) + \mathbb{E}\int_0^t e^{\lambda s}(\lambda+\kappa)U\mathrm{d}s \\ &= \mathbb{E}V(x_0,0,i_0) + \mathbb{E}\int_0^t e^{\lambda s}(\lambda+\kappa)V\mathrm{d}s \\ &\le \mathbb{E}V(x_0,0,i_0) + \mathbb{E}\int_0^t e^{\lambda s}(\lambda+\kappa)b_2|X_t|^p\mathrm{d}s.\end{aligned}$$

Thus we obtain that

$$b_1 e^{\eta t}\mathbb{E}|X_t|^p \le V(x_0,0,i_0) + \mathbb{E}\int_0^t e^{\lambda s}b_2(\lambda+\kappa)|X_t|^p\mathrm{d}s, \quad (28)$$

Dividing both sides of **Eq. 28** by $b_1 e^{\eta t}$, noting that $b_2(\lambda+\kappa) < 0$, we get

$$\mathbb{E}|X_t|^p \le \frac{e^{-\eta t}}{b_1}\mathbb{E}V(X_0,0,r_0) + \frac{e^{-\eta t}}{b_1}\mathbb{E}\int_0^t e^{\lambda s}b_2(\lambda+\kappa)|X_t|^p\mathrm{d}s \le \frac{e^{-\eta t}}{b_1}\mathbb{E}V(X_0,0,r_0).$$

Therefore, we obtain the required assertion

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|X_t|^p) < 0.$$

The proof is complete.

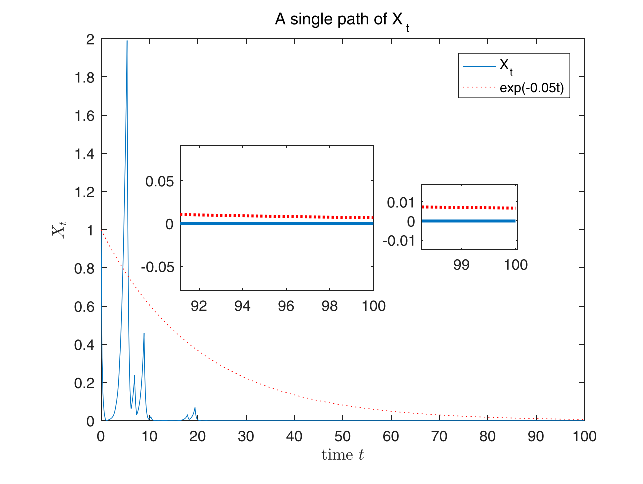

FIGURE 1 | A single path of solution.

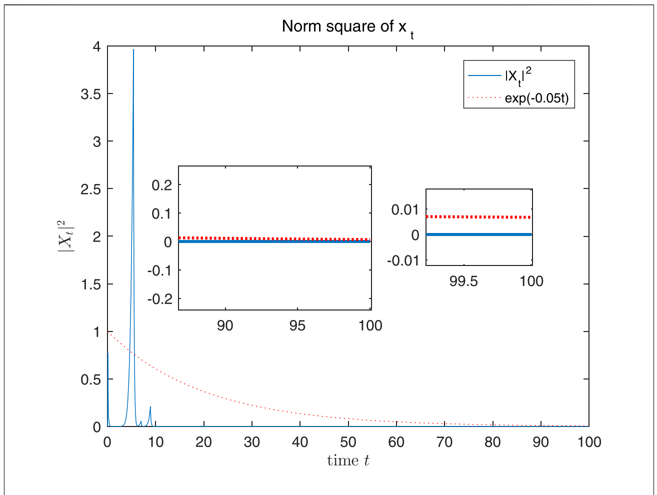

FIGURE 2 | Norm square trajectory.

6 EXAMPLE

In this section we give a numerical example to illustrate our results.

Example 1. Let $\{r_t\}_{t\geq 0}$ be a right-continuous Markov chain taking values in $\mathbb{S} = \{1, 2\}$ with invariant probability measure $\mu_1 = \mu_2 = \frac{1}{2}$.

Consider a risky asset, with the price dynamics:

$$\begin{cases} dX_t = f(X_t, t, r_r)dt + \sigma(t, r_t)X_t dB_t^H, \\ X_0 = 1, \end{cases} \tag{29}$$

on $t \geq 0$. Here we take $H = 0.7$ and

$$\begin{cases} f(x,t,i) = -4x, \ \ \sigma(t,i) = \dfrac{1}{t+1}, \ \text{if } i = 1, \\ f(x,t,i) = [2 - \sin(x)]x, \ \ \sigma(t,i) = e^{-t}, \ \text{if } i = 2. \end{cases}$$

Note that for all $i \in \mathbb{S}$, $dX_t = f(X_t, t, i)dt + \sigma(t, i)X_t dB_t^H$ satisfy the hypothesises (i)-(v). Then, by **Lemma 3.1**, it is easy to show that **Eq. 29** has a unique solution $\{X_t\}_{t\geq 0}$ as well. Set $V(x, t, i) = x^2$, for $i = 1, 2$.

Noting that for some $t_0 > 0$ sufficiently large and all $t > t_0$, we have

$$\begin{aligned} \mathcal{L}^{(1)}V(x,t,1) &= V_x(x,t,1)f(X_t,t,1) + V_{xx}(x,t,1)\frac{1}{t+1}xD_s^\phi x \\ &\leq -8x^2 + 2\frac{1}{t+1}x\left[xHt^{2H-1}\right] \\ &= -8x^2 + o(1)x^2 := \beta_1 x^2, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}^{(2)}V(x,t,2) &= V_x(x,t,2)f(X_t,t,2) + V_{xx}(x,t,2)e^{-t}xD_s^\phi x \\ &= 2x^2[2 - \sin(x)] + o(1)x^2 \\ &\leq 6x^2 + o(1)x^2 := \beta_2 x^2. \end{aligned}$$

Compute

$$\sum_{i\in\mathbb{S}} \mu_i \beta_i = \frac{1}{2}(-8 + 6) + o(1) < 0.$$

By **Theorem 5.2**, it's clear that the solution of **Eq. 29** is second moment exponential stable. **Figures 1**, **2** show a single path of the solution and the solution's norm square, respectively.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

WP contributed to conception and design of the study. WP wrote the first draft of the manuscript. ZZ and WP wrote sections of the manuscript. All authors contributed to manuscript revision, read, and approved the submitted version.

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