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Multivariate tempered stable model with long-range dependence and time-varying volatility

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High-frequency financial return time series data have stylized facts such as the long-range dependence, fat-tails, asymmetric dependence, and volatility clustering. In this paper, a multivariate model which describes those stylized facts is presented. To construct the model, a multivariate ARMA-GARCH model is considered along with fractional Lévy process. The fractional Lévy process in this paper is defined by the stochastic integral with a tempered stable driving process. Parameters of the new model are fit to high-frequency returns for five U.S stocks. Approximated form of portfolio value-at-risk and average value-at-risk are provided and portfolio optimization is discussed under the model.

Keywords: multivariate fractional normal tempered stable process, long-range dependence, fractional Brownian motion, fractional, Lévy processes, high-frequency market, intraday trading, volatility clustering, asymmetric dependence

JEL: C13, C32, C58, C61, G11, G32

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1. Introduction

The long-range dependence, fat-tail property, and volatility clustering effect are important issues for modeling high-frequency return time series in finance. The fractional Brownian motion introduced by Mandelbrot and Ness [1] can explain short-range or long-range dependence but it cannot explain the volatility clustering effect. The volatility clustering effect can be captured by the autoregressive conditional heteroskedastic (ARCH) and the generalized ARCH (GARCH) models formulated by Engle [2] and Bollerslev [3], respectively. However, GARCH models based on the normal distribution have not performed well in explaining high-frequency data analysis (see [4] and [5]), since the normal distribution does not capture the fat-tail property in the empirical innovation. To capture the fat-tail property in high-frequency return data, [5] suggested the univariate ARMA-GARCH model with tempered stable innovations without considering long-range dependence. The long-range dependence in high-frequency data is empirically reported in Sun et al. [6]. Sun et al. [4] provide a univariate model having long-range dependence, fat-tail property, and volatility clustering by taking the ARMA-GARCH model with fractional stable noise residuals exhibits, and show that the model has superior performance in high-frequency returns.

In this paper, a new multivariate market model that describes the long-range dependence, fat-tail property, and volatility clustering effect is developed. The new market model is constructed by taking the fractional tempered stable innovations on the multivariate ARMA-GARCH model. Univariate fractional tempered stable process was defined by the stochastic integral for the Volterra kernel in Houdre and Kawai [7] based on subclasses of Rosinski's tempered stable processes [8]. In order to construct multivariate model, we would better use normal tempered stable (NTS) process rather than Rosinski's tempered stable processes, since the NTS process is defined by the time-changed Brownian motion, and hence we can easily obtain a multivariate model which allows fatter tails than the multivariate Gaussian distribution and an asymmetric dependence structure. The NTS process has been discussed in many literatures including [9–12], and [13]. Although NTS is not included in Rosinski's tempered stable processes (see [14]), the fractional NTS process is redefined in this paper. The fractional NTS process is a multivariate process having the long-range dependence in time and asymmetric dependence between elements[1].

To verify the performance of the new model, empirical illustration is provided using high-frequency stock return data. Useful simulation and parameter estimation methods are provided, and the goodness-of-fit tests are performed for the estimated parameters. The long-range dependence and fat-tail property of the high frequency stock return data are observed by this investigation.

To apply the new model for the financial risk management and portfolio management, portfolio value-at-risk (VaR), average VaR (AVaR), and portfolio optimization based on the new market model are discussed. VaR has a number of well-known limitations as a risk measure, nevertheless the VaR measure has been popularly used as a standard risk measure in the financial industry. The AVaR is the average of VaRs exceeding the VaR for a given confidence level[2] AVaR is a superior alternative to VaR because it is a coherent risk measure[3]. and it is consistent with preference relations of risk-averse investors (see [20]).

A major contribution to the portfolio theory is the mean-variance model presented by Markowitz [21]. The importance of the model cannot be overstated, but some of assumptions underlying the model have been challenged since its introduction. One of the assumptions is that asset returns follow a Gaussian (normal) distribution and another is that the variance is a measure of risk ignoring higher-order moments. In this paper, the portfolio optimization is discussed based on the Markowitz's theory, but the Gaussian assumption is replaced by the ARMA-GARCH model with fractional NTS innovations and the variance risk measure is superseded by VaR and AVaR risk measures.

The remainder of this paper is organized as follows. The NTS process is reviewed in Section 2. The definition of the multivariate fractional NTS process and its simulation method are presented in Sections 3, 4, respectively. Multivariate ARMA-GARCH model having long-range dependence is defined in Section 5 along with the empirical illustration. In Section 6, portfolio risk will be assessed and the optimal portfolio will be found based on the new ARMA-GARCH model using high-frequency market data. In Section 7, the principal findings are summarized. Volterra kernel is briefly reviewed in the appendix.

[1] Kim [15] defines another different multivariate fractional NTS process using the time-changed fractional Brownian motion.

[2] AVaR is also known as conditional value-at-risk (CVaR). See Pflug [16] and Rockafellar and Uryasev [17, 18].

[3] VaR is not is not a coherent risk measure. The notion of a coherent risk measure was introduced by Artzner et al. [19].

2. Normal Tempered Stable Process

In the remainder of this paper, we assume that N is a positive integer standing for the dimension and $\alpha \in (0,2)$, $\theta > 0$, $\beta = (\beta_1, \beta_2, \cdots, \beta_N)^\mathsf{T}$, $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_N)^\mathsf{T}$ such that $\gamma_n > 0$ for $n \in \{1, 2, \cdots, N\}$, and $\Sigma = [\sigma_{m,n}]_{m,n\in\{1,2,\cdots,N\}}$ is a given correlation matrix such that $\sigma_{n,n} = 1$ for $n \in \{1, 2, \cdots, N\}$. The pure jump Lévy process $T = (T(t))_{t\geq 0}$ whose characteristic function $\phi_{T(t)}$ is equal to

$$\phi_{T(t)}(u) = \exp\left(\frac{\theta^{1-\frac{\alpha}{2}}t}{\Gamma\left(1-\frac{\alpha}{2}\right)}\int_0^\infty (e^{iux}-1)\frac{e^{-\theta x}}{x^{\alpha/2+1}}dx\right).$$

is a subordinator and referred to as the *tempered stable subordinator* with parameters (α, θ). Solving the integration in the last Equation, we can obtain the following formula,

$$\phi_{T(t)}(u) = \exp\left(-\frac{2\theta^{1-\frac{\alpha}{2}}t}{\alpha}\left((\theta - iu)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right)\right). \quad (1)$$

Consider a N-dimensional Brownian motion $B = (B(t))_{t\geq 0}$ such that $B(t) = (B_1(t), B_2(t), \cdots, B_N(t))^\mathsf{T}$, and suppose that

$$\mathrm{cov}(B_m(t), B_n(t)) = \sigma_{m,n}t$$

for all $m, n \in \{1, 2, \cdots, N\}$.

Suppose T is independent of B. Consider a N-dimensional process $Z = (Z(t))_{t\geq 0}$ such that $Z(t) = (Z_1(t), Z_2(t), \cdots, Z_N(t))^\mathsf{T}$. For $n \in \{1, 2, \cdots, N\}$, define $(Z_n(t))_{t\geq 0}$ by the time-changed Brownian motion as

$$Z_n(t) = \beta_n(T(t) - t) + \gamma_n B_n(T(t)). \quad (2)$$

Then the process Z is referred to as the *N-dimensional NTS* process with parameters $(\alpha, \theta, \beta, \gamma, \Sigma)$ and we denote by $Z \sim \mathrm{NTS}_N$ $(\alpha, \theta, \beta, \gamma, \Sigma)$.

By composing characteristic functions of $B(t)$ and $T(t)$, we obtain the characteristic function of $Z_n(t)$ as follows:

$$\phi_{Z_n(t)}(u) = \exp\left(-\beta_n uti - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}t\left(\left(\theta - i\beta_n u + \frac{\gamma_n^2 u^2}{2}\right)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right)\right). \quad (3)$$

The mean of $Z_n(t)$ is equal to zero for $n \in \{1, 2, \cdots, N\}$. Covariance between $Z_m(t)$ and $Z_n(t)$ is given by

$$\mathrm{cov}(Z_m(t), Z_n(t)) = t\gamma_m\gamma_n\sigma_{m,n} + t\beta_m\beta_n\left(\frac{2-\alpha}{2\theta}\right) \quad (4)$$

for $m, n \in \{1, 2, \cdots, N\}$. Moreover the variance of $Z_n(t)$ is equal to

$$\mathrm{Var}(Z_n(t)) = t\left(\gamma_n^2 + \beta_n^2\left(\frac{2-\alpha}{2\theta}\right)\right) \quad (5)$$

The linear combination of elements of Z is also NTS as follows.

Proposition 1. *Let $w = (w_1, w_2, \cdots, w_N)^T \in \mathbb{R}^N$, $Z \sim \text{NTS}_N$ $(\alpha, \theta, \beta, \gamma, \Sigma)$, and $Y(t) = \sum_{n=1}^{N} w_n Z_n(t)$. Then $(Y(t))_{t\geq 0} \sim \text{NTS}_1(\alpha, \theta, \bar{\beta}, \bar{\gamma}, 1)$, where*

$$\bar{\beta} = \sum_{n=1}^{N} w_n\beta_n, \text{ and } \bar{\gamma} = \sqrt{\sum_{m=1}^{N}\sum_{n=1}^{N} w_m w_n \gamma_m \gamma_n \sigma_{m,n}}.$$

Proof. See [12].

If $\gamma_n = \sqrt{1-\beta_n^2\left(\frac{2-\alpha}{2\theta}\right)}$ and $|\beta_n| < \sqrt{\frac{2\theta}{2-\alpha}}$ for $n \in \{1, 2, \cdots, N\}$ then $\text{Var}(Z_n(t)) = t$. In this case the process Z is referred to as the N-dimensional *standard NTS process* with parameters $(\alpha, \theta, \beta, \Sigma)$ and denoted by $Z \sim \text{stdNTS}_N\ (\alpha, \theta, \beta, \Sigma)$.

3. Fractional Normal Tempered Stable Process

Let $K_H(t, s)$ be the Voltera kernel and $Z \sim \text{NTS}_N\ (\alpha, \theta, \beta, \gamma, \Sigma)$. The N-dimensional *fractional NTS (fNTS) process* generated by Z is defined by the process of vector $X = (X(t))_{t\geq 0}$ with $X(t) = (X_1(t), X_2(t), \cdots, X_N(t))^T$ such that

$$X_n(t) = \lim_{||P||\to 0} \sum_{j=1}^{M} K_H(t, t_{j-1})\left(Z_n(t_j) - Z_n(t_{j-1})\right)$$

in distribution sense for $n \in \{1, 2, \cdots, N\}$, where

$$P: 0 = t_0 < t_1 < \cdots < t_M = t$$

is a partition of the interval $[0, t]$ and

$$||P|| = \max\{t_j - t_{j-1} | j = 1, 2, \cdots, M\}.$$

In this case we denote that $X_n(t) = \int_0^t K_H(t, s)dZ_n(s)$ and $X \sim fNTS_N\ (H, \alpha, \theta, \beta, \gamma, \Sigma)$. Since we have

$$\begin{aligned}
E&\left[\exp\left(iu\sum_{j=1}^{M} K_H(t, t_{j-1})\left(Z_n(t_j) - Z_n(t_{j-1})\right)\right)\right] \\
&= \prod_{j=1}^{M} E\left[\exp\left(iuK_H(t, t_{j-1})\left(Z_n(t_j) - Z_n(t_{j-1})\right)\right)\right] \\
&= \prod_{j=1}^{M} \exp\Bigg(-\beta_n uiK_H(t, t_{j-1})(t_j - t_{j-1}) \\
&\quad - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\left(\left(\theta - i\beta_n uK_H(t, t_{j-1}) + \frac{\gamma_n^2 u^2 (K_H(t, t_{j-1}))^2}{2}\right)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right)(t_j - t_{j-1})\Bigg) \\
&= \exp\Bigg(-\beta_n ui\sum_{j=1}^{M} K_H(t, t_{j-1})(t_j - t_{j-1}) \\
&\quad - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\sum_{j=1}^{M}\left(\left(\theta - i\beta_n uK_H(t, t_{j-1}) + \frac{\gamma_n^2 u^2 (K_H(t, t_{j-1}))^2}{2}\right)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right)(t_j - t_{j-1})\Bigg),
\end{aligned}$$

the characteristic function of $X_n(t)$ is given by

$$\begin{aligned}
\phi_{X_n(t)}(u) &= \lim_{||P||\to 0} \exp\Bigg(-\beta_n ui\sum_{j=1}^{M} K_H(t, t_{j-1})(t_j - t_{j-1}) \\
&\quad - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\sum_{j=1}^{M}\left(\left(\theta - i\beta_n uK_H(t, t_{j-1}) + \frac{\gamma_n^2 u^2 (K_H(t, t_{j-1}))^2}{2}\right)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right)(t_j - t_{j-1})\Bigg) \\
&= \exp\Bigg(-\beta_n ui\int_0^t K_H(t, s)ds \\
&\quad - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\int_0^t\left(\left(\theta - i\beta_n uK_H(t, s) + \frac{\gamma_n^2 u^2 (K_H(t, s))^2}{2}\right)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right)ds\Bigg).
\end{aligned}$$

Proposition 2. *For $n \in \{1, 2, \cdots, N\}$, the covariance between $X_n(s)$ and $X_n(t)$ is equal to*

$$\begin{aligned}
&\text{cov}(X_n(s), X_n(t)) \\
&= \frac{1}{2}\left(\gamma_n^2 + \beta_n^2\left(\frac{2-\alpha}{2\theta}\right)\right)\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s, t > 0.
\end{aligned} \tag{6}$$

Proof. Let P be a partition such that

$$\begin{aligned}
P:\ 0 &= t_0 < t_1 < \cdots < t_{M-1} < t_M \\
&= s \wedge t < t_{M+1} < \cdots < t_{M^*} = s \vee t.
\end{aligned}$$

Then we have

$$\begin{aligned}
\text{cov}(X_n(s), X_n(t)) &= E[X_n(s)X_n(t)] \\
&= \lim_{||P||\to 0} E\Bigg[\sum_{j=1}^{M} K_H(s, t_{j-1})\left(Z_n(t_j) - Z_n(t_{j-1})\right) \\
&\qquad \sum_{k=1}^{M^*} K_H(t, t_{k-1})\left(Z_n(t_k) - Z_n(t_{k-1})\right)\Bigg] \\
&= \lim_{||P||\to 0} \sum_{j=1}^{M}\sum_{k=1}^{M^*} K_H(s, t_{j-1})K_H(t, t_{k-1}) \\
&\qquad E\left[\left(Z_n(t_j) - Z_n(t_{j-1})\right)\left(Z_n(t_k) - Z_n(t_{k-1})\right)\right]
\end{aligned}$$

By the property of the NTS process Z_n, we have

$$E\left[\left(Z_n(t_j) - Z_n(t_{j-1})\right)\left(Z_n(t_k) - Z_n(t_{k-1})\right)\right] = \begin{cases}(t_j - t_{j-1})\mathrm{Var}(Z_n(1)) & \text{if } j = k \\ 0 & \text{if } j \neq k\end{cases}$$

Hence, we obtain

$$\begin{aligned}\mathrm{cov}(X_n(s), X_n(t)) &= \lim_{||P||\to 0}\sum_{j=1}^{M} K_H(s, t_{j-1})K_H(t, t_{j-1}) \\ &\quad (t_j - t_{j-1})\mathrm{Var}(Z_n(1)) \\ &= \mathrm{Var}(Z_n(1))\int_0^{s\wedge t} K_H(s, u)K_H(t, u)du.\end{aligned}$$

Hence we obtain Equation (6) by Equation (5) and Equation (21) in Appendix.

Proposition 3. *For $m, n \in \{1, 2, \cdots, N\}$, the covariance between $X_m(t)$ and $X_n(t)$ is equal to*

$$\mathrm{cov}(X_m(t), X_n(t)) = t^{2H}\left(\sigma_{m,n}\gamma_m\gamma_n + \beta_m\beta_n\left(\frac{2-\alpha}{2\theta}\right)\right), \quad t > 0. \tag{7}$$

Proof. Let P be a partition such that

$$P : 0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = t.$$

We have

$$\begin{aligned}\mathrm{cov}(X_m(t), X_n(t)) &= E[X_m(t)X_n(t)] \\ &= \lim_{||P||\to 0} E\left[\sum_{j=1}^{M} K_H(t, t_{j-1})\left(Z_m(t_j) - Z_m(t_{j-1})\right)\right. \\ &\quad \left.\sum_{k=1}^{M} K_H(t, t_{k-1})\left(Z_n(t_k) - Z_n(t_{k-1})\right)\right] \\ &= \lim_{||P||\to 0}\sum_{j=1}^{M}\sum_{k=1}^{M} K_H(t, t_{j-1})K_H(t, t_{k-1}) \\ &\quad E\left[\left(Z_m(t_j) - Z_m(t_{j-1})\right)\left(Z_n(t_k) - Z_n(t_{k-1})\right)\right]\end{aligned}$$

By the property of the NTS process Z_n, we have

$$E\left[\left(Z_m(t_j) - Z_m(t_{j-1})\right)\left(Z_n(t_k) - Z_n(t_{k-1})\right)\right] = \begin{cases}(t_j - t_{j-1})\mathrm{cov}(Z_m(1), Z_n(1)) & \text{if } j = k \\ 0 & \text{if } j \neq k\end{cases}$$

Hence, we obtain

$$\mathrm{cov}(X_m(t), X_n(t)) = \mathrm{cov}(Z_m(1), Z_n(1))\int_0^t (K_H(t, u))^2 du.$$

Hence we obtain Equation (7) by Equation (4) and Equation (22) in Appendix.

For a given stochastic process $Y = (Y(t))_{t\geq 0}$, the summation

$$\sum_{j=1}^{\infty} E[(Y(1) - Y(0))(Y(j+1) - Y(j))]$$

diverges, then we say that Y exhibits *long-range dependence* (See [22]). By Proposition 2 and L'Hopital's rule, we have

$$\begin{aligned}&E[X_n(1)(X_n(j+1) - X_n(j))] \\ &= \frac{\nu}{2}\left((j+1)^{2H} - 2j^{2H} + (j-1)^{2H}\right) \\ &= \frac{\nu}{2} j^{2H-2}\left(j^2\left(\left(1+\frac{1}{j}\right)^{2H} - 2 + \left(1-\frac{1}{j}\right)^{2H}\right)\right) \\ &\to \nu H(2H-1)j^{2H-2} \text{ as } j \to \infty,\end{aligned}$$

where $\nu = \left(\gamma_n^2 + \beta_n^2\left(\frac{2-\alpha}{2\theta}\right)\right)$. Hence, $\sum_{j=1}^{\infty} E[X_n(1)(X_n(j+1) - X_n(j))]$ diverges, i.e., the process $(X_n(t))_{t\geq 0}$ has long-range dependence, when $\frac{1}{2} < H < 1$.

Since the NTS has an asymmetric dependence structure, X has also asymmetric dependence. By Proposition 1, we can prove the following proposition.

Proposition 4. *Let $w = (w_1, w_2, \cdots, w_N)^T \in \mathbb{R}^N$, $X \sim$ fNTS$_N$ $(H, \alpha, \theta, \beta, \gamma, \Sigma)$, and $Y(t) = \sum_{n=1}^{N} w_n X_n(t)$. Then $(Y(t))_{t\geq 0} \sim$ fNTS$_1(H, \alpha, \theta, \bar{\beta}, \bar{\gamma}, 1)$, where*

$$\bar{\beta} = \sum_{n=1}^{N} w_n\beta_n, \text{ and } \bar{\gamma} = \sqrt{\sum_{m=1}^{N}\sum_{n=1}^{N} w_m w_n\gamma_m\gamma_n\sigma_{m,n}}.$$

When $Z \sim$ stdNTS$_N$ $(\alpha, \theta, \beta, \Sigma)$, the multivariate fractional NTS process X generated by Z is referred to as the *fractional standard NTS process*, and we denote that $X \sim$ fstdNTS$_N$ $(H, \alpha, \theta, \beta, \Sigma)$. In this case, we have

$$\mathrm{cov}(X_n(s), X_n(t)) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s, t > 0. \tag{8}$$

and

$$\begin{aligned}&\mathrm{cov}(X_m(t), X_n(t)) \\ &= t^{2H}\left(\sigma_{m,n}\sqrt{1-\beta_m^2\left(\frac{2-\alpha}{2\theta}\right)}\sqrt{1-\beta_n^2\left(\frac{2-\alpha}{2\theta}\right)}\right. \\ &\quad \left. + \beta_m\beta_n\left(\frac{2-\alpha}{2\theta}\right)\right), \quad t > 0.\end{aligned} \tag{9}$$

By Proposition 4, we can prove the following corollary.

Corollary 5. *Let $w = (w_1, w_2, \cdots, w_N)^T \in \mathbb{R}^N$, $X \sim$ fstdNTS$_N$ $(H, \alpha, \theta, \beta, \Sigma)$, and $Y(t) = \sum_{n=1}^{N} w_n X_n(t)$. Then $(Y(t))_{t\geq 0} \sim$ fNTS$_1(H, \alpha, \theta, \bar{\beta}, \bar{\gamma}, 1)$, where*

$$\bar{\beta} = \sum_{n=1}^{N} w_n\beta_n,$$

and

$$\bar{\gamma} = \sqrt{\sum_{m=1}^{N}\sum_{n=1}^{N} w_m w_n \sigma_{m,n} \sqrt{1-\beta_m^2\left(\frac{2-\alpha}{2\theta}\right)}\sqrt{1-\beta_n^2\left(\frac{2-\alpha}{2\theta}\right)}}.$$

4. Simulation

In this section, a numerical method is provided to generate the sample path for the multivariate fractional normal tempered stable process. By Theorem 5.3 (i) in Rosiński [8][4], we obtain series representation for the tempered stable subordinator T as follows:

$$T(t) = \lim_{M\to\infty} \sum_{j=1}^{M} 1_{(0,t)}(\tau_j)\left(\left(\frac{\alpha\xi_j\Gamma\left(-\frac{\alpha}{2}\right)}{2\theta^{1-\frac{\alpha}{2}}\mathcal{T}}\right)^{-\frac{2}{\alpha}} \wedge \frac{e_j u_j^{\frac{2}{\alpha}}}{\theta}\right), \quad t\in[0,\mathcal{T}], \quad (10)$$

where

- $\{u_j\}$ is an iid sequence of random variables on (0, 1),
- $\{e_j\}$ and $\{e'_j\}$ are an iid sequence of exponential random variables with parameter 1,
- $\xi_j = e'_1 + e'_2 + \cdots + e'_j$,
- $\{\tau_j\}$ be an independent and identically distributed uniform random variable in $[0,\mathcal{T}]$, where $\mathcal{T} > 0$ is fixed.
- and assume that $\{u_j\}$, $\{e_j\}$, $\{e'_j\}$ and $\{\tau_i\}$ are independent.

Let L_Σ be the lower triangular matrix obtained by the Cholesky decomposition for Σ with $\Sigma = L_\Sigma L_\Sigma^{\mathsf{T}}$, where Σ is the correlation matrix in Equation (2). Then we have $B(t) = L_\Sigma \bar{B}(t)$ where $\bar{B}(t) = (\bar{B}_1(t), \bar{B}_2(t), \cdots, \bar{B}_N(t))^{\mathsf{T}}$ is a mutually independent vector of Brownian motions.

Sample path of $Z \sim \mathrm{NTS}_N\ (\alpha,\theta,\beta,\gamma,\Sigma)$ is generated as follows. For a given partition $\{t_0, t_1, \cdots, t_M\}$ of the interval $[0,\mathcal{T}]$ with $t_0 = 0$, $t_M = \mathcal{T}$ and $t_j < t_k$ for $j < k$, we have

$$\bar{B}_n(T(t_j)) - \bar{B}_n(T(t_{j-1})) = \sqrt{T(t_j) - T(t_{j-1})}\epsilon_{j,n}, \quad n\in\{1,2,\cdots,N\}$$

where $\epsilon_{j,n} \sim N(0,1)$. Therefore, we have

$$Z(t_k) = (T(t_k) - t_k)\beta + \mathrm{diag}(\gamma)\sum_{j=1}^{k}\sqrt{T(t_j) - T(t_{j-1})}L_\Sigma\epsilon_j, \quad (11)$$

where $\epsilon_j = (\epsilon_{j,1}, \epsilon_{j,2}\cdots,\epsilon_{j,N})^{\mathsf{T}}$, $\epsilon_{j,n} \sim N(0,1)$, and $\epsilon_{j,m}$ is independent of $\epsilon_{j,n}$ for all $m,n \in \{1,2,\cdots,N\}$, and $j \in \{1,2,\cdots,M\}$.

[4]The tempered stable subordinator T with parameter (α,θ) is included in the class of tempered stable processes provided by Rosinski [8]. The parameter $\alpha/2$ of the tempered stable subordinator corresponds to the parameter α in Rosinski's tempered stable class.

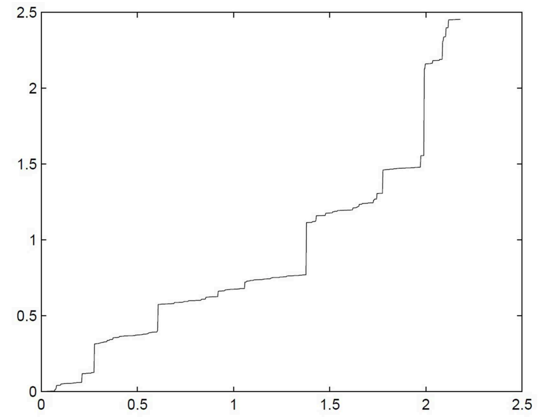

FIGURE 1 | Simulated sample path of the tempered stable subordinator T with parameters $\alpha = 1.2$ and $\theta = 0.8$.

Finally, sample paths of $X \sim \mathrm{fNTS}_N\ (H,\alpha,\theta,\beta,\gamma,\Sigma)$ is generated as follows:

$$X(t_k) = \sum_{j=0}^{k-1} K_H(t_k,t_j)(Z(t_{j+1}) - Z(t_j)), \quad k\in 1,2,\cdots,M. \quad (12)$$

Figure 1 presents one example sample path of the tempered stable subordinator T with parameters $\alpha = 1.2$ and $\theta = 0.8$. **Figure 2** exhibits one pair of simulated sample path of the 2 dimensional standard fractional NTS process $X = (X_1(t), X_2(t))_{t\ge0}$ with parameters $\alpha = 1.2$, $\theta = 0.8$, $\beta = (0.1, -0.3)$, $\sigma_{1,2} = \sigma_{2,1} = 0.7$, and $H = 0.6525$. In the process **Figure 2A** $(X_1(t))_{t\ge0}$ and $(X_2(t))_{t\ge0}$ are drawn on the time line. **Figure 2B** presents 2 dimensional movements of X.

5. ARMA-GARCH Model with fNTS Innovations and Empirical Illustration

Let $X \sim \mathrm{fstdNTS}_N\ (H,\alpha,\theta,\beta,\Sigma)$ generated by $Z \sim \mathrm{stdNTS}_N\ (\alpha,\theta,\beta,\Sigma)$. A N-dimensional discrete time process $Y = (Y(k))_{k\in\{0,1,2,\cdots\}}$ with $Y(k) = (Y_1(k), Y_2(k), \cdots, Y_N(k))$ is referred to as the N-dimensional *ARMA-GARCH model with fNTS innovations* when it is given by the ARMA(1,1) - GARCH(1,1) model as follows: $Y_n(0) = 0$, $\varepsilon_n(0) = 0$, and

$$\begin{cases} Y_n(k+1) = c_n + a_n Y_n(k) + b_n\sigma_n(k)\varepsilon_n(k) \\ \qquad\qquad + \sigma_n(k+1)\varepsilon_n(k+1) \\ (\sigma_n(k+1))^2 = \kappa_n + \xi_n(\sigma_n(k)\varepsilon_n(k))^2 + \zeta_n(\sigma_n(k))^2 \end{cases}, \quad (13)$$

where $\varepsilon_n(k+1) = X_n(k+1) - X_n(k)$ and $n \in \{1,2,\cdots,N\}$. This model describes volatility clustering effect by GARCH(1,1) model, the fat-tails and the asymmetric dependence between elements by the standard NTS process Z, and the long-range dependence by fractional NTS process X.

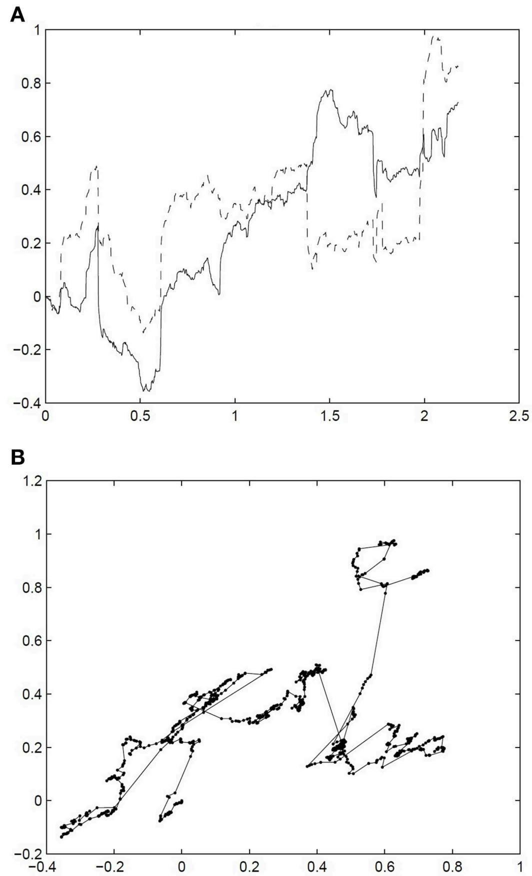

FIGURE 2 | Simulated sample path of the 2 dimensional standard fractional NTS process $X = (X_1(t), X_2(t))_{t \geq 0}$ with parameters $\alpha = 1.2$, $\theta = 0.8$, $\beta = (0.1, -0.3)$, $\sigma_{1,2} = \sigma_{2,1} = 0.7$, and $H = 0.6525$.

Since we have

$$X_n(k) \approx \sum_{j=0}^{k-1} K_H(k,j)(Z_n(j+1) - Z_n(j))$$

The increment $X_n(k+1) - X_n(k)$ can be approximated as follows:

$$\begin{aligned} & X_n(k+1) - X_n(k) \\ & \approx K_H(k+1,k)(Z_n(k+1) - Z_n(k)) \\ & \quad + \sum_{j=0}^{k-1}(K_H(k+1,j) - K_H(k,j))(Z_n(j+1) - Z_n(j)) \end{aligned} \tag{14}$$

Let M be the number of time steps in the sample and N be the number of assets in the portfolio. ARMA(1,1)-GARCH(1,1) model is fit to the data and extract $\varepsilon_n(t_k)$, for $n = 1, 2, \cdots, N$ and $k = 1, \cdots, M$. Since Equation (8) is the same as the covariance of the fractional Brownian motion, set $W_n(t_k) = \sum_{j=1}^{k} \varepsilon_n(t_j)$ and estimate the Hurst index H_n of the process $(W_n(t_k))_{t_k \geq 0}$ using the wavelet details regression estimator method by Flandrin [23] and Abry et al. [24]. The parameter H is obtained finally as the mean of H_n for $n = 1, 2, \cdots, N$.

TABLE 1 | Estimated Hurst index H_n.

Apple Inc.	$H_1 = 0.5631$
Google Inc.	$H_2 = 0.5241$
IBM Co.	$H_3 = 0.6986$
AT&T	$H_4 = 0.5303$
Wal-Mart Stores Inc.	$H_5 = 0.5870$
Mean	$H = 0.5806$

Suppose we have estimated Hurst index H. We estimate the parameters of the model as follows in this investigation.

1. Estimate ARMA(1,1)-GARCH(1,1) parameters a_n, b_n, c_n, κ_n, ξ_n, and ζ_n with standard normal innovations by maximum likelihood estimation (MLE) with assumption $(\sigma_n(0))^2 = \kappa_n/(1 - \xi_n - \zeta_n)$ for $n = 1, 2, \cdots, N$.
2. Extract residuals using the estimated parameters.
3. Put $X_n(t_k) = \sum_{j=1}^{k} \varepsilon_n(t_j)$ and extract $\{Z_n(t_k) | k = 1, 2, \cdots, M\}$ for $n = 1, 2, \cdots, N$ as follows.

   $Z_n(1) = X_n(1)/K_H(1,0)$ and

$$\begin{aligned} Z_n(k) = Z_n(k-1) + \frac{X_n(k) - X_n(k-1)}{K_H(k,k-1)} \\ - \sum_{j=0}^{k-2} \frac{K_H(k,j) - K_H(k-1,j)}{K_H(k,k-1)}(Z_n(j+1) - Z_n(j)) \end{aligned} \tag{15}$$

   for $k = 2, 3, \cdots, M$.
4. Estimate parameters α_n, θ_n, and β_n of the standard NTS process using $\{Z_n(k) | k = 1, 2, \cdots, M\}$ extracted in the step 4 by curve-fitting in least-squares sense. Set $\alpha = \sum_{n=1}^{N} \frac{\alpha_n}{N}$ and $\theta = \sum_{n=1}^{N} \frac{\theta_n}{N}$ Estimate parameters β_n again using $\{Z_n(k)| k = 1, 2, \cdots, M\}$ by means of MLE for $n = 1, 2, \cdots, N$.
5. Calculate the covariance between $(Z_m(1))$ and $(Z_n(1))$ for $m,n \in \{1, 2, \cdots, N\}$ using data $\{(Z_m(k), Z_n(k)) | k = 1, 2, \cdots, M\}$ extracted in the step 4. Estimate

$$\Sigma = [\sigma_{m,n}]_{m,n \in \{1,2,\cdots,N\}}$$

   by Equation (4) and $\mathrm{cov}(Z_m(1), Z_n(1))$.

The parameter H is estimated using 2,158 observed 1 min returns for five stocks (Apple Inc., Google Inc., IBM Co., AT&T, Wal-Mart Stores Inc.) from November 21 to November 29, 2011. Set $dt = 1/390$, since 1 day has 6 h and 30 min (from 9:30 to 16:00) trading time, that is 390 min, on New York Stock Exchange. The parameters H_n are presented in **Table 1** and parameter $H = 0.5806$ is obtained finally as the mean of H_n.

The ARMA(1,1)-GARCH(1,1) parameters and estimated fNTS parameters α, θ and β are reported in **Table 2** for each stocks. The estimated matrix Σ is presented in **Table 3**.

The Kolmogorov-Smirnov (KS) and Anderson-Darling (AD) tests are used for goodness-of-fit tests. The KS and AD^2 statistic are given by

$$KS = \sum_x |\hat{F}(x) - F(x)|, \quad AD^2 = s \int_{-\infty}^{\infty} \frac{(\hat{F}(x) - F(x))^2}{F(x)(1 - F(x))} dF(x),$$

TABLE 2 | Parameters the ARMA-GARCH model with (non-fractional) NTS innovations and fNTS innovations.

	Apple Inc. $n = 1$	Google Inc. $n = 2$	IBM Co. $n = 3$	AT&T $n = 4$	Wal-Mart Stores Inc. $n = 5$
ARMA(1,1)					
a_n	0.3244	0.0808	−0.0223	−0.1271	−0.1351
b_n	−0.5054	−0.3608	−0.5177	0.2376	−0.0445
c_n	$0.1560 \cdot 10^{-4}$	$0.4607 \cdot 10^{-4}$	$0.6596 \cdot 10^{-4}$	$0.6311 \cdot 10^{-4}$	0.2862
GARCH(1,1)					
ζ_n	0.6703	0.9949	0.3078	0.8529	0.7860
ξ_n	0.3297	0.0051	0.6922	0.1471	0.2140
κ_n	$0.1479 \cdot 10^{-6}$	$0.3412 \cdot 10^{-6}$	$0.1127 \cdot 10^{-6}$	$0.1353 \cdot 10^{-6}$	0.0535
NON-FRACTIONAL stdNTS, $\alpha = 1.5750$, $\theta = 50.23$					
β_n	0.5545	1.4326	9.0557	4.6653	0.0427
FRACTIONAL stdNTS, $H = 0.5806$, $\alpha = 1.6601$, $\theta = 17.61$					
β_n	1.3929	1.7403	6.7900	3.6981	−0.2057

TABLE 3 | Estimated $\Sigma = [\sigma_{m,n}]_{m,n \in \{1,2,\cdots,N\}}$.

	$\sigma_{m,n}$	Apple Inc. $n = 1$	Google Inc. $n = 2$	IBM Co. $n = 3$	AT&T $n = 4$	Wal-Mart Stores Inc. $n = 5$
Apple Inc.	$m = 1$	1.0000	0.6414	0.7109	0.4373	0.4290
Google Inc.	$m = 2$	0.6414	1.0000	0.7869	0.4773	0.4914
IBM Co.	$m = 3$	0.7109	0.7869	1.0000	0.5676	0.7603
AT&T	$m = 4$	0.4373	0.4773	0.5676	1.0000	0.5741
Wal-Mart Stores Inc.	$m = 5$	0.4290	0.4914	0.7603	0.5741	1.0000

where $\hat{F}(x)$ is the empirical sample distribution, $F(x)$ is the estimated theoretical distribution, and s is the number of observed samples. According to the p-values of KS statistic and AD^2 statistic, estimated parameters are not rejected. Calculating p-values for KS and AD^2 statistic are explained in Marsaglia et al. [25] and Marsaglia and Marsaglia [26]. The KS and AD^2 statistics are calculated between the standard NTS distribution with estimated parameters $(\alpha, \theta, \beta_n)$, and the empirical cumulative distribution of $\{Z_n(t_k) - Z_n(t_{k-1}) | \, k = 1, 2, \cdots, M\}$ where $Z_n(t_k)$ is the extracted process by Equation (15). According to the p-values of KS and AD^2 statistic in **Table 4**, all estimated parameters are not rejected at the 1% significance level for the five stock returns investigated.

If $H = 0.5$ then the ARMA-GARCH model with fNTS innovations becomes the ARMA-GARCH model with non-fractional NTS innovations. The estimated parameters of the non-fractional standard NTS process are presented also in **Table 2** for each stocks. To compare the performance of parameter estimation for ARMA-GARCH model with fNTS innovations with non-fractional NTS innovation, we provide KS and AD^2 statistic with p-values for the ARMA-GARCH model with non-fractional NTS innovations in **Table 4** as well. Unfortunately, the estimated parameters for IBM Co. and AT&T are rejected by KS and AD tests at the 1% significance level.

If we assume that $(\varepsilon_n(t_k))_{k=1,2,\cdots,M}$ is independent and identically distributed and $\varepsilon_n(t_k) \sim N(0, 1)$, then the Y_n follows the ARMA-GARCH model with normal innovations. The ARMA-GARCH model with normal innovations is rejected by both KS test and and AD test for all considered stocks at the 1% significance level.

Therefore, we can conclude that the ARMA-GARCH model with fNTS innovations describes behavior of the high frequency return time series investigated in this section. While, the ARMA-GARCH models with non-fractional NTS innovations and normal innovations do not perfectly explain the behavior of the high frequency return data.

6. Assessment Risk on the ARMA-GARCH Model with fNTS Innovations

In this section, portfolio VaR and AVaR on the ARMA-GARCH model with fNTS innovations are discussed, and they are applied to the portfolio optimization.

Let $\mathcal{T} > 0$ be a time horizon. Assume that we have a portfolio with N stocks. Suppose the stock return vector process $Y = (Y(t_k))_{k \in \{0,1,2,\cdots,M\}}$ with $Y(t_k) = (Y_1(t_k), Y_2(t_k), \cdots, Y_N(t_k))$ follows the ARMA-GARCH model with fNTS innovations as defined in Section 5 and $X \sim \text{fstdNTS}_N(H, \alpha, \theta, \beta, \Sigma)$ generated by $Z \sim \text{stdNTS}_N(\alpha, \theta, \beta, \Sigma)$, where $\{t_0, t_1, \cdots, t_M\}$ is a given discrete time such that $t_k = k \cdot \Delta t$ with $\Delta t = \mathcal{T}/M$ for $k \in \{0, 1, 2, \cdots, M\}$. Then a portfolio return process $R = (R(t_k))_{k \in \{0,1,2,\cdots,M\}}$ with allocation weight vector $w = (w_1, w_2, \cdots, w_N)^\mathsf{T}$ with $\sum_{n=1}^N w_n = 1$ is given by $R(t_k) = \sum_{n=1}^N w_n Y_n(t_k)$.

TABLE 4 | Goodness-of-fit test for the ARMA-GARCH model with normal innovations.

	Apple Inc.	Google Inc.	IBM Co.	AT&T	Wal-Mart Stores Inc.
ARMA-GARCH MODEL WITH fNTS INNOVATIONS					
KS	0.0314	0.0431	0.0579	0.0458	0.0431
(p-value)	(0.4186)	(0.1068)	(0.0103)	(0.0741)	(0.1069)
AD^2	1.1206	1.5869	3.5267	1.7146	1.6252
(p-value)	(0.2997)	(0.1570)	(0.0149)	(0.1326)	(0.1492)
ARMA-GARCH MODEL WITH (NON-FRACTIONAL) NTS INNOVATIONS					
KS	0.0309	0.0463	0.0845	0.0594	0.0485
(p-value)	(0.4372)	(0.0683)	(0.0000)	(0.0078)	(0.0490)
AD^2	1.1722	2.5099	8.0708	3.9592	1.8555
(p-value)	(0.2784)	(0.0490)	(0.0001)	(0.0091)	(0.1105)
ARMA-GARCH MODEL WITH NORMAL INNOVATIONS					
KS	0.0683	0.0858	0.1788	0.1129	0.0800
(p-value)	(0.0013)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
AD^2	671.3620	685.7825	895.2760	810.3928	665.1581
(p-value)	(0.0013)	(0.0000)	(0.0000)	(0.0000)	(0.0000)

We have

$$\begin{aligned} R(k+1) &= \sum_{n=1}^{N} w_n Y_n(k+1) \\ &= \sum_{n=1}^{N} w_n(c_n + a_n Y_n(k) + b_n\sigma_n(k)\varepsilon_n(k)) \\ &\quad + \sum_{n=1}^{N} w_n\sigma_n(k+1)\varepsilon_n(k+1) \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\sum_{n=1}^{N} w_n\sigma_n(k+1)\varepsilon_n(k+1) \\ &\quad = \sum_{n=1}^{N} w_n\sigma_n(k+1)\frac{X_n(k+1)-X_n(k)}{(t_{k+1}-t_k)^H} \\ &\quad = (\Delta t)^{-H}\sum_{n=1}^{N} w_n\sigma_n(k+1)(X_n(k+1) - X_n(k)). \end{aligned}$$

By Equation (14), we obtain

$$\begin{aligned} &\sum_{n=1}^{N} w_n\sigma_n(k+1)\varepsilon_n(k+1) \\ &\approx (\Delta t)^{-H}K_H(t_{k+1}, t_k)\sum_{n=1}^{N} w_n\sigma_n(k+1)(Z_n(k+1) - Z_n(k)) \\ &\quad + (\Delta t)^{-H}\sum_{n=1}^{N}\sum_{j=0}^{k-1} w_n\sigma_n(k+1)(K_H(t_{k+1}, t_j) \\ &\quad - K_H(t_k, t_j))(Z_n(t_{j+1}) - Z_n(t_j)) \end{aligned} \tag{17}$$

Let $(\mathcal{F}(t_k))_{k\in\{1,2,\cdots,M\}}$ be the natural filtration generated by Y. Then $\sigma_n(k+1)$ and

$$\sum_{n=1}^{N}\sum_{j=0}^{k-1} w_n\sigma_n(k+1)(K_H(t_{k+1}, t_j) - K_H(t_k, t_j))(Z_n(t_{j+1}) - Z_n(t_j))$$

are $\mathcal{F}(t_k)$-measurable. Moreover, since Z_n has stationary increments, we have

$$\sum_{n=1}^{N} w_n\sigma_n(k+1)(Z_n(k+1) - Z_n(k))\Big|_{\mathcal{F}(k)} \overset{\mathrm{d}}{=} \sum_{n=1}^{N} w_n\sigma_n(k+1)Z_n(\Delta t)$$

Hence, we have

$$\sum_{n=1}^{N} w_n\sigma_n(k+1)(Z_n(k+1) - Z_n(k))\Big|_{\mathcal{F}(k)} \overset{\mathrm{d}}{=} \Xi(\Delta t), \tag{18}$$

where $(\Xi(t))_{t\geq 0} \sim NTS_1(\alpha, \theta, \bar{\beta}, \bar{\gamma}, 1)$ with

$$\bar{\beta} = \sum_{n=1}^{N} w_n\sigma_n(k+1)\beta_n,$$

$$\bar{\gamma} = \sqrt{\sum_{m=1}^{N}\sum_{n=1}^{N} w_m w_n\sigma_m(k+1)\sigma_n(k+1)\sigma_{m,n}\gamma_m\gamma_n},$$

and

$$\gamma_n = \sqrt{1-\beta_n^2\left(\frac{1-\alpha}{2\theta}\right)}, \quad n = 1, 2, \cdots, N,$$

by Proposition 1. Therefore, we can discuss VaR and AVaR as follows. The VaR and AVaR for $R(k+1)$ with the significance level η under information until time t_k are defined by

$$\mathrm{VaR}_\eta(R(k+1)\big|\mathcal{F}(k)) = -\inf\{x|\mathbb{P}[R(k+1) \leq x\big|\mathcal{F}(k)] > \eta\}.$$

and

$$\text{AVaR}_\eta(R(k+1)\big|\mathcal{F}(k)) = \frac{1}{\eta}\int_0^\eta \text{VaR}_x(R(k+1)\big|\mathcal{F}(k))dx.$$

respectively. By Equation (16), we have

$$\begin{aligned}
&\text{VaR}_\eta(R(k+1)\big|\mathcal{F}(k)) \\
&\quad = -\sum_{n=1}^{N} w_n(c_n + a_n Y_n(k) + b_n\sigma_n(k)\varepsilon_n(k)) \\
&\quad + \text{VaR}_\eta\left(\sum_{n=1}^{N} w_n\sigma_n(k+1)\varepsilon_n(k+1)\big|\mathcal{F}(k)\right),
\end{aligned}$$

and by Equation (17), we obtain

$$\begin{aligned}
&\text{VaR}_\eta(R(k+1)\big|\mathcal{F}(k)) \\
&\quad \approx -\sum_{n=1}^{N} w_n(c_n + a_n Y_n(k) + b_n\sigma_n(k)\varepsilon_n(k)) \\
&\quad - (\Delta t)^{-H}\sum_{n=1}^{N}\sum_{j=0}^{k-1} w_n\sigma_n(k+1)(K_H(t_{k+1}, t_j) \\
&\quad - K_H(t_k, t_j))(Z_n(t_{j+1}) - Z_n(t_j)) \\
&\quad + (\Delta t)^{-H}K_H(t_{k+1}, t_k)\text{VaR}_\eta \\
&\quad \left(\sum_{n=1}^{N} w_n\sigma_n(k+1)(Z_n(k+1) - Z_n(k))\big|\mathcal{F}(k)\right)
\end{aligned}$$

Thus, by Equation (18) we obtain

$$\begin{aligned}
&\text{VaR}_\eta(R(k+1)\big|\mathcal{F}(k)) \\
&\quad \approx -\sum_{n=1}^{N} w_n(c_n + a_n Y_n(k) + b_n\sigma_n(k)\varepsilon_n(k)) \\
&\quad - (\Delta t)^{-H}\sum_{n=1}^{N}\sum_{j=0}^{k-1} w_n\sigma_n(k+1)(K_H(t_{k+1}, t_j) \\
&\quad - K_H(t_k, t_j))(Z_n(t_{j+1}) - Z_n(t_j)) \\
&\quad + (\Delta t)^{-H}K_H(t_{k+1}, t_k)\text{VaR}_\eta\left(\Xi(\Delta t)\right), \quad (19)
\end{aligned}$$

By the same argument, we obtain AVaR as follows

$$\begin{aligned}
&\text{AVaR}_\eta(R(k+1)\big|\mathcal{F}(k)) \\
&\quad \approx -\sum_{n=1}^{N} w_n(c_n + a_n Y_n(k) + b_n\sigma_n(k)\varepsilon_n(k)) \\
&\quad - (\Delta t)^{-H}\sum_{n=1}^{N}\sum_{j=0}^{k-1} w_n\sigma_n(k+1)(K_H(t_{k+1}, t_j) \\
&\quad - K_H(t_k, t_j))(Z_n(t_{j+1}) - Z_n(t_j)) \\
&\quad + (\Delta t)^{-H}K_H(t_{k+1}, t_k)\text{AVaR}_\eta\left(\Xi(\Delta t)\right), \quad (20)
\end{aligned}$$

The closed-form solutions of $\text{VaR}_\eta\left(\Xi(\Delta t)\right)$ and $\text{AVaR}_\eta\left(\Xi(\Delta t)\right)$ for NTS process $(\Xi(t))_{t\geq 0}$ is presented in Kim et al. [27].

The **Figures 3A,B** exhibit the forecasted 1 min ahead VaR and AVaR values, respectively, for an equally weighted portfolio for the five stocks in this study based on the parameters in **Table 2** and information we discussed in the Section 5. The equally weighted portfolio is the portfolio having allocation weight as $w = (1/N, 1/N, \cdots, 1/N)$, where N is the number of stocks in the portfolio. The forecasted 1-min VaR and AVaR values are calculated by Equation (19) and Equation (20), respectively, at confidence levels from 0.5 to 10%. The VaR and AVaR values for the portfolio based on the ARMA-GARCH model with non-fractional NTS innovations are presented in the figure. The VaR (AVaR) values of the ARMA-GARCH model with fNTS innovations are larger than VaR (AVaR) values of the model with non-fractional NTS innovations.

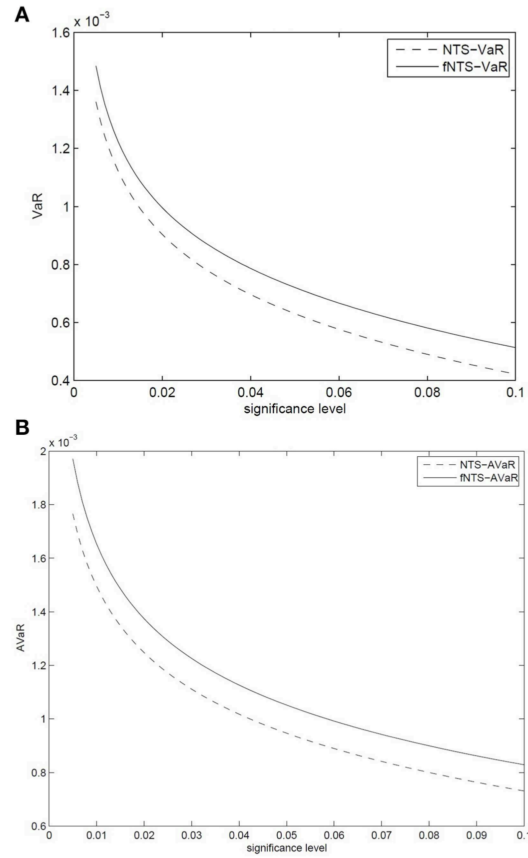

FIGURE 3 | Forecasted 1 min (A) VaR and (B) AVaR under the ARMA-GARCH model with non-fractional NTS and fNTS innovations. VaR and AVaR values for the ARMA-GARCH model with fNTS innovations are given by solid curves, and those values for the ARMA-GARCH model with non-fractional NTS innovations are given by dash curves.

7. Portfolio Optimization and Risk Budgeting on the ARMA-GARCH Model with fNTS Innovations

Using VaR and AVaR values by Equation (19) and Equation (20), we can find the optimal portfolio. The VaR minimizing portfolio is obtained by solving the following optimization problem:

TABLE 5 | Optimal allocation weight for the portfolio with the five stocks and performance measures.

	w_n	Equally weighted portfolio	VaR minimizing portfolio	AVaR minimizing portfolio
Apple Inc.	w_1	0.2	0.0641	0.0229
Google Inc.	w_2	0.2	0.0001	0.0069
IBM Co.	w_3	0.2	0.2330	0.2905
AT&T	w_4	0.2	0.3195	0.3030
Wal-Mart Stores Inc.	w_5	0.2	0.3833	0.3766
$E[R(k+1)\|\mathcal{F}(t_k)]$		$2.5924 \cdot 10^{-4}$	$2.6435 \cdot 10^{-4}$	$2.8388 \cdot 10^{-4}$
$VaR_{0.01}(R(k+1)\|\mathcal{F}(t_k))$		$1.2243 \cdot 10^{-3}$	$0.9455 \cdot 10^{-3}$	$0.9501 \cdot 10^{-3}$
$AVaR_{0.01}(R(k+1)\|\mathcal{F}(t_k))$		$1.6534 \cdot 10^{-3}$	$1.2686 \cdot 10^{-3}$	$1.2645 \cdot 10^{-3}$
VaRRatio(1%)		0.2117	0.2742	0.2988
STARR(1%)		0.1568	0.2044	0.2245

$$\min_w \text{VaR}_\eta(R(k+1)\big|\mathcal{F}(k)) \quad \text{s.t.} \quad \begin{cases} E[R(k+1)\big|\mathcal{F}(k)] \geq \mu_0, \\ \sum_{n=1}^{N} w_n = 1, \\ w_n \geq 0.0001, \text{ for} \\ \qquad\qquad n = 1, 2, \cdots, N \end{cases}$$

where μ_0 is the benchmark expected return. Similarly, the AVaR minimizing portfolio is obtained by solving the following optimization problem:

$$\min_w \text{AVaR}_\eta(R(k+1)\big|\mathcal{F}(k)) \quad \text{s.t.} \quad \begin{cases} E[R(k+1)\big|\mathcal{F}(k)] \geq \mu_0, \\ \sum_{n=1}^{N} w_n = 1, \\ w_n \geq 0.0001, \text{ for} \\ \qquad\qquad n = 1, 2, \cdots, N \end{cases}$$

Table 5 presents the VaR and AVaR minimizing portfolios with the benchmark return $\mu_0 = 2.5924 \cdot 10^{-4}$. To measure performance of the optimal portfolio, we use the VaR ratio and stable tail-adjusted return ratio (STARR), which are defined by

$$\text{VaR Ratio}(\eta) = \frac{E[R(k+1)|\mathcal{F}(t_k)]}{\text{VaR}_\eta(R(k+1)|\mathcal{F}(t_k))}$$

and

$$\text{STARR}(\eta) = \frac{E[R(k+1)|\mathcal{F}(t_k)]}{AVaR_\eta(R(k+1)|\mathcal{F}(t_k))},$$

respectively[5]. The following is concluded from the results reported in **Table 5**:

- The VaR minimizing portfolio has the best expected return among the three portfolios.
- The AVaR minimizing portfolio has smaller VaR than the case of the VaR minimizing portfolio, but that is not surprising since the AVaR minimizing portfolio has less expected return than the case of the VaR minimizing portfolio.
- The AVaR minimizing portfolio has the largest VaR Ratio among the three VaR ratios.
- The AVaR minimizing portfolio has the largest STARR among the three STARR.

8. Conclusion

The multivariate ARMA-GARCH model with fNTS innovations exhibits fat-tails, asymmetric dependence, volatility clustering, and long-range dependence. Comparing with two ARMA-GARCH models with non-fractional NTS innovation and normal innovations, the ARMA-GARCH model with fNTS innovations has better performance in parameter estimation for 1-min stock return data investigated in this paper. That means the fNTS process describes the behavior of the residual process of 1-min returns better than the non-fractional NTS process or Brownian motion. The portfolio VaR and AVaR are calculated by the approximation method under the model, and those risk measures are used for portfolio optimization. In this investigation, we obtain the fact that the risk measures of the ARMA-GARCH model with fNTS innovations are more conservative than those of the model with non-fractional NTS innovations. The AVaR minimizing portfolio performs better than the VaR minimizing portfolio for the case considered in this paper.

[5]Many literatures define the VaR ratio and STARR as

$$\text{VaR Ratio}(\eta) = \frac{E[R(k+1) - R_f(k+1)|\mathcal{F}(t_k)]}{\text{AVaR}_\eta(R(k+1) - R_f(k+1)|\mathcal{F}(t_k))}$$

and

$$\text{STARR}(\eta) = \frac{E[R(k+1) - R_f(k+1)|\mathcal{F}(t_k)]}{AVaR_\eta(R(k+1) - R_f(k+1)|\mathcal{F}(t_k))},$$

respectively, where R_f is a market index return. In this paper, we assume $R_f = 0$ for considering absolute performance.

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Appendix

To define a process with long-range dependence, we use the Volterra kernel $K_H : [0,\infty) \times [0,\infty) \to [0,\infty)$, given by

$$K_H(t,s) = c_H \left(\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right) 1_{[0,t]}(s)$$

with

$$c_H = \left(\frac{H(1-2H)\Gamma(\frac{1}{2}-H)}{\Gamma(2-2H)\Gamma(H+\frac{1}{2})} \right)^{\frac{1}{2}}.$$

and $H \in (0,1)$. According to Nualart [28] and Houdre and Kawai [7], we have the following facts:

- We have

$$\int_0^{t\wedge s} K_H(t,u)K_H(s,u)du = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \tag{21}$$

and

$$\int_0^t K_H(t,s)^2 ds = t^{2(H-\frac{1}{2})+1}. \tag{22}$$

- If $H \in (\frac{1}{2}, 1)$ then

$$K_H(t,s) = c_H \left(H-\frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du 1_{[0,t]}(s).$$

Moreover, $K_{\frac{1}{2}}(t,s) = 1_{[0,t]}(s)$.

- Let $t > 0$ and let $p \geq 2$. $K_H(t,\cdot) \in L^p([0,t])$ if and only if $H \in \left(\frac{1}{2}-\frac{1}{p}, \frac{1}{2}+\frac{1}{p}\right)$. When $K_H(t,\cdot) \in L^p([0,t])$, we have

$$\int_0^t K_H(t,s)^p ds = C_{H,p} t^{p(H-\frac{1}{2})+1},$$

where

$$C_{H,p} = c_H^p \int_0^1 v^{p(\frac{1}{2}-H)} \times \left[(1-v)^{H-\frac{1}{2}} - \left(H-\frac{1}{2}\right) \int_v^1 w^{H-\frac{3}{2}} (w-v)^{H-\frac{1}{2}} dw \right]^p dv. \tag{23}$$