frontiers in Applied Mathematics and Statistics

ORIGINAL RESEARCH
published: 13 September 2016
doi: 10.3389/fams.2016.00013

On Singular Interval-Valued Iteration Groups

Marek C. Zdun *

Institute of Mathematics, Pedagogical University of Cracow, Kraków, Poland

Let $I = (a, b)$ and L be a nowhere dense perfect set containing the ends of the interval I and let $\varphi : I \to \mathbb{R}$ be a non-increasing continuous surjection constant on the components of $I \setminus L$ and the closures of these components be the maximal intervals of constancy of φ. The family $\{F^t, t \in \mathbb{R}\}$ of the interval-valued functions $F^t(x){:} = \varphi^{-1}[t + \varphi(x)]$, $x \in I$ forms a set-valued iteration group. We determine a maximal dense subgroup $T \subsetneq \mathbb{R}$ such that the set-valued subgroup $\{F^t, t \in T\}$ has some regular properties. In particular, the mappings $T \ni t \to F^t(x)$ for $t \in T$ possess selections $f^t(x) \in F^t(x)$, which are disjoint group of continuous functions.

Keywords: iteration group, set-valued functions, simultaneous functional equations, Cantor set, singular Lebesgue function

OPEN ACCESS

Edited by:
Witold Jarczyk, University of Zielona Góra, Poland

Reviewed by:
Ludwig Reich, University of Graz, Austria
Krzysztof Cieplinski, AGH University of Science and Technology, Poland

***Correspondence:**
Marek C. Zdun
mczdun@up.krakow.pl

Specialty section:
This article was submitted to Dynamical Systems, a section of the journal Frontiers in Applied Mathematics and Statistics

Received: *24 May 2016*
Accepted: *30 August 2016*
Published: *13 September 2016*

Citation:
Zdun MC (2016) On Singular Interval-Valued Iteration Groups. Front. Appl. Math. Stat. 2:13. doi: 10.3389/fams.2016.00013

1. INTRODUCTION

A family of functions $\{f^t : I \to I, t \in \mathbb{R}\}$ such that $f^t \circ f^s = f^{t+s}$, $t, s \in \mathbb{R}$ is said to be an *iteration group*, however a family of set-valued functions $\{F^t : I \to 2^I, t \in \mathbb{R}\}$ such that $F^t \circ F^s = F^{t+s}$, $t, s \in \mathbb{R}$ is said to be a *set-valued iteration group* (abbreviated to *s-v iteration group*). The notion of an iteration semigroup of set-valued functions was introduced and studied by Smajdor [1] and then studied in some classes of set-valued functions (see e.g., [2], [3], [4], [5]). The fundamental problem in the theory of multivalued iteration semigroups is the problem of existence and regularity properties of continuous selections. In this note we considered particular set-valued iteration groups whose values are the intervals or singletons. The presented results complete and generalize some of the topics from Zdun [6]. The considered s-v iteration groups have the very irregular properties. For every such s-v iteration group $\{F^t : I \to 2^I, t \in \mathbb{R}\}$ we find a special maximal additive subgroup $T \subset \mathbb{R}$ such that group $\{F^t : I \to 2^I, t \in T\}$ has several "regular" properties.

2. MATERIALS AND METHODS

Let $I = (a, b)$ and $\varphi : I \to \mathbb{R}$ be a surjection. Define the set-valued functions

$$F^t(x){:} = \varphi^{-1}[\varphi(x) + t], \quad t \in \mathbb{R}, \quad x \in I. \tag{1}$$

The surjection φ is said to be the generating function of the family $\{F^t\}$.

THEOREM 1
The family $\{F^t : I \to 2^I\}$ is a set-valued iteration group, i.e.,

$$F^t \circ F^s = F^{t+s}, \quad t, s \in \mathbb{R},$$

where

$$F^t \circ F^s(x) = \bigcup_{y \in F^s(x)} F^t(y) \;\; x \in I.$$

Moreover, $x \notin F^t(x)$ for $t \neq 0$.

Proof. Fix an $x \in I$. Let $z \in F^t \circ F^s(x)$. Then there exists a $y \in F^s(x)$ such that $z \in F^t(y)$. This means that $\varphi(y) = \varphi(x) + s$ and $\varphi(z) = \varphi(y) + t$, which gives that $\varphi(z) = \varphi(x) + t + s$. Hence $z \in F^{t+s}(x)$. Similarly we prove the converse inclusion. □

If φ is a homeomorphism then Equation (1) defines the general form of continuous iteration groups such that $F^1(x) \neq x$ for $x \in I$.

If φ is non-injective then s-v iteration group generated by φ has very irregular properties and we will call this group *singular*. The purpose of this paper is the study of these "singularities."

Obviously the set-valued functions F^t, $t \in \mathbb{R}$ pairwise commute. This property is not transferible on the continuous selections of these set-valued mappings.

Let us assume that there exist F^u, F^v with $\frac{u}{v} \notin \mathbb{Q}$ which possess homeomorphic commuting selections f and g, that is $f(x) \in F^u(x)$ and $g(x) \in F^v(x)$ for $x \in (a, b)$ and $f \circ g = g \circ f$. Then the generating function φ satisfies the equations $\varphi(f(x)) = \varphi(x) + u$ and $\varphi(g(x)) = \varphi(x) + v$. Note that then f, g are iteratively incommensurable, i.e.,

$$f^n(x) \neq g^m(x), \;\; n, m \in \mathbb{Z}, \; |n| + |m| > 0, \; x \in I,$$

where f^n denotes the n-th iterate of function f and $f^0 = id$. Define

$$L_{f,g} := \{f^n \circ g^m(x), \; n, m \in \mathbb{Z}\}^d.$$

The set $L_{f,g}$ does not depend on x and either this set is the interval $\mathrm{cl}\, I$ or $L_{f,g}$ is a nowhere dense perfect set in I (see Zdun [7]). If the generating function φ is continuous at least at one point of $L_{f,g}$ then it is continuous and it is monotonic (see [8]).

We have more

THEOREM 2
If f and g are commuting iteratively incommensurable homeomorphisms, then there exist infinitely many s-v iteration groups $\{F^t, t \in \mathbb{R}\}$ of type (1) such that $f(x) \in F^1(x)$ and $g(x) \in F^s(x)$ for an $s \notin \mathbb{Q}$, but the only one of them has a monotonic generating function φ. Then the generating function φ is continuous and $\varphi[L_{f,g}] = \mathbb{R}$.

The proof is a simple consequence of Theorem 2 and Corollary 1 in Zdun [8].

The family $\{F^t, t \in \mathbb{R}\}$ is a single-valued iteration group if and only if $L_{f,g} = [a, b]$. Then φ is strictly monotonic (see Zdun [8]).

In this paper we consider the case where $L_{f,g} \neq [a, b]$, that is $\{F^t : t \in \mathbb{R}\}$ is a proper set-valued iteration group.

In the next section we will consider the more general case.

3. RESULTS

Assume the following general hypothesis:

(H) $\varphi : I \to \mathbb{R}$ is a non-decreasing and non-injective surjection.

Then the function φ is continuous and the values of F^t are closed intervals or singletons. Denote by $\{I_\alpha, \alpha \in A\}$ a family of the intervals of constancy of φ. These intervals are closed. Put

$$L^* := I \setminus \bigcup_{\alpha \in A} I_\alpha$$

and

$$L := I \setminus \bigcup_{\alpha \in A} \mathrm{Int}\, I_\alpha. \tag{2}$$

Note that $\varphi_{|L^*}$ is strictly increasing, $\varphi[I_\alpha]$ are singletons and if $I_\alpha < I_\beta$ then $\varphi[I_\alpha] < \varphi[I_\beta]$.

It is easy to verify that the s-v iteration group $\{F^t : I \to cc[I], t \in \mathbb{R}\}$ generated by φ has the following properties.

PROPOSITION 1
(i) For every $x \in I$ $F^t(x)$ either is a closed proper interval I_α or a singleton belonging to L^*;
(ii) for every $x \in I$ the s-v function $t \to F^t(x)$ is strictly decreasing, i.e., if $s < t$ then for every $u \in F^s(x)$ and $v \in F^t(x)$, $u < v$;
(iii) for every $x \in I$ $\bigcup_{t \in \mathbb{R}} F^t(x) = I$;
(iv) every s-v function F^t is constant on the intervals I_α;
(v) if $s \neq t$ then $F^t(x) \cap F^s(x) = \emptyset$ for $x \in I$, that is the group $\{F^t, t \in \mathbb{R}\}$ is disjoint.

The conditions (i), (ii), (iii) characterize the interval-valued iteration groups. We have the following.

PROPOSITION 2
If an s-v iteration group $\{F^t, t \in \mathbb{R}\}$ satisfies conditions (i), (ii), and (iii), where $\{I_\alpha, \alpha \in A\}$ is a given family of closed disjoint proper intervals, then there exists a function φ satisfying (H) such that F^t are given by the formula (1).

Proof. Define

$$\mathcal{X} := \{\{x\}, x \in L^*\} \cup \{I_\alpha, \alpha \in A\}.$$

Let $x_0 \in I$ and put $h(t) := F^t(x_0)$. Note that h is a bijection from $\mathbb{R}$ onto $\mathcal{X}$. Define φ by the following way: if $x \in I_\alpha$ for an $\alpha \in A$ then $\varphi(x) := h^{-1}(I_\alpha)$, if $x \in L^*$ then $\varphi(x) := h(\{x\})$. It is easy to see that φ is a non-decreasing surjection of I onto $\mathbb{R}$ constant on the intervals I_α and

$$\varphi[h(t)] = t, \;\; t \in \mathbb{R}.$$

Since $F^t \circ F^s(x_0) = F^{t+s}(x_0)$ we have

$$F^t[h(s)] = F^{t+s}(x_0) = h(s + t), \;\; s, t \in \mathbb{R}.$$

Hence

$$\varphi[F^t(h(s))] = \varphi[h(s+t)] = s + t.$$

Let $x \in I$. Then, by (iii), there exists an $s \in \mathbb{R}$ such that $x \in h(s)$. Hence $\varphi(x) \in \varphi[h(s)] = s$, thus $\varphi(x) = s$. This gives that $\varphi[F^t(x)] \subset \varphi[F^t(h(s)] = \varphi(x) + t$, so

$$\varphi[F^t(x)] = \varphi(x) + t. \tag{3}$$

Since $F^t(x) \subset \varphi^{-1}[\varphi[F^t(x)]]$ we have $F^t(x) \subset \varphi^{-1}[\varphi(x) + t]$. Note that $\varphi^{-1}[\varphi(x) + t]$ is a singleton or equals to one of the intervals I_α. If $F^t(x)$ is a singleton then, by (i), $F^t(x) \notin I_\alpha$ for any $\alpha \in A$. Thus $\varphi^{-1}[\varphi(x) + t]$ is not any of the intervals I_α, so it is a singleton. If $F^t(x)$ is an interval I_α, then $\varphi^{-1}[\varphi(x+t)]$ must be also the same interval. This gives equality $F^t(x) = \varphi^{-1}[\varphi(x+t)]$. □

PROPOSITION 3
Let a family of set-valued function F^t be given by (1), where φ satisfies (H). Define

$$f^t_-(x) := \inf F^t(x), \ \ f^t_+(x) := \sup F^t(x)$$

for $t \in \mathbb{R}$, $x \in I$. Then

(i) the families $\{f^t_-, t \in \mathbb{R}\}$ and $\{f^t_+, t \in \mathbb{R}\}$ are iteration groups;
(ii) f^t_- and f^t_+ for $t \in \mathbb{R}$ are non-decreasing discontinuous functions constant on the intervals of constancy of φ;
(iii) the mappings $t \to f^t_\pm(x)$ are strictly decreasing;
(iv) $f^t_-[I] \subset L, f^t_+[I] \subset L, t \in \mathbb{R}$;
(v) $F^t(x) = [f^t_-(x), f^t_+(x)]$, $t \in \mathbb{R}$.

Proof. (i) Fix an $x \in I$. Note that $f^t_-(x), \ f^t_+(x) \in F^t(x)$ since the sets $F^t(x)$ are closed. Hence, by Equation (1),

$$\varphi(f^t_\pm(x)) = \varphi(x) + t, \tag{4}$$

so $\varphi(f^t_\pm(f^s_\pm(x))) = \varphi(x) + t + s = \varphi(f^{t+s}_\pm(x))$. This implies that

$$f^t_\pm(f^s_\pm(x)) \in I_\alpha \text{ and } f^{t+s}_\pm(x) \in I_\alpha = F^{t+s}(x)$$

for an $\alpha \in A$ or both belong to L^*, since I_α for $\alpha \in A$ are the intervals of constancy of φ. Obviously, in the second case, both values are equal. However, at the first case, $f^t_+(f^s_+(x)) \leq \sup I_\alpha = f^{t+s}_+(x)$ and $f^{t+s}_-(x) = \inf I_\alpha \leq f^t_-(f^s_-(x))$. On the other hand, putting $f^s_+(x) = :y$ we have that $f^t_+(y) \in I_\alpha$ and $f^t_+(y) \in F^t(y)$. Hence $F^t(y) = I_\alpha$ and $f^t_+(y) = \sup I_\alpha \geq f^{t+s}_+(x)$. This gives that

$$f^s_+(f^t_+(x)) = f^{t+s}_+(x).$$

Similarly we prove that

$$f^s_-(f^t_-(x)) = f^{t+s}_-(x).$$

(iv) Proving (i) we have shown that $f^t_\pm(x)$ either belong to L^* or equals to one of the ends of the interval I_α which belong to L. Both cases give that $f^t_\pm(x) \in L$.

The remaining assertions are the simple consequences of formula (Equation 1). □

Let φ be non-decreasing and non-injective surjection. Define the following family of functions

$$\mathrm{Realm}(\varphi) := \{f : I \to I : \exists_{c_f} \forall_{x \in I} \ \varphi(f(x)) = \varphi(x) + c_f\}.$$

The index c_f is uniquely determined. This allows us to define

$$\mathrm{ind} f := c_f.$$

As a particular case of Proposition 2.2 in Farzadfard and Zdun [9] we get the following

LEMMA 1
If $f \in \mathrm{Realm}(\varphi)$ then the following conditions are equivalent:

(i) $\varphi[L^*] = \varphi[L^*] + \mathrm{ind} f$;
(ii) $\varphi[I \setminus L^*] = \varphi[I \setminus L^*] + \mathrm{ind} f$;
(iii) $f[L^*] = L^*$;
(iv) f maps each I_α into another one; moreover for every I_β there exists I_α such that $f[I_\alpha] \subset I_\beta$.

Let φ satisfy (H) and define

$$T := \{t \in \mathbb{R} : \varphi[I \setminus L^*] + t = \varphi[I \setminus L^*]\}. \tag{5}$$

If $T \neq \{0\}$, then T is a countable Abelian subgroup of group $(\mathbb{R}, +)$.

In fact, since φ is constant in the intervals I_α, we have $\varphi[I \setminus L^*] = \{\varphi[I_\alpha], \alpha \in A\}$. It is easy to see that this set is unbounded above and below thus it is infinite and, consequently, countable since the intervals $\{I_\alpha, \alpha \in A\}$ are pairwise disjoint.

DEFINITION 1
A subgroup T given by Equation (5) is said to be a *supporting group* of the s-v iteration group $\{F^t : t \in \mathbb{R}\}$.

THEOREM 3
Let $T \neq \{0\}$ be a supporting group of s-v iteration group $\{F^t : t \in \mathbb{R}\}$ generated by a function φ satisfying (H). Then

(i) if $t \in T$ then for every $x \in L^*$ $F^t(x)$ is a single point and $F^t(x) \in L^*$;
(ii) if $t \in T$ then for every $\alpha \in A$ there exists $\beta \in A$ such that $F^t(x) = I_\beta$ for $x \in I_\alpha$;
(iii) if $t \in T$ then for every $\beta \in A$ there exists $\alpha \in A$ such that $F^t(x) = I_\beta$ for $x \in I_\alpha$;
(iv) if $F^t[L^*] = L^*$ then $t \in T$.

Proof. (i) By Equation (2) $f^t_-, f^t_+ \in \mathrm{Realm}(\varphi)$, $\mathrm{ind} f^t_\pm = t$ for $t \in \mathbb{R}$ and $\varphi(f^t_-(x)) = \varphi(f^t_+(x))$. By Lemma 1 $f^t_\pm(x) \in L^*$ for $x \in L^*$. Since $\varphi_{|I_\alpha}$ is injective $f^t_-(x) = f^t_+(x)$ for $x \in L^*$. Thus, by Proposition 3 (v), $F^t(x)$ is a singleton belonging to L^*.

(ii) Let $x \in I_\alpha$. By Lemma 1 $f^t_\pm(x) \in I_\beta$ for a $\beta \in A$. Thus $F^t(x) \subset I_\beta$. If $F^t(x)$ is a singleton then, by Proposition 1 (i), $F^t(x)$ belongs to L^*, so $f^t_\pm(x) \in L^*$, but this is a contradiction. Thus $F^t(x)$ is a proper interval, so $F^t(x) = I_\beta$.

(iii) Fix a $\beta \in A$. Since $\varphi[I_\beta]$ is a singleton and φ is a surjection from I onto $\mathbb{R}$ there exists an $x \in I$ such that $\varphi[I_\beta] = t + \varphi(x)$, that is $F^t(x) = I_\beta$. Suppose $x \in L^*$. Then, by Lemma 1, $f^t_\pm(x) \in L^*$, but this is a contradiction since $f^t_{\pm}(x) \in F^t(x) = I_\beta$, so there exists an $\alpha \in A$ such that $x \in I_\alpha$.

(iv) Since φ satisfies relation Equation (3) we have $\varphi[L^*] = \varphi[F^t[L^*]] = \varphi[L^*] + t$, so, by Lemma 1, $t \in T$. □

Directly by Theorem 3 we get the following

COROLLARY 1
Let $T \neq \{0\}$ be the supporting group of the s-v group $\{F^t : t \in \mathbb{R}\}$ with generating function satisfying (H). Then

(i) $T = \{t \in \mathbb{R} : \forall_{\omega\in A}\ \exists_{\overline{\omega}\in A}\ F^t[I_\omega] = I_{\overline{\omega}}\}$;
(ii) $T = \{t \in \mathbb{R} : \forall_{x\in L^*}\ F^t(x)$ is a singleton$\}$;
(iii) $T = \{t \in \mathbb{R} : F^t[\mathrm{L}^*] = \mathrm{L}^*\}$.

DEFINITION 2
A family of continuous mappings $\{f^t : I \to I, t \in T\}$ such that $f^t \circ f^s = f^{t+s}$ for $t, s \in T$ is said to be a *T-iteration group*.

Now we consider the problems connected with continuous selections of s-v iteration groups. The iteration groups $\{f^t_-, t \in \mathbb{R}\}$ and $\{f^t_+, t \in \mathbb{R}\}$ are the monotonic selections of s-v group $\{F^t, t \in \mathbb{R}\}$ that is $f^t_{\pm}(x) \in F^t(x)$, but they are discontinuous.

Let φ satisfies (H) and $I_\alpha = :[a_\alpha, b_\alpha]$ for $\alpha \in A$ be the intervals of constancy of φ. For $t \in T$ define the affine mappings $q_{t,\alpha} : [a_\alpha,\ b_\alpha] \to I$ such that

$$q_{t,\alpha}(a_\alpha) = f^t_-(a_\omega) \text{ and } q_{t,\alpha}(b_\alpha) = f^t_+(b_\alpha).$$

For every $t \in T$ define the following mapping

$$q^t(x) := \begin{cases} q_{t,\alpha}(x), & x \in I_\alpha \\ f^t_+(x), & x \in L^*. \end{cases} \tag{6}$$

LEMMA 2
If $T \neq \{0\}$ is the supporting group of s-v group $\{F^t : t \in \mathbb{R}\}$ generated by a function satisfying condition (H), then $\{q^t : I \to I, t \in T\}$ is a T-iteration group of continuous functions. Moreover, $q^t(x) \in F^t(x)$ for $t \in T$ and $x \in I$.

Proof. Note that $q_{t,\alpha}[I_\alpha] = F^t[I_\alpha]$ and $F^t[I_{\alpha_1}] < F^t[I_{\alpha_2}]$ if $I_{\alpha_1} < I_{\alpha_2}$. Hence, by Theorem 3, it follows that the mappings q^t are strictly increasing surjections and, consequently, they are continuous.

It follows that for every $t, s \in T$, $q^t \circ q^s[I_\alpha] = q^t[F^s[I_\alpha]] = F^t[F^s[I_\alpha]] = F^{t+s}[I_\alpha] = q^{t+s}[I_\alpha]$. Since the composition of affine functions is an affine function and there exists a unique increasing affine function mapping I_α onto the interval $F^{t+s}[I_\alpha]$ we get that $q^t \circ q^s = q^{t+s}$ on I_α. Now it is easy to see that Proposition 3 implies our assertion. □

THEOREM 4
If s-v group $\{F^t : t \in \mathbb{R}\}$ generated by a function satisfying condition (H) has a non trivial supporting group T, then there exists infinitely many disjoint T-iteration groups $\{f^t, t \in T\}$ of continuous functions such that $f^t(x) \in F^t(x)$ for $t \in T$ and $x \in I$. T is a maximal additive group with this property.

Proof. Let $\gamma : I \to I$ be a homeomorphism such that $\gamma(x) = x$ for $x \in L$ and for every $\alpha \in A$ $\gamma[I_\alpha] = I_\alpha$. Put

$$f^t := \gamma^{-1} \circ q^t \circ \gamma, \quad t \in T.$$

It follows, by Lemma 2, that $\{f^t, t \in T\}$ is a T-iteration group and $f^t(x) \in F^t(x)$.

Let F^t have a continuous and strictly increasing selection f. Since for every $\alpha \in A$, $f[I_\alpha]$ is a proper interval, $F^t[I_\alpha]$ is also an interval. Thus, by Corollary 1, $t \in T$. □

Let us make the following assumptions.

(i) Let L be a Cantor set in I, that is L is a nowhere dense perfect set in $I = (a, b)$ and $a, b \in L$.
(ii) Let I_ω, $\omega \in \mathbb{Q}$ be open pairwise disjoint intervals such that

$$I \setminus L =: \bigcup_{\omega\in\mathbb{Q}} I_\omega.$$

(iii) Let $\varphi : I \to \mathbb{R}$ be a Lebesgue function which lives on a set L that is φ is a continuous non-increasing surjection constant on cl I_ω and, let cl I_ω be the maximal intervals of constancy of φ.

The conditions (i), (ii), and (iii) imply that φ is continuous and

$$\varphi[L] = \mathbb{R}.$$

THEOREM 5
Let T be the supporting group of s-v group $\{F^t : t \in \mathbb{R}\}$ generated by a function φ satisfying condition (H). If the group T is acyclic then the set L defined by (2) is a Cantor set and φ is a Lebesgue function which lives on L.

Proof. By Lemma 2 the family of mappings $\{q^t, t \in T\}$ defined by Equation (6) is a disjoint T-iteration group. Denote by L_T the set of limit points of the orbits $O(x) = \{q^t(x) : t \in T\}$, i.e., $L_T = O(x)^d$. In Zdun [10] (see Th.1) it is proved that the set L_T does not depend on x and L_T is either a Cantor set in I or $L_T = [a, b]$ or $L_T = \{a, b\}$. Moreover, $L_T = \{a, b\}$ if and only if $\{q^t, t \in T\}$ is a cyclic group (see [10] Theorem 2).

Since $q^t(x) \in F^t(x)$ we have $\varphi(q^t(x)) = \varphi(x) + t$ for $x \in I$. $L_T \neq [a, b]$. In fact, suppose that $L_T = [a, b]$. Fix an $x \in I$ and an interval I_α. By the density of the orbit $O(x)$ there exist $u, v \in \mathbb{R}$ such that $u \neq v$ and $q^u(x), q^v(x) \in I_\alpha$. Hence $\varphi(x) + u = \varphi(q^u(x)) = \varphi(q^v(x)) = \varphi(x) + v$ what is a contradiction.

By Proposition 1 (ii) and Lemma 2 the mapping $\Phi(t) := q^t$ is an isomorphism of T onto the group $\{q^t, t \in T\}$. Thus T is cyclic if and only if $\{q^t, t \in T\}$ is cyclic, so T is cyclic if and only if $L_T = \{a, b\}$. Hence T is acyclic if and only if L_T is a Cantor set.

If T is acyclic then φ lives on L_T. Let $x \in L_T$ and $t \in T$. Then $q^t(x) = f^t_+(x) \in L$. Thus $O(x) \subset L$ and, consequently, $L_T \subset L$, so L is also a Cantor set. By the assumption φ lives on L, however by the definition of q^t φ lives on L_F. Thus we get $L_F = L$. □

THEOREM 6
If f, g are commuting, iteratively incommensurable homeomorphisms and $L_{f,g} \neq \operatorname{cl} I$, then f and g are embeddable in a non-extensible disjoint T-iteration group $\{f^t, t \in T\}$, where T is a dense, countable subgroup of $\mathbb{R}$.

Proof. By Theorem 2 there exists an s-v iteration group $\{F^t : t \in \mathbb{R}\}$ with continuous non-decreasing generating function φ such that $f(x) \in F^1(x)$ and $g(x) \in F^s(x)$ for an $s \notin \mathbb{Q}$ and $\varphi[L_{f,g}] = \mathbb{R}$. Since $L_{f,g} \neq \operatorname{cl} I$, φ is a Lebesgue function which lives on $L_{f,g}$. Define T by Equation (5). By Theorem 5 f and g are embeddable in a T-iteration group $\{f^t, t \in T\}$. Since $1, s \in T$ the group T is dense. □

4. DISCUSSION

In this note we consider the relation between the iteration groups of monotonic functions and the interval-valued iteration groups. These groups are still poorly investigated.

In Section 2 we indicate a desirability of the generalization of classical iteration groups in the real case. It is known that not all commutable iteratively incommensurable homeomorphisms are embeddable in an iteration group. However, Theorem 2 shows that the embeddabilty is always possible for s-v iteration groups.

Propositions 1 and 2 characterize s-v iteration groups of the form Equation (1). It is shown that, in our investigations, the form Equation (1) of s-v iteration groups are quite natural. Proposition 3 shows how s-v iteration groups of the form Equation (1) determine iterations groups of non-decreasing functions which are not injective.

A key concept of the paper is the supporting group T defined by Equation (5). If T is non-trivial additive group then it is countable and the set of all intervals of constancy of the generating function φ is also countable. Theorem 3 and Corollary 1 explain the meaning of the supporting group T. The restricted s-v group $\{F^t : t \in T\}$ has a property that s-v functions F^t transform the intervals of constancy of the generating function φ onto itself and the points from its complement, that is the set L^*, onto singletons in L^*. Moreover, Theorem 4 and Corollary 1 show that each s-v function F^t for $t \in T$ has continuous selection f^t such that family $\{f^t : t \in T\}$ forms a group. Moreover, any F^t for $t \notin T$ has no continuous selection.

We have also proved that supporting group T is acyclic if and only if the generating function φ is a Lebesgue function which lives in a Cantor set.

The presented results may be helpful in the constructions of different iteration groups of non-decreasing functions.

AUTHOR CONTRIBUTIONS

MZ conceived the study and prepared the manuscript.

REFERENCES

1. Smajdor A. Iterations of multifunctions. *Pr Nauk Uniw Slask Katowice* (1985) **759**:1–55.
2. Olko J. On indexed functions generated by families of functions. *Aequat Math.* (2014) **88**:125–35. doi: 10.1007/s00010-013-0239-1
3. Olko J. Selections of an iteration semigroup of linear set-valued functions. *Aequat Math.* (1998) **56**:157–68. doi: 10.1007/s000100050052
4. Łydzińska G. On lower semicontinuity of some set-valued iteration semigroups. *Nonlinear Anal.* (2009) **71**:5644–54. doi: 10.1016/j.na.2009.04.051
5. Piszczek M. On multivalued iteration semigroup. *Aequat Math.* (2011) **81**:97–108. doi: 10.1007/s00010-010-0034-1
6. Zdun MC. On set-valued iteration groups generated by commuting functions. *J Math Anal Appl.* (2013) **398**:638–48. doi: 10.1016/j.jmaa.2012.09.016
7. Zdun MC. Note on commutable functions. *Aequat Math.* (1988) **36**:153–64.
8. Zdun MC. On simultaneous Abel equations. *Aequat Math.* (1989) **38**:163–77.
9. Farzadfard H, Zdun MC. On a limit formula for regular iterations. *J Math Anal Appl.* (2016) **443**:947–58. doi: 10.1016/j.jmaa.2016.05.068
10. Zdun MC. On the orbits of disjoint groups of continuous functions. *Rad Mat.* (1992) **8**:95–104.

Conflict of Interest Statement: The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.