The Fractional Laguerre Equation: Series Solutions and Fractional Laguerre Functions

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In this paper, we propose a fractional generalization of the well-known Laguerre differential equation. We replace the integer derivative by the conformable derivative of order $0 < \alpha < 1$. We then apply the Frobenius method with the fractional power series expansion to obtain two linearly independent solutions of the problem. For certain eigenvalues, the infinite series solution truncate to obtain the singular and non-singular fractional Laguerre functions. We obtain the fractional Laguerre functions in closed forms, and establish their orthogonality result. The applicability of the new fractional Laguerre functions is illustrated.

Keywords: fractional differential equations, Laguerre equation, conformable fractional derivative, series solution, Frobenius method

1. INTRODUCTION

In recent years, there are interests in studying fractional Sturm-Liouville eigenvalue problems. For instance, the fractional Bessel equation with applications was investigated in Okrasinski and Plociniczak [1, 2], where the fractional derivative is of the Riemann-Liouville type. In Abu Hammad and Khalil [3] the authors solved the fractional Legendre equation with conformable derivative and established the orthogonality property of the fractional Legendre functions. The applications of the fractional Legendre functions in solving fractional differential equations, were illustrated in Kazema et al. [4] and Syam and Al-Refai [5]. In this project we propose the following fractional generalization of the well-known Laguerre differential equation

$$x^{\alpha} D_0^{\alpha} D_0^{\alpha} y + (1 - x^{\alpha}) D_0^{\alpha} y + \lambda y = 0, \quad \frac{1}{2} < \alpha < 1, \quad x > 0, \tag{1.1}$$

where D_0^{α} is the conformable derivative of order α. The conformable derivative was introduced recently in Khalil et al. [6], and below are the definition and main properties of the derivative.

Definition 1.1. For a function $f : (0, \infty) \to \mathbb{R}$, the conformable derivative of order $0 < \alpha \le 1$ of f at $x > 0$, is defined by

$$(D_0^{\alpha} f)(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon},$$

and the derivative at $x = 0$ is defined by $(D_0^{\alpha} f)(0) = \lim_{x \to 0^+} (D_0^{\alpha} f)(x)$.

The conformable derivative is a local derivative which has a physical and a geometrical interpretations and potential applications in physics and engineering [7, 8]. It satisfies the nice

properties of the integer derivative such as, the product rule, the quotient rule, and the chain rule, and it holds that

1. $D_0^\alpha C = 0, \ \ C \in \mathbb{R}$,
2. $D_0^\alpha x^p = p \, x^{p-\alpha}$,
3. $D_0^\alpha \sin(\frac{1}{\alpha} x^\alpha) = \cos(\frac{1}{\alpha} x^\alpha)$,
4. $D_0^\alpha \cos(\frac{1}{\alpha} x^\alpha) = -\sin(\frac{1}{\alpha} x^\alpha)$,
5. $D_0^\alpha e^{\frac{1}{\alpha} x^\alpha} = e^{\frac{1}{\alpha} x^\alpha}$.
6. $\int_0^a f(x) d\alpha(x) = \int_0^a x^{\alpha-1} f(x) dx$.

For more details about the conformable derivative we refer the reader to Abdeljawad [9] and Khalil et al. [6]. We mention here that even though the conformable is a nonlocal derivative (see [10, 11]), the simplicity and applications of the derivative make it of interests. Also, the applications of the obtained Fractional Leguerre functions are indicated in this manuscript. The rest of the paper is organized as follows: In section 2, we apply the Frobenius method together with the fractional series solution to solve the above equation and to obtain the fractional Laguerre functions. In section 3, we establish the orthogonality result of the fractional Laguerre functions and present the fractional Laguerre functions for several eigenvalues. Finally, we close up with some concluding remarks in section 4.

2. THE SERIES SOLUTION

The series solution is commonly used to solve various types of fractional differential equations (see [12–16]). Since $x = 0$, is α-regular singular point of Equation (1.1), see [17], we apply the well-known Frobenius method to obtain a solution of the form

$$y = \sum_{n=0}^\infty a_n x^{\alpha(n+r)},$$

where the values of r will be determined. We have

$$D_0^\alpha y = \sum_{n=0}^\infty \alpha(n+r) a_n x^{\alpha(n+r-1)},$$

$$= \alpha a_0 r x^{\alpha(r-1)} + \sum_{n=0}^\infty \alpha(n+r+1) a_{n+1} x^{\alpha(n+r)},$$

$$x^\alpha D_0^\alpha y = \sum_{n=0}^\infty \alpha(n+r) a_n x^{\alpha(n+r)},$$

$$D_0^\alpha D_0^\alpha y = \sum_{n=0}^\infty \alpha^2(n+r)(n+r-1) a_n x^{\alpha(n+r-2)},$$

$$x^\alpha D_0^\alpha D_0^\alpha y = \sum_{n=0}^\infty \alpha^2(n+r)(n+r-1) a_n x^{\alpha(n+r-1)},$$

$$= \alpha^2 r(r-1) a_0 x^{\alpha(r-1)}$$

$$+ \sum_{n=0}^\infty \alpha^2(n+r+1)(n+r) a_{n+1} \, x^{\alpha(n+r)}.$$

By substituting the above results in Equation (1.1) we have

$$0 = \alpha^2 r(r-1) a_0 x^{\alpha(r-1)} + \sum_{n=0}^\infty \alpha^2(n+r+1)(n+r) a_{n+1} \, x^{\alpha(n+r)}$$

$$+ \alpha a_0 r x^{\alpha(r-1)} + \sum_{n=0}^\infty \alpha(n+r+1) a_{n+1} x^{\alpha(n+r)}$$

$$- \sum_{n=0}^\infty \alpha(n+r) a_n x^{\alpha(n+r)} + \lambda \sum_{n=0}^\infty a_n x^{\alpha(n+r)}.$$

The coefficients of $x^{\alpha(r-1)}$ will lead to

$$a_0 \alpha r \Big(\alpha(r-1) + 1 \Big) = 0. \tag{2.1}$$

Because $\alpha \neq 0$, and $a_0 = 0$, will lead to the zero solution, we have

$$r = 0, \ r = 1 - \frac{1}{\alpha}. \tag{2.2}$$

We start with $r = 0$, we have

$$\alpha^2 n(n+1) a_{n+1} + \alpha(n+1) a_{n+1} - \alpha n a_n + \lambda a_n = 0,$$

or

$$a_{n+1} = \frac{\alpha n - \lambda}{\alpha(n+1)(\alpha n + 1)} \, a_n, \ \ n \geq 0. \tag{2.3}$$

Lemma 2.1. *The coefficients a_n in Equation (2.3) satisfy*

$$a_{n+1} = \frac{\displaystyle\prod_{j=0}^n (j\alpha - \lambda)}{\alpha^{n+1}(n+1)! \displaystyle\prod_{j=0}^n (j\alpha + 1)} \, a_0, \ \ n \geq 0. \tag{2.4}$$

Proof: The proof can be easily obtained by iterating the recursion in (2.3) and applying induction arguments. □

Remark 2.1. *For $\alpha = 1$, the recursion relation in (2.4) will reduce to*

$$a_{n+1} = \frac{-\lambda(1-\lambda)(2-\lambda)\cdots(n-\lambda)}{[(n+1)!]^2} \, a_0, \tag{2.5}$$

which is exactly the recursion relation that has been obtained in solving the Laguerre equation with integer derivative.

For $r = -\frac{1}{\alpha} + 1$, we have for $n \geq 0$,

$$a_{n+1} = \frac{\alpha(n+r) - \lambda}{\alpha(n+r+1)(\alpha[n+r]+1)} \, a_n,$$

$$= \frac{\alpha(n+1) - (\lambda+1)}{\alpha(n+1)(\alpha[n+2]-1)} \, a_n. \tag{2.6}$$

By iterating the recursion in (2.6) and applying induction arguments, we have

Lemma 2.2. *The coefficients a_n in Equation (2.6) satisfy*

$$a_{n+1} = \frac{\prod_{j=1}^{n+1}(j\alpha - [\lambda+1])}{\alpha^{n+1}(n+1)! \prod_{j=2}^{n+2}(j\alpha - 1)} a_0, \quad n \geq 0. \quad (2.7)$$

Remark 2.2. *By applying the Frobenius method to the regular Laguerre equation with integer derivative $\alpha = 1$, we obtain only one value of $r = 0$, which produces only one solution. Here with the fractional case, we obtain two values of $r = 0, 1 - \frac{1}{\alpha}$, that will produce two linearly independent solutions of the problem as we will see later.*

Now, in Equation (2.4), if we choose $\alpha = \alpha_m$ and $\lambda = \lambda_m$ such that

$$m\alpha_m = \lambda_m,$$

for some integer m, then

$$a_{m+1} = a_{m+2} = \cdots = 0,$$

and the infinite series solution will truncate to obtain the finite sum

$$u(x) = \sum_{n=0}^{m} a_n x^{n\alpha_m} = a_0 \left(1 + \sum_{n=1}^{m} \frac{\prod_{j=0}^{n-1}(j\alpha_m - \lambda_m)}{\alpha_m^n n! \prod_{j=0}^{n-1}(j\alpha_m + 1)} x^{n\alpha_m} \right)$$

$$= a_0 L_{m,\alpha_m}^0(x),$$

where $L_{m,\alpha_m}^0(x)$ is the non-singular fractional Laguerre function of order m. Since

$$\prod_{j=0}^{n-1}(j\alpha_m - \lambda_m) = \prod_{j=0}^{n-1}(j\alpha_m - m\alpha_m) = \prod_{j=0}^{n-1}\alpha_m \prod_{j=0}^{n-1}(j-m)$$

$$= \alpha_m^n \prod_{j=0}^{n-1}(j-m),$$

then

$$L_{m,\alpha_m}^0(x) = 1 + \sum_{n=1}^{m} \frac{\prod_{j=0}^{n-1}(j-m)}{n! \prod_{j=0}^{n-1}(j\alpha_m + 1)} x^{n\alpha_m}. \quad (2.8)$$

Analogously, in Equation (2.7), if we choose $\alpha = \alpha_m$ and $\lambda = \lambda_m$ such that

$$m\alpha_m = \lambda_m + 1,$$

then

$$a_m = a_{m+1} = \cdots = 0,$$

and the infinite series solution will truncate to obtain the solution

$$u(x) = \sum_{n=0}^{m-1} a_n x^{\alpha_m(n - \frac{1}{\alpha_m} + 1)} = x^{-1} \sum_{n=0}^{m-1} a_n x^{\alpha_m(n+1)}$$

$$= a_0 L_{m-1,\alpha_m}^1(x),$$

where

$$L_{m-1,\alpha_m}^1(x) = x^{-1} \left(x_m^\alpha + \sum_{n=1}^{m-1} \frac{\prod_{j=1}^{n}(j\alpha_m - (\lambda_m + 1))}{\alpha_m^n n! \prod_{j=2}^{n+1}(j\alpha_m - 1)} x^{\alpha_m(n+1)} \right),$$

$$= x^{\alpha_m - 1} \left(1 + \sum_{n=1}^{m-1} \frac{\prod_{j=1}^{n}(j\alpha_m - m\alpha_m)}{\alpha_m^n n! \prod_{j=2}^{n+1}(j\alpha_m - 1)} x^{\alpha_m n} \right),$$

$$= x^{\alpha_m - 1} \left(1 + \sum_{n=1}^{m-1} \frac{\prod_{j=1}^{n}(j - m)}{n! \prod_{j=2}^{n+1}(j\alpha_m - 1)} x^{\alpha_m n} \right), \quad (2.9)$$

is the fractional singular Laguerre function of order $m - 1$.

Remark 2.3. *If we substitute $\alpha_m = 1$, then*

$$L_{m,1}^0(x) = L_{m,1}^1(x) = 1 + \sum_{n=1}^{m} \frac{\prod_{j=0}^{n-1}(j-m)}{n! \prod_{j=0}^{n-1}(j+1)}.$$

Since $\prod_{j=0}^{n-1}(j+1) = n!$, and $\prod_{j=0}^{n-1}(j-m) = (-1)^n \frac{m!}{(m-n)!}$, we have

$$L_{m,1}^0(x) = L_{m,1}^1(x) = 1 + \sum_{n=1}^{m} \frac{(-1)^m m!}{(n!)^2 (m-n)!},$$

which is the expansion of the Laguerre polynomial $L_m(x)$.

3. THE FRACTIONAL LAGUERRE FUNCTIONS

We start with the orthogonality property of the fractional Laguerre functions $\left(L_{m,\alpha_m}(x) \right)$, $m = 0, 1, 2, \cdots$. Here by $L_{m,\alpha_m}(x)$ we mean the non-singular and singular Laguerre functions obtained in (2.8) and (2.9).

Theorem 3.1. *The fractional Laguerre functions $\left(L_{m,\alpha_m}(x) \right)$, $m = 0, 1, 2, \cdots$ are orthogonal on $(0, \infty)$ with respect to the weight function $\mu(x) = e^{-\frac{x^\alpha}{\alpha}}$, i.e.,*
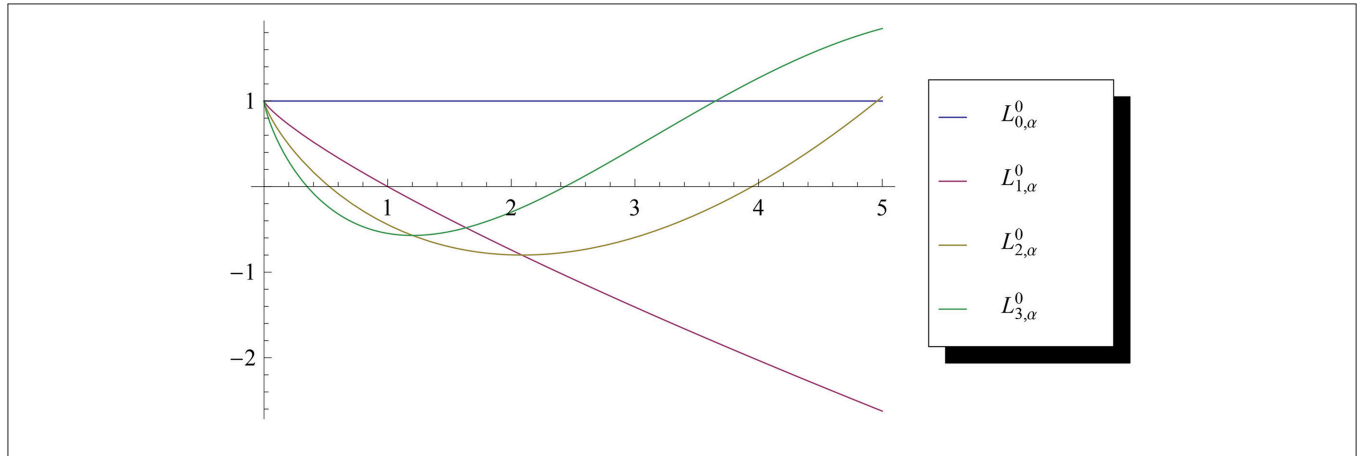
$$\int_0^\infty e^{-\frac{x^\alpha}{\alpha}} L_{m,\alpha_m}(x) L_{n,\alpha_n}(x) dx = 0, \quad m \neq n.$$

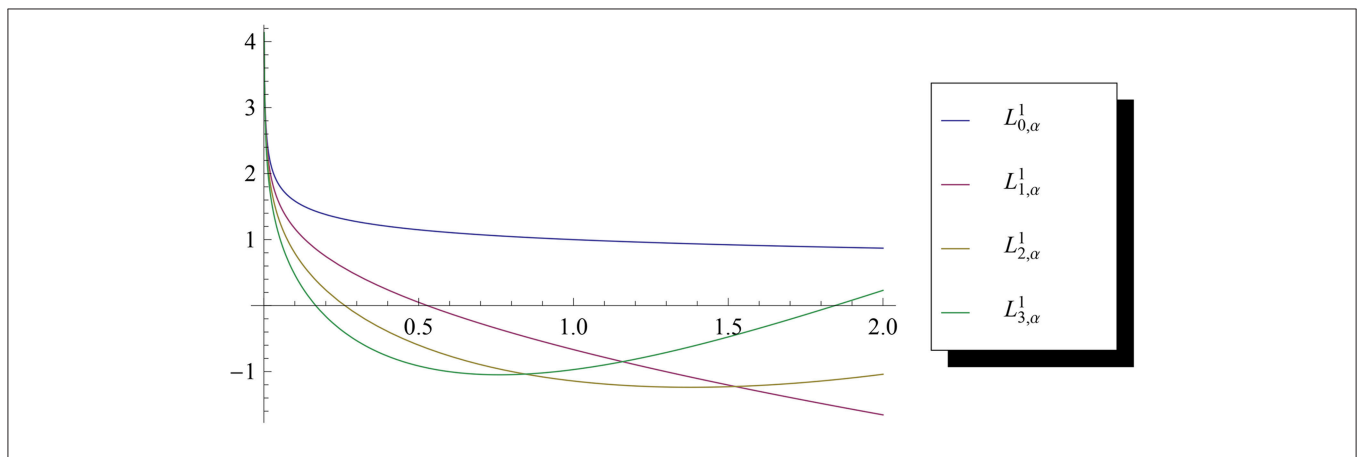*Proof:* One can easily prove that Equation (1.1) can be written as

$$D_0^\alpha \left(xe^{-\frac{x^\alpha}{\alpha}} D_0^\alpha y \right) = -\lambda x^{1-\alpha} e^{-\frac{x^\alpha}{\alpha}} y. \qquad (3.1)$$

Thus, the equation is of a special type of the fractional Sturm-liouville eigenvalue problem
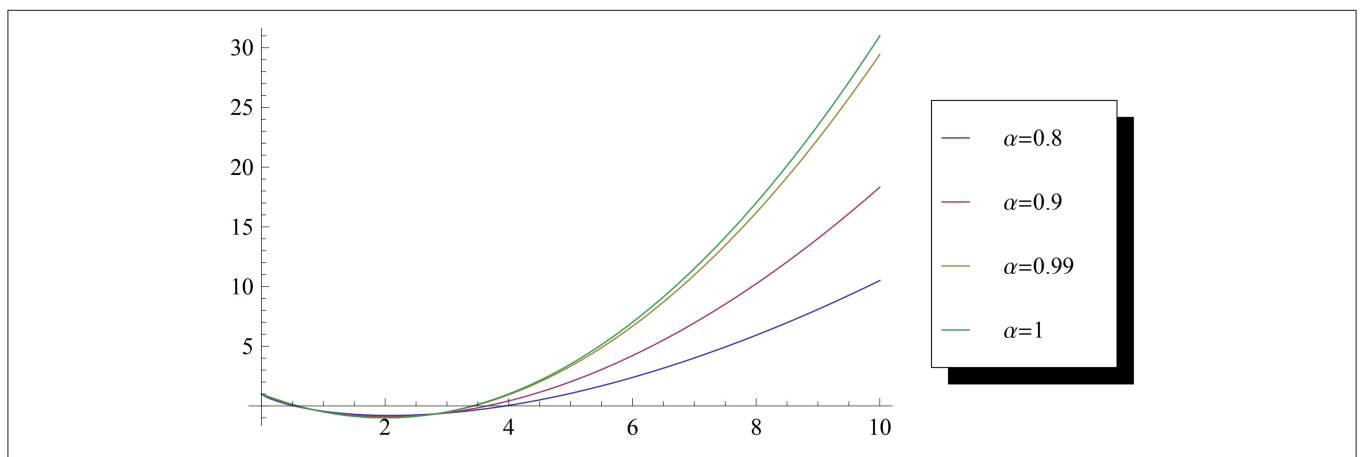
$$D_0^\alpha \left(p(x) D_0^\alpha y \right) + q(x)y = -\lambda w(x)y,$$

FIGURE 1 | A plot of $L_{0,\alpha}^0, L_{1,\alpha}^0, L_{2,\alpha}^0, L_{3,\alpha}^0$ for $\alpha = 0.8$.

FIGURE 2 | A plot of $L_{0,\alpha}^1, L_{1,\alpha}^1, L_{2,\alpha}^1, L_{3,\alpha}^1$ for $\alpha = 0.8$.

FIGURE 3 | A plot of $L_{2,\alpha}^0$, for $\alpha = 0.8, 0.9, 0.99, 1$.

where $p(x) = xe^{-\frac{x^\alpha}{\alpha}}$, $q(x) = 0$ and $w(x) = x^{1-\alpha}e^{-\frac{x^\alpha}{\alpha}}$. Using the fractional Lagrange Identity obtained in Al-Refai and Abdeljawad [18], we have

$$-(\lambda_m - \lambda_n)\int_0^\infty w(x)L_{m,\alpha_m}L_{n,\alpha_n}d\alpha(x)$$

$$= p(x)\left(L_{n,\alpha_n}D_0^\alpha L_{m,\alpha_m} - L_{m,\alpha_m}D_0^\alpha L_{n,\alpha_n}\right)\Big|_0^\infty. \quad (3.2)$$

We have $p(0) = 0$, and

$$\lim_{x\to\infty} x^{1-\alpha}e^{-\frac{x^\alpha}{\alpha}}\left(L_{n,\alpha_n}D_0^\alpha L_{m,\alpha_m} - L_{m,\alpha_m}D_0^\alpha L_{n,\alpha_n}\right)(x) = 0.$$

Thus the right hand side of Equation (3.2) equals zero which together with $\lambda_m \neq \lambda_n$ will lead to

$$\int_0^\infty w(x)L_{m,\alpha_m}L_{n,\alpha_n}d\alpha(x) = \int_0^\infty x^{1-\alpha}e^{-\frac{x^\alpha}{\alpha}}L_{m,\alpha_m}(x)L_{n,\alpha_n}(x)x^{\alpha-1}dx$$

$$= \int_0^\infty e^{-\frac{x^\alpha}{\alpha}}L_{m,\alpha_m}(x)L_{n,\alpha_n}(x)dx = 0, \quad (3.3)$$

and hence the result. $\square$

Remark 3.1. *Since the fractional Laguerre functions are orthogonal, they can be used as a basis of the spectral method to study fractional differential equations analytically and numerically. They also can be used as a basis of the fractional Gauss-Laguerre quadrature for approximating the value of integrals of the form*

$$\int_0^\infty e^{-\frac{x^\alpha}{\alpha}}f(x)dx.$$

Remark 3.2. *New types of improper integrals are determined using the orthogonality property which are not known before, such as*

$$\int_0^\infty (1-x^\alpha)e^{-\frac{x^\alpha}{\alpha}}dx = 0, \quad L_{0,\alpha_0}^0(x) = 1, L_{1,\alpha_1}^0(x) = 1 - x^{\alpha_1},$$

$$\int_0^\infty x^{2(\alpha-1)}(1 - \frac{1}{2\alpha-1}x^\alpha)e^{-\frac{x^\alpha}{\alpha}}dx = 0,$$

$$L_{0,\alpha_0}^1(x) = x^{\alpha_0-1}, L_{1,\alpha_1}^1(x) = x^{\alpha_1-1}(1 - \frac{1}{2\alpha_1-1}x^{\alpha_1}),$$

In the following we present the singular and non-singular fractional Laguerre functions of several orders.

$$L_{0,\alpha_0}^0(x) = 1,$$

$$L_{1,\alpha_1}^0(x) = 1 - x^{\alpha_1}$$

$$L_{2,\alpha_2}^0(x) = 1 - 2x^{\alpha_2} + \frac{1}{\alpha_2+1}x^{2\alpha_2}$$

$$L_{3,\alpha_3}^0(x) = 1 - 3x^{\alpha_3} + \frac{3}{\alpha_3+1}x^{2\alpha_3} - \frac{1}{(\alpha_3+1)(2\alpha_3+1)}x^{3\alpha_3}.$$

$$L_{0,\alpha_0}^1(x) = x^{\alpha_0-1},$$

$$L_{1,\alpha_1}^1(x) = x^{\alpha_1-1}(1 - \frac{1}{2\alpha_1-1}x^{\alpha_1}),$$

$$L_{2,\alpha_2}^1(x) = x^{\alpha_2-1}(1 - \frac{2}{2\alpha_2-1}x^{\alpha_2} + \frac{1}{(2\alpha_2-1)(3\alpha_2-1)}x^{2\alpha_2}),$$

$$L_{3,\alpha_3}^1(x) = x^{\alpha_3-1}\left(1 - \frac{3}{2\alpha_3-1}x^{\alpha_3} + \frac{3}{(2\alpha_3-1)(3\alpha_3-1)}x^{2\alpha_3}\right.$$

$$\left. - \frac{1}{(2\alpha_3-1)(3\alpha_3-1)(4\alpha_3-1)}x^{3\alpha_3}\right).$$

Figures 1, **2** depict the non-singular and singular fractional Laguerre functions of several orders for $\alpha = 0.8$. **Figure 3** depicts $L_{2,\alpha}^0$ for several values of α. One can see that, as α approaches 1, the non-singular fractional Laguerre functions approach the Laguerre polynomial of degree 2.

4. CONCLUSION

We have considered the fractional Laguerre equation with conformable derivative. We obtained two linearly independent solutions using the fractional series solution and Frobenius method. The first non-singular solution is analytic on $(0, \infty)$, and the second singular solution has a singularity at $x = 0$. For certain eigenvalues, these infinite solutions truncate to obtain the fractional Laguerre functions. Because of the orthogonality property of the fractional Laguerre functions, they can be used as a basis of the spectral method to study fractional differential equations, or as a basis of the Gauss-Laguerre quadrature for evaluating certain integrals. The obtained results coincide with the ones of the regular Laguerre polynomials as the derivative α approaches 1.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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