frontiers in Applied Mathematics and Statistics

ORIGINAL RESEARCH
published: 06 October 2021
doi: 10.3389/fams.2021.631052

Some New Solutions of Non Linear Evolution Equations With Mutable Coefficients

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We construct the traveling wave solutions of some NonLinear Evolution Equations (NLEEs) with mutable coefficients arising in different branches of physics and mathematics. we apply a novel $(\frac{G'}{G})$-formalism to construct more general solitary traveling wave solutions of NLEEs such as Sharma-Tasso-Olver with mutable coefficients and Zakharov Kuznetsov equation. Interesting solutions of NLEEs are investigated by traveling wave solutions which are in form of trigonometric, rational, and hyperbolic functions. This may build more unified new solutions for different kinds of such NLEEs with mutable coefficients arising in mathematics and physics. Wolfram Mathematica 11 is used to perform the computation work and their corresponding plots and counter graphs are plotted. This method is found to be more useful and efficient for searching the exact solutions of NLEEs.

Keywords: nonlinear evolution equations, solitons, solitary wave solutions, explicit solution PACS no: 0230lk, 0365Fd, 0320+i

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Specialty section:
This article was submitted to Mathematical and Statistical Physics, a section of the journal Frontiers in Applied Mathematics and Statistics

Received: *19 November 2020*
Accepted: *26 July 2021*
Published: *06 October 2021*

Citation:
Virdi JS (2021) Some New Solutions of Non Linear Evolution Equations With Mutable Coefficients. Front. Appl. Math. Stat. 7:631052. doi: 10.3389/fams.2021.631052

1 INTRODUCTION

NonLinear Evolution Equations (NLEEs) plays a significant role in the analysis of mathematical modeling and soliton theory. These NLEEs, which are primarily studied in mathematics and physics play an important role and character in various branches of science and technology, such as propagation of shallow water waves, population statistics physics, fluid dynamics, condensed matter physics, computational physics, and geophysics. NLEEs also appear and are very important in many fields such as wave mechanics, dissipation mechanics, dispersion in optics, reaction and convection equations. Over the past few decades, many compelling methodologies for extracting exact solutions of NLEEs have been formulated. However, it is more difficult to solve the NLEEs but, various methods have been tried for solving NLEEs, such as the Hirota's bilinear operations [1] truncated Painleve expansion [2], inverse scattering transform [3], extended tanh-function method [4], F-expansion method [5], tanh-coth method [6], Jacobi-elliptic function expansion [7], homogenous balance method [8], sub ODE method [9], Rank analysis method [10], Extended and modified direct algebraic method [11], extended mapping method [12, 13] and Seadawy techniques to find solutions for some nonlinear partial differential equations [14] and many other ansatzes comprising exponential and hyperbolic functions are accurately used for the analytic analysis of NLEEs. Recently a few other well-known methods are used to extract explicit solutions of soliton equations, as example the adomionos decomposition method [15], Darboux transformation (DT) [16], Hirota technique [17] etc. In early 1990s, Wang et al. [18] familiarize a new formalism called extended $(\frac{G'}{G})$-expansion method for a decisive treatment of NLEEs. After that further applications of this method have also been proclaimed [19, 20]. Therefore, further to expand the domain of applications of extended $(\frac{G'}{G})$-expansion formalism, in this work we examine the nonlinear Sharma-Tasso-Olver with mutable coefficients (STO) [21, 22] equation exact solutions are attained which are in form of hyperbolic, trigonometric and rational functions. In works of literature, many times it appears as an evolution equation acquiring many symmetries. The extended $(\frac{G'}{G})$-expansion formalism has also been applied to the Zakharov Kuznetsov (ZK) equation [23] and abundant exact solutions are derived which included the trigonometric, rational, and hyperbolic functions. First of all, we describe the method and applied it to two given equations, all different possible wave solutions are presented by their 3D plots and their corresponding contour plots. At the end of this article, discussion and conclusion are given in detail in the last section.

2 THE METHOD AND ITS APPLICATIONS

In order to solve NLEEs, defined with two independent variables x and t, in this section we highlights the important points of the extended $(\frac{G'}{G})$-expansion formalism. Suppose a NLEEs is of the form

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0, \tag{1}$$

where $u = u\ (x,\ t)$ and P is polynomial in $u = u\ (x,\ t)$

Point 1: Let us introduce the wave variable

$$\xi = x - ct, \tag{2}$$

so that

$$u(x,t) = u(\xi). \tag{3}$$

This leads to NonLinear ordinary differential equation (NLODE) as

$$P(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \ldots) = 0, \tag{4}$$

where u_ξ denotes differentiation of u wrt ξ.

Integrating the ODE (4) many times with setting the constant of integration to be zero.

Point 2: The solution of **Eq. 4** can be expressed by a polynomial in extended $(\frac{G'}{G})$ i.e.

$$u(\xi) = \delta_m\left(\frac{G'}{G}\right)^m + \delta_{m-1}\left(\frac{G'}{G}\right)^{m-1} + \ldots\ldots = \sum_{i=-m}^{i=m} \delta_i\left(\frac{G'}{G}\right)^i, \tag{5}$$

where $G = G(\xi)$ entertain following ODE of the form

$$G'' + \gamma G' + \rho G = 0, \tag{6}$$

where δ_m, δ_{m-1}, . . .,δ_0, γ and ρ are constants to be found out later and $\delta_m \neq 0$.

Point 3: Replacing **Eq. 5** into **Eq. 4** and using **Eq. 6**, and assembling all terms with the equal order of $(\frac{G'}{G})$ together, and then equating each participating and the resulting polynomial to be zero yields a set of mathematical statement for δ_m, δ_{m-1}, . . .,δ_0, c, γ and ρ.

Point 4: Since the general solutions of **Eq. 6** have been well known for us, then substituting δ_m, δ_{m-1}, . . .,δ_0 and c and the general solutions of **Eq. 6** into **Eq. 5** we obtain more solitary wave solutions of NLEEs **Eq. 1**.

With this detailed mathematical explanation of the extended $(\frac{G'}{G})$-expansion formalism, we now try to solve the NLEEs of physical importance as discussed in previous section.

3 SHARMA-TASSO-OLVER EQUATION

Nonlinear Sharma-Tasso-Olver (STO) the mutable coefficients have discussed in many branches of mathematical physics, science and engineering. STO equation reads as [21].

$$u_t + f(t)\left(uu_x + \frac{1}{3}u^3\right)_x + g(t)u_{xxx} = 0. \tag{7}$$

This equation contains both linear dispersive term u_{xxx} and the double nonlinear terms uu_x and u_t. Here the parameters $f(t) \neq 0$, $g(t) \neq 0$ and are both temporal variable. Using a transformation STO (7) with new variable reads

$$u(x,t) = u(\xi), \xi = x + \frac{w}{\zeta}\int_0^t g(t)dt. \tag{8}$$

provided $f(t)$ and $g(t)$ in **Eq. 7** should hold the condition $f(t) = 3g(t)$. Integration **Eq. 7**, shorten to

$$\frac{w}{\zeta}u_\xi + u_{\xi\xi\xi} + 3\left(uu_\xi + \frac{1}{3}u^3\right)_\xi. \tag{9}$$

Homogeneous balance between linear dispersive term $u_{\xi\xi}$ and the double nonlinear terms u^3 solution are as suggested by formalism, we find $n = 1$

$$u(\xi) = \delta_0 + \delta_1\left(\frac{G'}{G}\right) + \delta_{-1}\left(\frac{G'}{G}\right)^{-1}, \tag{10}$$

where δ_0, δ_1, δ_{-1} are constants. Replacing **Eq. 10** with **Eq. 6** into **Eq. 7**, accumulating the coefficients of $(\frac{G'}{G})$ we attain a set of mathematical statement for δ_0, δ_1, δ_{-1}, w and ζ and solving set of equation by mathematical software, we have.

Case (1)

$$\delta_{-1} = \rho, \delta_0 = 0, \zeta = \zeta, \delta_1 = 1, \gamma = -\gamma^2\zeta + 4\zeta\rho, \tag{11}$$

Case (2)

$$\delta_{-1} = 0, \delta_0 = \frac{\gamma}{2}, \zeta = \zeta, \delta_1 = 1, \gamma = -\frac{1}{4}\gamma^2\zeta + \zeta\rho, \tag{12}$$

so as per the first conditions the **Eq. 10** will give

$$u_1(\xi) = \left(\frac{G'}{G}\right) - \rho\left(\frac{G'}{G}\right)^{-1}, \quad \xi = x + \frac{-\gamma^2\zeta + 4\zeta\rho}{\zeta}\int_0^t g(t)dt. \tag{13}$$

For the second case **Eq. 10** will give

$$u_2(\xi) = \frac{\gamma}{2} + \left(\frac{G'}{G}\right), \quad \xi = x + \frac{\frac{-\gamma^2}{4}\zeta + \zeta\rho}{\zeta}\int_0^t g(t)dt. \tag{14}$$

with the aid of **Eq. 6**, solutions of **Eqs 13, 14** are the solitary wave solutions as trigonometric, rational and hyperbolic functions.

Solution of first kind (1): when $\sqrt{\gamma^2 - 4\rho}$ greater than 0

$$u_1(\xi) = \left[(\sqrt{\gamma^2-4\rho})\left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi + A_2\cosh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi}{2A_1\cosh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi + 2A_2\sinh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi}\right) - \frac{\gamma}{2}\right] - \rho\left[(\sqrt{\gamma^2-4\rho})\left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi + A_2\cosh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi}{2A_1\cosh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi + 2A_2\sinh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi}\right) - \frac{\gamma}{2}\right]^{-1}. \quad (15)$$

$$u_2(\xi) = \frac{\gamma}{2} + \left[(\sqrt{\gamma^2-4\rho})\left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi + A_2\cosh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi}{2A_1\cosh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi + 2A_2\sinh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi}\right) - \frac{\gamma}{2}\right]. \quad (16)$$

Solution of second kind (2): when $\sqrt{\gamma^2-4\rho}$ less than 0

$$u_1(\xi) = \left[\left(\sqrt{4\rho-\gamma^2}\right) - \left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi + A_2\cosh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi}{2A_1\cosh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi + 2A_2\sinh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi}\right) - \frac{\gamma}{2}\right] - \rho\left[(\sqrt{4\rho-\gamma^2}) - \left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi + A_2\cosh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi}{2A_1\cosh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi + 2A_2\sinh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi}\right) - \frac{\gamma}{2}\right]^{-1} \quad (17)$$

$$u_2(\xi) = \frac{\gamma}{2} + \left[(\sqrt{4\rho-\gamma^2})\left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi + A_2\cosh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi}{2A_1\cosh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi + 2A_2\sinh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi}\right) - \frac{\gamma}{2}\right] \quad (18)$$

Solution of third kind (3): when $\sqrt{\gamma^2-4\rho}$ equal to 0

$$u_1(\xi) = \left[\frac{A_1}{A_1+A_2\xi} - \frac{\gamma}{2}\right] - \rho\left[\frac{A_1}{A_1+A_2\xi} - \frac{\gamma}{2}\right]^{-1},$$
$$u_2(\xi) = \frac{\gamma}{2} + \left[\frac{A_1}{A_1+A_2\xi} - \frac{\gamma}{2}\right]^{-1}. \quad (19)$$

where A_1 and A_2 are integration constants. All different possible traveling wave solutions are presented by their 3D plots and their corresponding contour plots (right side figures) (**Figure 1**). For simplicity of the plot, we have taken all required vales of δ_0, δ_1, δ_{-1}, $\zeta(t)$, A_1, and A_2 and plotted for only the first solution $u_1(\chi)$ only. Important results are discussed in concluding remarks.

4 GENERALIZED ZAKHAROV KUZNETSOV EQUATION

Generalized Zakharov Kuznetsov (ZK) equation with mutable coefficients describes wave features in plasma physics [23]. Particularly the ZK equation was a handful in for describing weakly nonlinear ion-acoustic waves in strongly magnetized lossless plasma in two dimensions. The ZK equation with dual power law nonlinearity is the main motivation of this work. The ZK equation reads as

$$u_t\delta(t)uu_x + \zeta^2(t)u^2u_x + u_{xxx} + \theta(t)u_{xyy} = 0. \quad (20)$$

where $\delta(t)$, $\zeta(t)$ and $\theta(t)$ are arbitrary function of t. For wave solutions for **Eq. 20**, use transformation

$$u(x,t) = u(\xi), \xi = kx + ly + \int_0^t \tau(t)dt. \quad (21)$$

where k, l are constants, $\tau(t)$ is an integrable function of t to be determined later. Substituting **Eq. 21** into **Eq. 20**, we have

$$\tau(t)u'(\xi) + \delta(t)ku'(\xi)u(\xi) + \zeta(t)ku^2(\xi)u'(\xi) + [k^3 + \theta(t)kl^2]u'''(\xi) = 0. \quad (22)$$

where the prime denotes the differential with respect to x. Formalism suggests to introduce the anstaz

$$u(\xi) = \delta_m\left(\frac{G'}{G}\right)^m + \delta_{m-1}\left(\frac{G'}{G}\right)^{m-1} + \ldots.. = \sum_{i=-m}^{i=m}\delta_i\left(\frac{G'}{G}\right)^i, \quad (23)$$

where δ_i are constants. Making the homogeneous balance between $u'''(\xi)$ and $u^2(\xi)$, $u'(\xi)$ in **Eq. 22**, yields $n = 1$, so solution of **Eq. 20** be as suggested by formalism

$$u(\xi) = \delta_0\left(\frac{G'}{G}\right)^m + \delta_1\left(\frac{G'}{G}\right)^1 + \delta_{-1}\left(\frac{G'}{G}\right)^{-1}. \quad (24)$$

Replacing **Eq. 24** with **Eq. 6** into **Eq. 22**, accumulating the coefficients of $\left(\frac{G'}{G}\right)$ for δ_0, δ_1, δ_{-1} and $\tau(t)$ and solving this system one can get.

Case (1)

$$\delta_0 = \frac{\delta_1(\zeta(t)\delta_1 + 6\theta(t)t^2\gamma + 6k^2\gamma)}{12(k^2+\theta(t)t^2)}, \delta_{-1} = \delta_0, \quad \delta_1 = \delta_1,$$
$$\eta = \eta, k = k, \eta(t) = -\frac{6(k^2+\theta(t)t^2)}{\delta_1^2},$$
$$\tau(t) = \frac{k[12\theta(t)t^4k^2 + 96\theta(t)t^4\rho - \zeta(t)^2\delta_1^2]}{24(k^2+\theta(t)t^2)}$$
$$\frac{12\theta(t)t^4k^2 + 96\theta(t)t^4\rho - \zeta(t)^2\delta_1^2}{24(k^2+\theta(t)t^2)}. \quad (25)$$

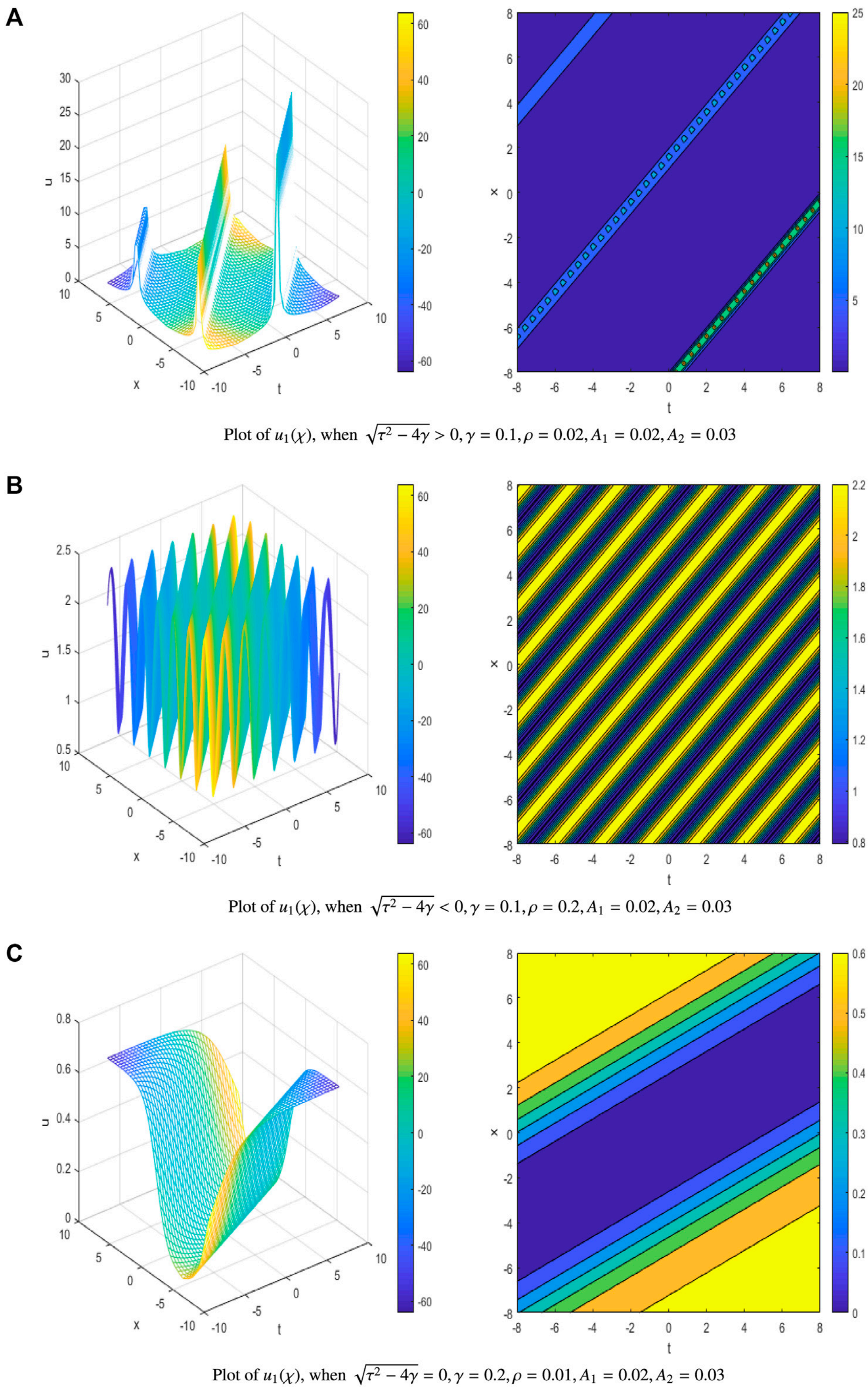

FIGURE 1 | Traveling wave solution corresponding to the Sharma-Tasso-Olver (STO) equation. **(A)** Plot of $u_1(\chi)$, when $\sqrt{\tau^2 - 4\gamma} > 0$, $\gamma = 0.1$, $\rho = 0.02$, $A_1 = 0.02$, $A_2 = 0.03$. **(B)** Plot of $u_1(\chi)$, when $\sqrt{\tau^2 - 4\gamma} < 0$, $\gamma = 0.1$, $\rho = 0.2$, $A_1 = 0.02$, $A_2 = 0.03$. **(C)** Plot of $u_1(\chi)$, when $\sqrt{\tau^2 - 4\gamma} = 0$, $\gamma = 0.2$, $\rho = 0.01$, $A_1 = 0.02$, $A_2 = 0.03$.

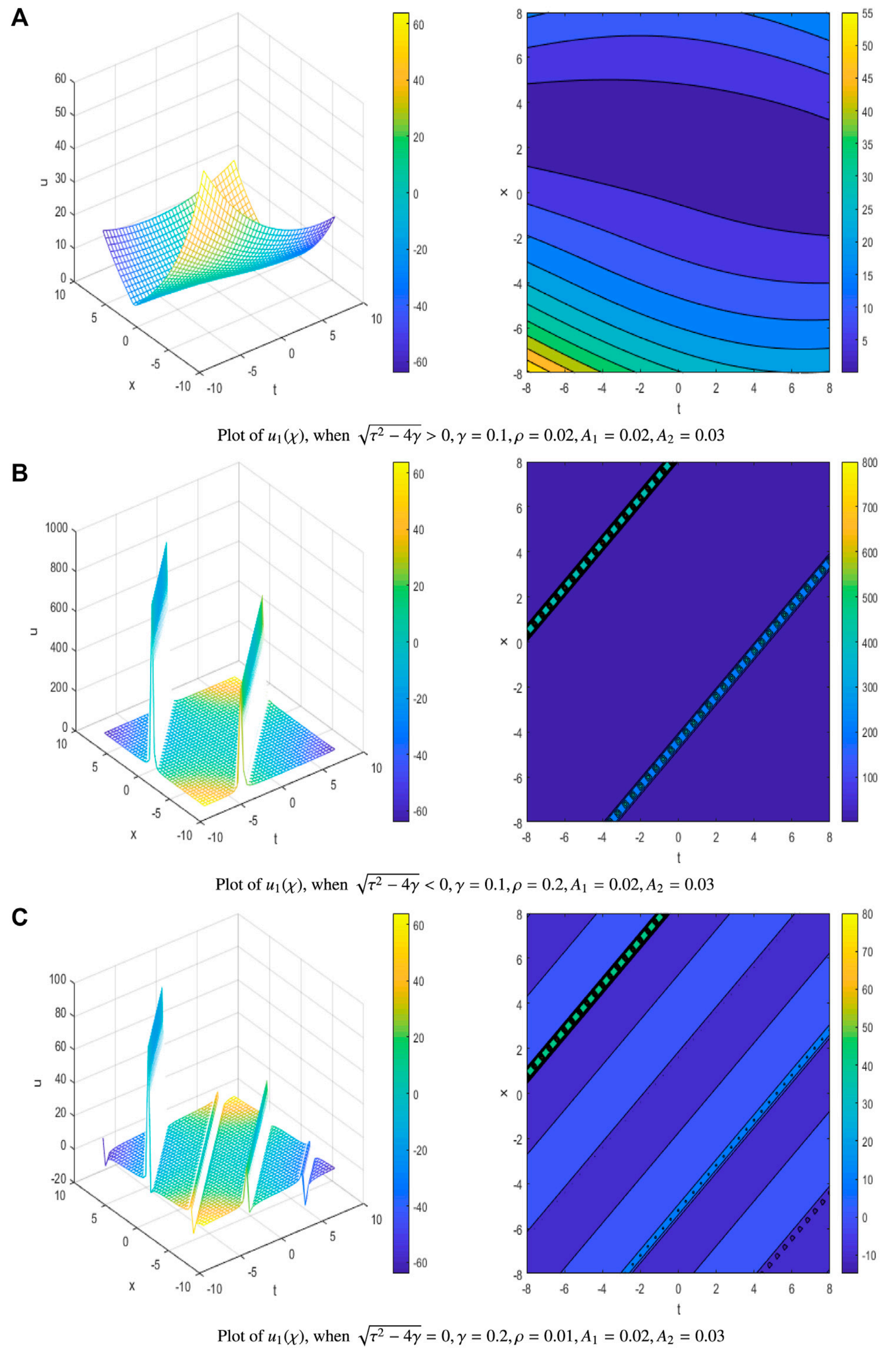

FIGURE 2 | Traveling wave solution corresponding to the Zakharov Kuznetsov equation (ZK) equation. **(A)** Plot of $u_1(\chi)$, when $\sqrt{\tau^2-4\gamma}>0, \gamma=0.1, \rho=0.02, A_1=0.02, A_2=0.03$. **(B)** Plot of $u_1(\chi)$, when $\sqrt{\tau^2-4\gamma}<0, \gamma=0.1, \rho=0.2, A_1=0.02, A_2=0.03$. **(C)** Plot of $u_1(\chi)$, when $\sqrt{\tau^2-4\gamma}=0, \gamma=0.2, \rho=0.01, A_1=0.02, A_2=0.03$.

Case (2)

$$\delta_0 = \frac{\delta_1(\zeta(t)\delta_1 + 6\theta(t)l^2\gamma + 6k^2\gamma)}{12(k^2+\theta(t)l^2)}, \delta_{-1} = 0, \quad \delta_1 = \delta_1,$$
$$\eta(t) = -\frac{6(k^2+\theta(t)l^2)}{\delta_1^2},$$
$$\tau(t) = \frac{k\{24\theta(t)k^2\gamma^2l^2 + 12k^4\gamma^2 + 12\theta(t)^2l^4\gamma^2 - 96\theta(t)l^2\rho k^2}{24(k^2+\theta(t)l^2)}$$
$$\frac{-48\theta(t)^2\rho l^4 - \zeta^2(t)\delta_1^2 - 48k^4\rho}{24(k^2+\theta(t)l^2)}. \tag{26}$$

by using above two conditions solutions can be written as

$$u_1(\xi) = \frac{\delta_1(\zeta(t)\delta_1 + 6\theta(t)t^2\gamma + 6k^2\gamma)}{12(k^2+\theta(t)t^2)} + \delta_1\left(\frac{G'}{G}\right)^1 + \delta_{-1}\left(\frac{G'}{G}\right)^{-1}$$
$$\xi = kx + ly + \int_0^t\left[\frac{k(12\theta t^4k^2 + 96\theta t^4\rho - \zeta^2\delta_1^2)}{24(k^2+\theta(t)t^2)}\right]dt + \int_0^t\left[\frac{k(12k^4\gamma^2 + 96k^4\rho + 24\theta t^2\gamma^2k^2 + 192\theta t^2\rho k^2)}{24(k^2+\theta(t)t^2)}\right]dt$$

similarly we have

$$u_1(\xi) = \frac{\delta_1(\zeta(t)\delta_1 + 6\theta(t)t^2\gamma + 6k^2\gamma)}{12(k^2+\theta(t)l^2)} + \delta_1\left(\frac{G'}{G}\right)^1 + \delta_{-1}\left(\frac{G'}{G}\right)^{-1}$$
$$\xi = kx + ly + \int_0^t\left[\frac{k\{24\theta k^2\gamma^2l^2 + 12k^4\gamma^2 + 12\theta^2l^4\gamma^2\}}{24(k^2+\theta(t)l^2)}\right]dt + \int_0^t\left[\frac{k\{96\theta l^2\rho k^2 - 48\theta^2\rho l^4 - \zeta^2\delta_1^2 - 48k^4\rho\}}{24(k^2+\theta(t)l^2)}\right]dt$$

with **Eq. 26** and **Eq. 6**, we have exponential and hyperbolic and rational functions types of solitary wave solutions for ZK equation with mutable coefficients **Eq. 20** as:

Solution of first kind (1): when $\sqrt{\gamma^2-4\rho} > 0$

$$u_1(\xi) = \left[(\sqrt{\gamma^2-4\rho})\left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi + A_2\cosh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi}{2A_1\cosh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi + 2A_2\sinh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi}\right) - \frac{\gamma}{2}\right] + \frac{\delta_1(\zeta(t)\delta_1 + 6\theta(t)t^2\gamma + 6k^2\gamma)}{12(k^2+\theta(t)l^2)} + \delta_1\left[(\sqrt{\gamma^2-4\rho})\left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi + A_2\cosh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi}{2A_1\cosh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi + 2A_2\sinh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi}\right) - \frac{\gamma}{2}\right]^{-1}. \tag{27}$$

$$u_2(\xi) = \delta_1\left[(\sqrt{\gamma^2-4\rho})\left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi + A_2\cosh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi}{2A_1\cosh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi + 2A_2\sinh\left(\frac{1}{2}\sqrt{\gamma^2-4\rho}\right)\xi}\right) - \frac{\gamma}{2}\right] + \frac{\delta_1(\zeta(t)\delta_1 + 6\theta(t)t^2\gamma + 6k^2\gamma)}{12(k^2+\theta(t)l^2)}. \tag{28}$$

Second kind (2): when $\sqrt{\gamma^2-4\rho} < 0$

$$u_1(\xi) = \delta_1\left[(\sqrt{4\rho-\gamma^2})-\left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi + A_2\cosh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi}{2A_1\cosh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi + 2A_2\sinh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi}\right) - \frac{\gamma}{2}\right] + \frac{\delta_1(\zeta(t)\delta_1 + 6\theta(t)t^2\gamma + 6k^2\gamma)}{12(k^2+\theta(t)l^2)} + \delta_1\left[\left(\sqrt{4\rho-\gamma^2}\right)-\left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi + A_2\cosh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi}{2A_1\cosh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi + 2A_2\sinh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi}\right) - \frac{\gamma}{2}\right]^{-1}. \tag{29}$$

$$u_2(\xi) = \delta_1\left[(\sqrt{4\rho-\gamma^2})\left(\frac{A_1\sinh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi + A_2\cosh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi}{2A_1\cosh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi + 2A_2\sinh\left(\frac{1}{2}\sqrt{4\rho-\gamma^2}\right)\xi}\right) - \frac{\gamma}{2}\right] + \frac{\delta_1(\zeta(t)\delta_1 + 6\theta(t)t^2\gamma + 6k^2\gamma)}{12(k^2+\theta(t)l^2)}. \tag{30}$$

Third kind (3): when $\sqrt{\gamma^2-4\rho} = 0$

$$u_1(\xi) = \delta_1\left[\frac{A_1}{A_1+A_2\xi} - \frac{\gamma}{2}\right] + \frac{\delta_1\{\zeta(t)\delta_1 + 6\theta(t)t^2\gamma + 6k^2\gamma\}}{12(k^2+\theta(t)l^2)} + \delta_1\left[\frac{A_1}{A_1+A_2\xi} - \frac{\gamma}{2}\right]^{-1}, \tag{31}$$

$$u_2(\xi) = \delta_1 + \left[\frac{A_1}{A_1+A_2\xi} - \frac{\gamma}{2}\right]^{-1} + \frac{\delta_1\{\zeta(t)\delta_1 + 6\theta(t)t^2\gamma + 6k^2\gamma\}}{12(k^2+\theta(t)l^2)}. \tag{32}$$

where A_1 and A_2 are integration constants. As results, many exact solitary wave solutions of the form hyperbolic, trigonometric and rational functions are obtained (**Figure 2**). We have taken all required vales of δ_0, δ_1, δ_{-1}, ρ, $\zeta(t)$, $\theta(t)$ k, l, A_1, A_2 and $\tau(t)$ and plotted for only the first solution $u_1(\chi)$ only.

5 CONCLUSION

Finally, it is worthwhile to mention these hyperbolic, trigonometric and rational function solutions are difficult to obtain by the methods mentioned in the introduction. The general solitary wave solutions can give soliton or periodic solutions under different parametric restrictions. These results mean that there are rich solitary wave patterns for the STO and ZK equation. To the best of our knowledge concern, this paper reports the aforementioned new solutions by this novel method. To our knowledge our solutions **Eq. 15** and **Eq. 17** of **Eqs 19, 27, 29** of **Eq. 32** are all new and not reported in literature. It is interesting to note that from the general results, one can easily recover solutions that are obtained from other methods. If we take the concrete parameters in a real model, we can give the corresponding representation of the solution. Comparing this work with the work in [6] where the tanh method were used, we find that the proposed method in this work presents more kink and solitons solutions compared to the work in [6]. Therefore, it is rather convenient for practice. We also presented three-dimensional plots and their corresponding contour plots of some of the solutions. This is direct and the concise method can further be used to explore more applications.

DATA AVAILABILITY STATEMENT

The raw data supporting the conclusion of this article will be made available by the authors, without undue reservation.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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Conflict of Interest: The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.