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Numerical treatment of singularly perturbed unsteady Burger-Huxley equation

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This article deals with the numerical treatment of a singularly perturbed unsteady Burger-Huxley equation. The equation is linearized using the Newton-Raphson-Kantorovich approximation method. The resulting linear equation is discretized using the implicit Euler method and an exponential spline method for time and space variables, respectively. Richardson's extrapolation technique is employed to increase the accuracy in the temporal direction. The stability and uniform convergence of the proposed scheme are investigated. The scheme is shown uniformly convergent with the order of convergence $O(\tau+\ell^2)$ and $O(\tau^2+\ell^2)$ before and after Richardson extrapolation, respectively. Several test examples are considered to validate the applicability and efficiency of the scheme. It is observed that the proposed scheme provides more accurate results than the methods available in the literature.

KEYWORDS

singular perturbation problem, burger-huxley equation, implicit Euler method, exponential fitted operator method, Richardson extrapolation

1. Introduction

Singular perturbed non-linear parabolic partial differential equations have many applications in real life scenarios. A good example of singularly perturbed nonlinear differential equations is the mathematical model for the prototype model used to describe the interaction between non-linear convection effects, reaction mechanisms, and diffusion transport. This equation has many intriguing phenomena such as bursting oscillation [1], interspike [2], population genetics [3], bifurcation, and chaos [4]. Several membrane models based on the dynamics of potassium and sodium ion fluxes were found in Lewis et al. [5].

Many researchers have made efforts to construct analytical and numerical methods for Burger equations, for instance, Ismail et al. [6], Javidi et al. [7], Estevez [8], Krisnangkura et al. [9], Satsuma et al. [10], Wang et al. [11], Hashim et al. [12, 13], Khattak [14], Krisnangkur et al. [9], Mohammadi [15], Sari et al. [16], Appadu et al. [17, 18], and Appadu and Tijani [19], and references therein. A Burger-Huxley equation in which the highest order derivative is multiplied by a perturbation parameter ε ($0 < \varepsilon << 1$) is termed the singularly perturbed Burger-Huxley equation (SPBHE). Due to the presence of the perturbation parameter and the nonlinearity in the problem, finding their solutions are tedious task. For instance, due to the presence of the perturbation parameter, the solution exposes boundary/shrill interior layer(s), and it is not easy to find a stable numerical approximation. The methods suggested in Ismail et al. [6], Javidi and Golbabai [7], Estevez [8], Krisnangkura et al. [9], Satsuma et al. [10], Wang et al. [11], Hashim et al. [12, 13], Khattak [14], Mohammadi [15], and Sari and Gürarslan [16] and other standard numerical methods on a uniform mesh is insufficient to approximate the SPBHE. Thus, to resolve this drawback, the fitting approaches are competitive computational techniques to overcome the limitations of classical numerical methods. Kaushik and Sharma [20] presented a uniformly convergent finite difference method (FDM) for the problem Equation (1). Gupta and Kadalbajoo [21] developed a numerical method that comprises an implicit-Euler method and a monotone hybrid finite difference operator with a Shishkin mesh type for the problem Equation (1). A robust numerical method that consists of a backward-Euler method and an upwind FDM on an adaptive nonuniform grid for Equation (1) is suggested by Liu et al. [22]. Kabato and Duressa [23] and Jima et al. [24] considered a similar problem as in Liu et al. [22] and proposed a parameter uniform numerical method based on fitted operator techniques. Kehinde et al. [25] developed a fitted operator FDM for the two-dimensional semilinear singularly perturbed convection-diffusion problem.

However, there has been little development in the numerical treatment of the problem Equation (1). This work aims to develop and analyze an ε, which is a uniformly convergent numerical method for solving the problem Equation (1). First, the nonlinear terms are linearized by using the Newton-Raphson-Kantorovich approximation method. Then, we approximate the resulting singularly perturbed parabolic convection-diffusion equations using the implicit Euler method for the time derivative and fitted the exponential spline method for the space derivative on a uniform step length. Finally, to enhance the order of convergence in the time variable, the Richardson extrapolation technique is applied to the temporal variable. The novelty of the presented method, unlike Shishkin and Bakhvalov mesh types, does not require *a priori* information about the location and width of the boundary layer.

2. Continuous problem

On the domain $\Upsilon = \chi_z \times \chi_t = (0,1) \times (0,T]$, we consider singularly perturbed Burger-Huxley equation of the form

$$\begin{cases} \pounds_\varepsilon \Im(z,t) = \frac{\partial \Im}{\partial t} - \varepsilon \frac{\partial^2 \Im}{\partial z^2} + \alpha \Im \frac{\partial \Im}{\partial z} - \theta\,(1-\Im)\,(\Im-\lambda) = 0, (z,t) \in \Upsilon, \\ \Im(z,0) = \Im_0(z), z \in \overline{\chi}_z, \\ \Im(0,t) = \Theta_0(t), \Im(1,t) = \Theta_1(t), t \in \chi_t, \end{cases} \tag{1}$$

Where $0 < \varepsilon \ll 1$ is a singular perturbation parameter and $\alpha \geq 1, \theta \geq 0$, and $\lambda \in (0,1)$. The functions $\Theta_0(t), \Theta_1(t)$, and $\Im_0(z)$ are assumed to be sufficiently smooth, bounded, and independent of ε.

Lemma 2.1 (Maximum principle). If $\Im \in C^{2,1}(\overline{\Upsilon})$, $\Im|_{\partial\Upsilon} \geq 0$, and $\pounds_\varepsilon \Im|_{\Upsilon} \geq 0$, then $\Im|_{\overline{\Upsilon}} \geq 0$.

Proof. See Gupta and Kadalbajoo [21].

An immediate consequence of Lemma 2.1 for the solution of problem Equation (1) resulted in Lemma 2.2.

Lemma 2.2 (Stability estimate). Let $\Im(z,t)$ be the solution of problem Equation (1), then we have,

$$\|\Im\|_{\overline{\Upsilon}} \leq T\,\|\Im_0\|_{\Gamma_i} + \|\Im\|_{\partial\Upsilon}\,.$$

Proof. See Gupta and Kadalbajoo [21].

3. Formulation of the numerical scheme

3.1. Quasi-linearization

The Equation (1) is rewritten as

$$\begin{cases} \pounds_\varepsilon \Im(z,t) = \left(\frac{\partial \Im}{\partial t} - \varepsilon \frac{\partial^2 \Im}{\partial z^2}\right)(z,t) = \zeta(z,t,\Im(z,t),\Im_z(z,t)), (z,t) \in \Upsilon, \\ \Im(z,0) = \Im_0(z), z \in \overline{\chi}_z, \\ \Im(0,t) = \Theta_0(t), \Im(1,t) = \Theta_1(t), t \in (0,T], \end{cases} \tag{2}$$

Where $\zeta(z,t,\Im(z,t),\Im_z(z,t)) = -\alpha\Im\frac{\partial \Im}{\partial z} + \theta\,(1-\Im)\,(\Im-\lambda)$ is the non-linear function of $z,t,\Im(z,t),\Im_z(z,t)$.

To linearize the semi-linear term of Equation (1), we choose the reasonable initial approximation for the function $\Im(z,t)$ in the term $\zeta(z,t,\Im(z,t),\Im_z(z,t))$ and denote it as $\Im^0(z,t)$ that satisfy both initial and boundary conditions which are obtained by the separation of variables method of the homogeneous part of the problem under consideration and is given by Kabeto and Duressa [23].

$$\Im^0(z,t) = \Im_0(z)\exp(-\pi^2 t) \tag{3}$$

Thus, the nonlinear term $\zeta(z,t,\Im(z,t),\Im_z(z,t))$ of Equation (2) can be linearized by applying the Newton-Raphson- Kantorovich approximation approach as

$$\begin{aligned} &\zeta(z,t,\Im^{(k+1)}(z,t),\Im_z^{(k+1)}(z,t)) \cong \zeta(z,t,\Im^{(k)}(z,t),\Im_z^{(k)}(z,t)) + \\ &(\Im^{(k+1)}(z,t) - \Im^{(k)}(z,t))\zeta_\Im|(\Im,t,\Im^{(k)}(z,t),\Im_z^{(k)}(z,t)) + \\ &(\Im_z^{(k+1)}(z,t) - \Im_z^{(k)}(\Im,t))\zeta_{\Im_z}|(z,t,\Im^{(k)}(z,t),\Im_z^{(k)}(z,t)) + \cdots, \end{aligned} \tag{4}$$

Where $\left\{\Im^{(k)}\right\}_{k=0}^{\infty}$ are a sequence of approximate solutions of $\zeta(z,t,\Im(z,t),\Im_z(z,t))$.
Substituting Equation (4) into Equation (2), we have

$$\begin{cases} \pounds_\varepsilon \Im^{(k+1)}(z,t) = \Im_t^{(k+1)}(z,t) - \varepsilon \Im_{zz}^{(k+1)}(z,t) \\ \qquad + \varpi(z,t)\Im_z^{(k+1)}(z,t) + q(z,t)\Im^{(k+1)}(z,t) \\ \qquad = v(z,t), \\ \Im^{(k+1)}(z,0) = \Im_0(z), z \in \overline{\chi}_z, \\ \Im^{(k+1)}(0,t) = \Theta_0(t), t \in \overline{\chi}_t, \\ \Im^{(k+1)}(1,t) = \Theta_1(t), t \in \overline{\chi}_t, \end{cases} \tag{5}$$

where

$$\begin{aligned} \varpi(z,t) &= -\zeta_{u_z}|(z,t,\Im^{(k)}(z,t),\Im_z^{(k)}(z,t)), \\ q(z,t) &= -\zeta_{\Im}|(z,t,\Im^{(k)}(z,t),\Im_z^{(k)}(z,t)), \\ v(z,t) &= \zeta(z,t,\Im^{(k)}(z,t),\Im_z^{(k)}(z,t)) \\ &\quad - \Im^{(k)}\zeta_{\Im}|(z,t,\Im^{(k)}(z,t),\Im_z^{(k)}(z,t)) \\ &\quad - \Im_z^{(k)}\zeta_{\Im_z}|(z,t,\Im^{(k)}(z,t),\Im_z^{(k)}(z,t)). \end{aligned}$$

3.2. Temporal semi-discretization

We define the uniform mesh for the domain χ_t as $\chi_t = \Upsilon_\tau^M = \left\{t_j = j\tau, \forall j = 0,2,3,\cdots,M, \tau = 1/M\right\}$ and we approximated the temporal derivative term of Equation (5) using implicit Euler method, which gives

$$\begin{cases} \left(1+\tau\pounds_\varepsilon^M\right)\hat{\Im}^{k+1}(z,t_{j+1}) = -\varepsilon\hat{\Im}_{zz}^{k+1}(z,t_{j+1}) \\ \qquad + \varpi(z,t_{j+1})\hat{\Im}_z^{k+1}(z,t_{j+1}) \\ \qquad + q(z,t_{j+1})\hat{\Im}^{k+1}(z,t_{j+1}) - v(z,t_{j+1}) \\ \qquad = \hat{\Im}^{k+1}(z,t_j), \\ \hat{\Im}^{k+1}(z,0) = \hat{\Im}_0^{k+1}(z), z \in \overline{\chi}_z, \\ \hat{\Im}^{k+1}(0,t_{j+1}) = \Theta_0(t_{j+1}), 0 \le j \le M-1, \\ \hat{\Im}^{k+1}(1,t_{j+1}) = \Theta_1(t_{j+1}), 0 \le j \le M-1. \end{cases} \tag{6}$$

Since $\varpi(z,t_{j+1}) \ge \varpi^* > 0$ and $q(z,t_{j+1}) \ge q^* > 0, z \in \overline{\chi}_z$ Equation (6) exhibits boundary located at $z = 1$ and admits a unique solution.

Lemma 3.1 (Local Error Estimate). In the solution of Equation (6), the local truncation error estimate is bounded by

$$\left\|LTE_{j+1}\right\|_\infty \le C^*(\tau)^2,$$

Where C^* is a positive constant independent of ε and τ.

Proof. On applying Taylor's series expansion on $\hat{\Im}^{k+1}(z,t_{j+1})$ yields

$$\hat{\Im}^{k+1}(z,t_{j+1}) = \hat{\Im}^{k+1}(z,t_j) + \tau\hat{\Im}_t^{k+1}(z,t_j) + O((\tau)^2).$$

This implies

$$\frac{\hat{\Im}^{k+1}(z,t_{j+1}) - \hat{\Im}^{k+1}(z,t_j)}{\tau} = \hat{\Im}_t^{k+1}(z,t_j) + O(\tau). \tag{7}$$

Substituting Equation (5) into Equation (7) we have

$$\begin{aligned} &\frac{\hat{\Im}^{k+1}(z,t_{j+1}) - \hat{\Im}^{k+1}(z,t_j)}{\tau} \\ &= -\left(-\varepsilon\hat{\Im}_{zz}^{k+1}(z,t_j) + \varpi(z,t_j)\hat{\Im}_z^{k+1}(z,t_j)\right. \\ &\left. + q(z,t_j)\hat{\Im}^{k+1}(z,t_j) - v(z,t_j)\right) + O(\tau) \\ &\Rightarrow \left(1+\tau\pounds_\varepsilon^M\right)\hat{\Im}^{k+1}(z,t_{j+1}) - \tau v(z,t_j) + \hat{\Im}^{k+1}(z,t_j) = O(\tau)^2). \end{aligned} \tag{8}$$

Subtracting Equation (6) from Equation (8), and the local truncation error $LTE_{j+1} = \hat{\Im}^{k+1}(z,t_{j+1}) - \hat{\Im}^{k+1}(z,t_{j+1})$ at $(j+1)th$ is the solution of a BVP

$$\left(1+\tau\pounds_\varepsilon^M\right)LTE_{j+1} = O((\tau)^2), LTE_{j+1}(0) = 0 = LTE_{j+1}(1). \tag{9}$$

Hence, applying the maximum principle on the operator gives

$$\left\|LTE_{j+1}\right\|_\infty \le C^*(\tau)^2.$$

Lemma 3.2. [Global Error Estimate (GEE)] The GEE in the temporal direction of Equation (6) is given by

$$\left\|E_j\right\|_\infty \le C(\tau),\ \forall\, j \le T/\tau.$$

Proof. Using Lemma 3.1, we obtain the following GEE at $(j)th$ time step

$$\begin{aligned} \left\|E_j\right\|_\infty &= \left\|\textstyle\sum_{i=1}^{j} LTE_i\right\|_\infty, \quad j \le T/\tau \\ &\le \|LTE_1\|_\infty + \|LTE_2\|_\infty + \|LTE_3\|_\infty + \cdots + \left\|e_j\right\|_\infty \\ &\le c_1 j(\tau)^2 \quad (\text{by} \quad \text{Lemma 3.1}) \\ &\le c_1\left(j\tau\right)(\tau) \\ &\le c_1 T(\tau) \quad \left(j\tau \le T\right) \\ &\le C(\tau), \end{aligned}$$

Where c_1, C are positive constants independent of ε and τ.

Lemma 3.3. The solution $\hat{\Im}^{k+1}(z)$ of the Equation (6) is bounded by

$$\left\|\frac{\partial^m\hat{\Im}^{k+1}(z,t_{j+1})}{\partial z^m}\right\|_{\overline{\chi}_z} \le C\left(1+\varepsilon^{-m}\exp\left(\frac{-(\varpi^*(1-z))}{\varepsilon}\right)\right), \quad 0 \le m \le 4.$$

Proof. See Gupta et al. [21].

3.2.1. Spatial semi-discretization

We define the uniform mesh for the space domain [0, 1] as

$$\Upsilon_\ell^N = \{z_i = i\ell, \forall i = 1,2,3,\cdots,N, z_0 = 0, z_N = 1, \ell = 1/N\}.$$

Equation (6) can be rewritten as

$$\begin{aligned} &-\varepsilon\hat{\Im}_{zz}^{k+1}(z,t_{j+1}) + \varpi(z,t_{j+1})\hat{\Im}_z^{k+1}(z,t_{j+1}) \\ &+ r(z,t_{j+1})\hat{\Im}^{k+1}(z,t_{j+1}) = \varphi(z,t_{j+1}), \\ &\hat{\Im}(0,t_{j+1}) = \Theta_0(t_{j+1}), \hat{\Im}(1,t_{j+1}) = \Theta_1(t_{j+1}), z \in \chi_z, \\ &0 \le j \le M-1, \end{aligned} \tag{10}$$

Where $r(z,t_{j+1}) = \left(q(z,t_{j+1}) + \frac{1}{\tau}\right)$ and $\varphi(z,t_{j+1}) = v(z,t_{j+1}) + \frac{\hat{\Im}^{k+1}(z,t_j)}{\tau}$.

For spatial discretization of Equation (10), we use an exponential cubic spline method.

The approximation solution $\hat{\Im}_i^{j+1}$ of Equation (10) obtained by the segment $S_\triangle(z)$ passing through the points $\left(z_i, \hat{\Im}_i^{j+1}\right)$ and $\left(z_{i+1}, \hat{\Im}_{i+1}^{j+1}\right)$. Each mixed spline segment $S_\triangle^{j+1}(z)$ has the form [26].

$$S_\triangle^{j+1}(z) = A_i e^{\psi(z-z_i)} + B_i e^{-\psi(z-z_i)} + C_i(z-z_i) + D_i, 0 \le i \le N, \tag{11}$$

Where A_i, B_i, C_i, and D_i are constants and $\psi \neq 0$ is a free parameter that is used to enhance the accuracy of the method. To find the values of A_i, B_i, C_i, and D_i in Equation (11), we denote

$$\begin{aligned} S_\triangle^{j+1}(z_i) &= \hat{\Im}_i^{j+1}, & S_\triangle^{j+1}(z_{i+1}) &= \hat{\Im}_{i+1}^{j+1}, \\ \frac{d^2 S_\triangle^{j+1}(z_i)}{dz^2} &= \Xi_i, & \frac{d^2 S_\triangle^{j+1}(z_{i+1})}{dz^2} &= \Xi_{i+1}. \end{aligned} \tag{12}$$

Differentiating twice both sides of Equation (11), we get

$$\frac{d^2 S_\triangle^{j+1}(z)}{dz^2} = \psi^2 \left(A_i e^{\psi(z-z_i)} + B_i e^{-\psi(z-z_i)}\right). \tag{13}$$

Using the relation in Equation (12) into Equation (13) at $z = z_i$, we get

$$\frac{d^2 S_\triangle^{j+1}(z_i)}{dz^2} = \Xi_i = \psi^2 (A_i + B_i)$$

and it yields

$$A_i = \frac{\ell^2 \Xi_i}{\varrho^2} - B_i. \tag{14}$$

Where $\varrho = \psi \ell$ and $i = 0, 1, 2, \cdots, N$.
Again substituting Equation (12) into Equation (13) at $z = z_{i+1}$, we obtain

$$\frac{d^2 S_\triangle^{j+1}(z_{i+1})}{dz^2} = \Xi_{i+1} = \psi^2 \left(A_i e^{\varrho} + B_i e^{-\varrho}\right)$$

and it gives

$$A_i = e^{-\varrho}\left(\frac{\ell^2 \Xi_{i+1}}{\varrho^2}\right) - B_i e^{-2\varrho}. \tag{15}$$

From Equations (14) and (15), we have

$$A_i = \frac{\ell^2 \left(\Xi_{i+1} - \Xi_i e^{-\varrho}\right)}{2\varrho^2 \sinh(\varrho)} \text{ and } B_i = \frac{\ell^2 \left(\Xi_i e^{\varrho} - \Xi_{i+1}\right)}{2\varrho^2 \sinh(\varrho)}. \tag{16}$$

Substituting Equation (12) into Equation (11) at $z = z_i$ and $z = z_{i+1}$, we obtain

$$\begin{aligned} S_\triangle^{j+1}(z_i) = \hat{\Im}_i^{j+1} &= A_i + B_i + D_i, \text{ and } S_\triangle^{j+1}(z_{i+1}) = \hat{\Im}_{i+1}^{j+1} \\ &= A_i e^{\varrho} + B_i e^{-\varrho} + C_i \ell + D_i, \end{aligned}$$

which implies

$$\begin{aligned} C_i &= \frac{\left(\hat{\Im}_{i+1}^{j+1} - \hat{\Im}_i^{j+1}\right)}{\ell} - \frac{\ell\left(\Xi_{i+1} - \Xi_i\right)}{\varrho^2} \text{ and} \\ D_i &= \hat{\Im}_i^{j+1} - \left(\frac{\ell^2}{\varrho^2}\Xi_i\right). \end{aligned} \tag{17}$$

The first derivative continuity condition at $z = z_i$, that is $S_{\triangle-1}(z_i) = S_\triangle(z_i)$ gives

$$\ell^2\left(\omega_1 \Xi_{i-1} + 2\omega_2 \Xi_i + \omega_1 \Xi_{i+1}\right) = \left(\hat{\Im}_{i-1}^{j+1} - 2\hat{\Im}_i^{j+1} + \hat{\Im}_{i+1}^{j+1}\right), \quad i = 1, 2, \cdots, N-1, \tag{18}$$

where

$$\omega_1 = \left(\frac{\sinh(\varrho) - \varrho}{\varrho^2 \sinh(\varrho)}\right) \text{ and } \omega_2 = \left(\frac{2\varrho\cosh(\varrho) - 2\sinh(\varrho)}{\varrho^2 \sinh(\varrho)}\right).$$

Now from Equation (10), we have

$$\begin{cases} \varepsilon \Xi_{i-1} = \varpi^{j+1}(z_{i-1})\hat{\Im}_z^{j+1}(z_{i-1}) + r^{j+1}(z_{i-1})\hat{\Im}^{j+1}(z_{i-1}) \\ \qquad - \varphi^{j+1}(z_{i-1}), \\ \varepsilon \Xi_i = \varpi^{j+1}(z_i)\hat{\Im}_z^{j+1}(z_i) + r^{j+1}(z_i)\hat{\Im}^{j+1}(z_i) - \varphi^{j+1}(z_i), \\ \varepsilon \Xi_{i+1} = \varpi^{j+1}(z_{i+1})\hat{\Im}_z^{j+1}(z_{i+1}) + r^{j+1}(z_{i+1})\hat{\Im}^{j+1}(z_{i+1}) \\ \qquad - \varphi^{j+1}(z_{i+1}), \end{cases} \tag{19}$$

Where we approximate $\hat{\Im}_z^{j+1}(z_{i-1}), \hat{\Im}_z^{j+1}(z_i)$ and $\hat{\Im}_z^{j+1}(z_{i+1})$ as

$$\begin{cases} \hat{\Im}_z^{j+1}(z_{i-1}) \cong \frac{-\hat{\Im}_{i+1}^{j+1} + 4\hat{\Im}_i^{j+1} - 3\hat{\Im}_{i-1}^{j+1}}{2\ell}, \\ \hat{\Im}_z^{j+1}(z_i) \cong \frac{\hat{\Im}_{i+1}^{j+1} - \hat{\Im}_{i-1}^{j+1}}{2\ell}, \\ \hat{\Im}_z^{j+1}(z_{i+1}) \cong \frac{3\hat{\Im}_{i+1}^{j+1} - 4\hat{\Im}_i^{j+1} + \hat{\Im}_{i-1}^{j+1}}{2\ell}. \end{cases} \tag{20}$$

Substituting Equation (20) into (19), we have

$$\begin{cases} \varepsilon \Xi_{i-1} = \varpi^{j+1}(z_{i-1})\left(\frac{-\hat{\Im}_{i+1}^{j+1} + 4\hat{\Im}_i^{j+1} - 3\hat{\Im}_{i-1}^{j+1}}{2\ell}\right) \\ \qquad + r^{j+1}(z_{i-1})\hat{\Im}^{j+1}(z_{i-1}) - \varphi^{j+1}(z_{i-1}), \\ \varepsilon \Xi_i = \varpi^{j+1}(z_i)\left(\frac{\hat{\Im}_{i+1}^{j+1} - \hat{\Im}_{i-1}^{j+1}}{2\ell}\right) + r^{j+1}(z_i)\hat{\Im}^{j+1}(z_i) - \varphi^{j+1}(z_i), \\ \varepsilon \Xi_{i+1} = \varpi^{j+1}(z_{i+1})\left(\frac{3\hat{\Im}_{i+1}^{j+1} - 4\hat{\Im}_i^{j+1} + \hat{\Im}_{i-1}^{j+1}}{2\ell}\right) \\ \qquad + r^{j+1}(z_{i+1})\hat{\Im}^{j+1}(z_{i+1}) - \varphi^{j+1}(z_{i+1}). \end{cases} \tag{21}$$

Substituting Equation (21) into Equation (18) and rearranging, we get

$$\begin{aligned}&\varepsilon\left(\frac{\hat{\Im}_{i-1}^{j+1}-2\hat{\Im}_{i}^{j+1}+\hat{\Im}_{i+1}^{j+1}}{\ell^2}\right)\\&=\left(-\frac{3\omega_1\varpi_{i-1}^{j+1}}{2\ell}-\frac{\omega_2\varpi_{i}^{j+1}}{\ell}+\frac{\omega_1\varpi_{i+1}^{j+1}}{2\ell}+\omega_1 r_{i-1}^{j+1}\right)\hat{\Im}_{i-1}^{j+1}+\\&\left(\frac{2\omega_1\varpi_{i-1}^{j+1}}{\ell}-\frac{2\omega_1\varpi_{i+1}^{j+1}}{\ell}+2\omega_2 r_{i}^{j+1}\right)\hat{\Im}_{i}^{j+1}+\\&\left(-\frac{\omega_1\varpi_{i-1}^{j+1}}{2\ell}+\frac{\omega_2\varpi_{i}^{j+1}}{\ell}+\frac{3\omega_1\varpi_{i-1}^{j+1}}{2\ell}+\omega_1 r_{i+1}^{j+1}\right)\hat{\Im}_{i+1}^{j+1}\\&=\left(\omega_1\varphi_{i-1}^{j+1}+\omega_2\varphi_{i}^{j+1}+\omega_1\varphi j+1_{i+1}\right).\end{aligned}\tag{22}$$

To grip the effect of the perturbation parameter ε, we multiply the perturbation parameter of Equation (22) by fitting factor $\sigma\,(\rho)$, we obtain

$$\begin{aligned}&\left(\frac{-\sigma(\rho)}{\rho}-\frac{3\omega_1\varpi_{i-1}^{j+1}}{2}-\omega_2\varpi_{i}^{j+1}+\frac{\omega_1\varpi_{i+1}^{j+1}}{2}+\ell\omega_1 r_{i-1}^{j+1}\right)\hat{\Im}_{i-1}^{j+1}+\\&\left(\frac{2\sigma(\rho)}{\rho}+2\omega_1\varpi_{i-1}^{j+1}-2\omega_1\varpi_{i+1}^{j+1}+2\ell\omega_2 r_{i}^{j+1}\right)\hat{\Im}_{i}^{j+1}+\\&\left(\frac{-\sigma(\rho)}{\rho}-\frac{\omega_1\varpi_{i-1}^{j+1}}{2}+\omega_2\varpi_{i}^{j+1}+\frac{3\omega_1\varpi_{i-1}^{j+1}}{2}+\ell\omega_1 r_{i+1}^{j+1}\right)\hat{\Im}_{i+1}^{j+1}\\&=\ell\left(\omega_1\varphi_{i-1}^{j+1}+\omega_2\varphi_{i}^{j+1}+\omega_1\varphi_{i+1}^{j+1}\right),\end{aligned}\tag{23}$$

Where $\rho=\frac{\ell}{\varepsilon}$.
Evaluating the limit of Equation (23) as $\ell\to 0$

$$\begin{aligned}&\lim_{\ell\to0}\left(\frac{\sigma(\rho)}{\rho}\right)\left(\hat{\Im}^{j+1}(i\ell-\ell)-2\hat{\Im}^{j+1}(i\ell)+\hat{\Im}^{j+1}(i\ell+\ell)\right)+\\&(\omega_1+\omega_2)\lim_{\ell\to0}\left(\varpi^{j+1}(i\ell)\right)\left(\hat{\Im}^{j+1}(i\ell+\ell)-\hat{\Im}^{j+1}(i\ell-\ell)\right)\\&=0.\end{aligned}\tag{24}$$

When the boundary layer is on the right side of the domain, from the theory of singular perturbation [27], the solution of Equation (10) is of the form

$$\begin{aligned}\hat{\Im}^{j+1}(z)&\approx\Im_{o}^{j+1}(z)+\frac{\varpi^{j+1}(1)}{\varpi^{j+1}(z)}(\Theta_1^{j+1}-\hat{\Im}_0^{j+1}(1))\\&\exp\left(-\varpi^{j+1}(z)\frac{(1-z)}{\varepsilon}\right)+O\left(\varepsilon\right),\end{aligned}\tag{25}$$

Where $\Im_0^{j+1}(z)$ is the solution of the reduced problem

$$\varpi^{j+1}(z)\frac{\partial\hat{\Im}_0^{j+1}(z)}{\partial z}+r^{j+1}(z)\hat{\Im}_0^{j+1}(z)=\varphi^{j+1}(z),\quad \text{with } \hat{\Im}_0^{j+1}(1)=\Theta_1^{j+1}.$$

Using Taylor's series expansion for $\varpi^{j+1}(z)$ about the point $z=1$ and restricting to their first terms, Equation (25) becomes

$$\begin{aligned}\hat{\Im}^{j+1}(z)&\approx\hat{\Im}_0^{j+1}(z)+(\Theta_1^{j+1}-\hat{\Im}_0^{j+1}(1))\\&\exp\left(-\varpi^{j+1}(1)\frac{(1-z)}{\varepsilon}\right)+O\left(\varepsilon\right).\end{aligned}\tag{26}$$

Equation (26) at $z_i=i\ell$ and as $\ell\to0$ becomes

$$\begin{cases}\lim_{\ell\to0}\hat{\Im}^{j+1}(i\ell)\approx\hat{\Im}_0^{j+1}(0)+\left(\Theta_1^{j+1}-\hat{\Im}_0^{j+1}(1)\right)\\\qquad\exp\left(-\varpi^{j+1}(1)\left(\frac{1}{\varepsilon}-i\rho\right)\right)+O(\varepsilon),\\\lim_{\ell\to0}\hat{\Im}^{j+1}((i-1)\ell)\approx\hat{\Im}_0^{j+1}(0)+\left(\Theta_1^{j+1}-\hat{\Im}_0^{j+1}(1)\right)\\\qquad\exp\left(-\varpi^{j+1}(1)\left(\frac{1}{\varepsilon}-i\rho+\rho\right)\right)+O(\varepsilon),\\\lim_{\ell\to0}\hat{\Im}^{j+1}((i+1)\ell)\approx\hat{\Im}_0^{j+1}(0)+\left(\Theta_1^{j+1}-\hat{\Im}_0^{j+1}(1)\right)\\\qquad\exp\left(-\varpi^{j+1}(1)\left(\frac{1}{\varepsilon}-i\rho-\rho\right)\right)+O(\varepsilon).\end{cases}$$

Plugging the above equations into Equation (24), we get

$$\sigma\,(\rho)=\varpi^{j+1}(0)\rho(\omega_1+\omega_2)\coth\left(\frac{\varpi^{j+1}(1)\rho}{2}\right).\tag{27}$$

Finally, from Equations (23) and (27), we get

$$\pounds_{\varepsilon}^{N,M}\hat{\Im}_i^{j+1}=H_i^{j+1},\; i=1,2,\cdots,N-1,\tag{28}$$

where

$$\begin{cases}\pounds_{\varepsilon}^{N,M}\hat{\Im}_i^{j+1}=\vartheta_i^{-}\hat{\Im}_{i-1}^{j+1}+\vartheta_i^{c}\hat{\Im}_i^{j+1}+\vartheta_i^{+}\hat{\Im}_{i+1}^{j+1},\\\vartheta_i^{-}=\frac{-\sigma(\rho)}{\rho}-\frac{3\omega_1\varpi_{i-1}^{j+1}}{2}-\omega_2\varpi_i^{j+1}+\frac{\omega_1\varpi_{i+1}^{j+1}}{2}+\ell\omega_1 r_{i-1}^{j+1},\\\vartheta_i^{c}=\frac{2\sigma(\rho)}{\rho}+2\omega_1\varpi_{i-1}^{j+1}-2\omega_1\varpi_{i+1}^{j+1}+2\ell\omega_2 r_i^{j+1},\\\vartheta_i^{+}=-\frac{-\sigma(\rho)}{\rho}-\frac{\omega_1\varpi_{i-1}^{j+1}}{2}+\omega_2\varpi_i^{j+1}+\frac{3\omega_1\varpi_{i-1}^{j+1}}{2}+\ell\omega_1 r_{i+1}^{j+1},\\H_i^{j+1}=\ell\left(\omega_1\varphi_{i-1}^{j+1}+\omega_2\varphi_i^{j+1}+\omega_1\varphi_{i+1}^{j+1}\right).\end{cases}$$

For sufficiently small mesh sizes, the above matrix is non-singular and $\left|\vartheta_i^c\right|\ge\left|\vartheta_i^-\right|+\left|\vartheta_i^+\right|$ (i.e., the matrix are diagonally dominant). Hence, by Nichols [28], the matrix ϑ is M-matrix and has an inverse. Therefore, the system of equations can be solved by matrix inverse with the given boundary conditions.

4. Convergence analysis

Lemma 4.1 (Discrete Maximum Principle). If the discrete function $\Im_i^{j+1}$ satisfies $\Im_i^{j+1}\ge0$, on $\partial\Upsilon$. Then $\pounds_{\varepsilon}^{N,M}\Im_i^{j+1}\ge0$ on $\Upsilon^{N,M}$ implies that $\Im_i^{j+1}\ge0$ at each point of $\overline{\Upsilon}^{N,M}$.

This lemma gives assurance for the presence of a unique discrete solution.

Lemma 4.2 (Discrete Uniform Stability). The solution $\hat{\Im}_i^{j+1}$ of the discrete problem (28) at $(j+1)th$ time level and $\eta=\min_{0\le i\le N}\left\{r_i^{j+1}\right\}$, where η is some positive constant that satisfies

$$\left\|\hat{\Im}_i^{j+1}\right\|\le\frac{\left\|\pounds_{\varepsilon}^{N,M}\hat{\Im}_i^{j+1}\right\|}{\eta}+\max\left\{\left|\hat{\Im}_0^{j+1}\right|,\left|\Im_N^{j+1}\right|\right\}.$$

Proof. We define barrier functions $\left(\Pi_i^{j+1}\right)^{\pm}$ as

$$\left(\Pi_i^{j+1}\right)^{\pm}=Z\pm\hat{\Im}_i^{j+1},$$

Where $Z = \dfrac{\left\| \pounds_\varepsilon^{N,M} \hat{\Im}_i^{j+1} \right\|}{\eta} + \max\left\{ \left|\hat{\Im}_0^{j+1}\right|, \left|\hat{\Im}_N^{j+1}\right| \right\}$.

On the boundary points, we obtain

$$\left(\Pi_0^{j+1}\right)^{\pm} = Z \pm \hat{\Im}_0^{j+1} = \frac{\left\| \pounds_\varepsilon^{N,M} \hat{\Im}_i^{j+1} \right\|}{\eta} + \max\left\{ \left|\hat{\Im}_0^{j+1}\right|, \left|\hat{\Im}_N^{j+1}\right| \right\} \pm \Theta^{j+1}(0) \geq 0,$$

$$\left(\Pi_N^{j+1}\right)^{\pm} = Z \pm \hat{\Im}_N^{j+1} = \frac{\left\| \pounds_\varepsilon^{N,M} \hat{\Im}_i^{j+1} \right\|}{\eta} + \max\left\{ \left|\hat{\Im}_0^{j+1}\right|, \left|\hat{\Im}_N^{j+1}\right| \right\} \pm \Theta^{j+1}(N) \geq 0.$$

Now, on the discretized spatial domain Υ_ℓ^N, we have

$$\begin{aligned}
\pounds_\varepsilon^{N,M}\left(\Pi_i^{j+1}\right)^{\pm} &= \pounds_\varepsilon^{N,M}\left(Z \pm \hat{\Im}_i^{j+1}\right)\\
&= \left(\frac{-\sigma(\rho)}{\rho} - \frac{3\omega_1 \varpi_{i-1}^{j+1}}{2} - \omega_2 \varpi_i^{j+1} + \frac{\omega_1 \varpi_{i+1}^{j+1}}{2} + \ell\omega_1 r_{i-1}^{j+1}\right)\left(Z \pm \hat{\Im}_{i-1}^{j+1}\right)\\
&+ \left(\frac{2\sigma(\rho)}{\rho} + 2\omega_1 \varpi_{i-1}^{j+1} - 2\omega_1 \varpi_{i+1}^{j+1} + 2\ell\omega_2 r_i^{j+1}\right)\left(Z \pm \hat{\Im}_i^{j+1}\right)\\
&+ \left(-\frac{-\sigma(\rho)}{\rho} - \frac{\omega_1 \varpi_{i-1}^{j+1}}{2} + \omega_2 \varpi_i^{j+1} + \frac{3\omega_1 \varpi_{i-1}^{j+1}}{2} + \ell\omega_1 r_{i+1}^{j+1}\right)\left(Z \pm \hat{\Im}_{i+1}^{j+1}\right),
\end{aligned}$$

$$\begin{aligned}
&= \pm\left(\frac{-\sigma(\rho)}{\rho} - \frac{3\omega_1 \varpi_{i-1}^{j+1}}{2} - \omega_2 \varpi_i^{j+1} + \frac{\omega_1 \varpi_{i+1}^{j+1}}{2} + \ell\omega_1 r_{i-1}^{j+1}\right)\Im_{i-1}^{j+1}\\
&\pm \left(\frac{2\sigma(\rho)}{\rho} + 2\omega_1 \varpi_{i-1}^{j+1} - 2\omega_1 \varpi_{i+1}^{j+1} + 2\ell\omega_2 r_i^{j+1}\right)\hat{\Im}_i^{j+1}\\
&\pm \left(-\frac{-\sigma(\rho)}{\rho} - \frac{\omega_1 \varpi_{i-1}^{j+1}}{2} + \omega_2 \varpi_i^{j+1} + \frac{3\omega_1 \varpi_{i-1}^{j+1}}{2} + \ell\omega_1 r_{i+1}^{j+1}\right)\Im_{i+1}^{j+1}\\
&+ \left(\ell\omega_1 r_{i-1}^{j+1} + 2\ell\omega_2 r_i^{j+1} + \ell\omega_1 r_{i+1}^{j+1}\right) Z \quad ,\\
&\pm \ell\left(\omega_1 \varphi_{i-1}^{j+1} + 2\omega_2 \varphi_i^{j+1} + \omega_1 \varphi_{i+1}^{j+1}\right) + \left(\ell\omega_1 r_{i-1}^{j+1} + 2\ell\omega_2 r_i^{j+1} + \ell\omega_1 r_{i+1}^{j+1}\right) Z,\\
&= \left(\ell\omega_1 r_{i-1}^{j+1} + 2\ell\omega_2 r_i^{j+1} + \ell\omega_1 r_{i+1}^{j+1}\right)\left(\frac{\left\| \pounds_\varepsilon^{N,M} \hat{\Im}_i^{j+1} \right\|}{\eta} + \max\left\{ \left|\hat{\Im}_0^{j+1}\right|, \left|\hat{\Im}_N^{j+1}\right| \right\}\right)\\
&\mp \ell\left(\omega_1 \varphi_{i-1}^{j+1} + 2\omega_2 \varphi_i^{j+1} + \omega_1 \varphi_{i+1}^{j+1}\right),\\
&\geq 0, \qquad \text{since } r^{j+1}(z_i) \geq \eta > 0.
\end{aligned}$$

By Lemma 4.1, we obtain $\left(\Pi_i^{j+1}\right)^{\pm} \geq 0, 0 \leq i \leq 1$. Hence, the required bound is obtained.

Lemma 4.3. The local truncation error in space semi-discretization of the discrete problem (28) is given as

$$\left\| \hat{\Im}^{j+1}(z_i) - \hat{\Im}_i^{j+1} \right\| \leq C\ell^2,$$

Where C is a positive constant independent of ε and ℓ.

Proof. From the truncation error of Equation (20), we have

$$\begin{cases}
e'_{i-1} = \dfrac{d\hat{\Im}^{j+1}(z_{i-1})}{dz} - \dfrac{d\hat{\Im}_{i-1}^{j+1}}{dz} = \dfrac{\ell^2}{3}\dfrac{d^3\hat{\Im}^{j+1}(z_i)}{dz^3} - \dfrac{\ell^3}{12}\dfrac{d^4\hat{\Im}^{j+1}(z_i)}{dz^4} + \dfrac{\ell^4}{30}\dfrac{d^5\hat{\Im}^{j+1}(\xi_i)}{dz^5},\\
e'_i = \dfrac{d\hat{\Im}^{j+1}(z_i)}{dz} - \dfrac{d\hat{\Im}_i^{j+1}}{dz} = -\dfrac{\ell^2}{6}\dfrac{d^3\hat{\Im}^{j+1}(z_i)}{dz^3} - \dfrac{\ell^4}{120}\dfrac{d^5\hat{\Im}^{j+1}(\xi_i)}{dz^5},\\
e'_{i+1} = \dfrac{d\hat{\Im}^{j+1}(z_{i+1})}{dz} - \dfrac{d\hat{\Im}_{i+1}^{j+1}}{dz} = \dfrac{\ell^2}{3}\dfrac{d^3\hat{\Im}^{j+1}(z_i)}{dz^3} + \dfrac{\ell^3}{12}\dfrac{d^4\hat{\Im}^{j+1}(z_i)}{dz^4} + \dfrac{\ell^4}{30}\dfrac{d^5\hat{\Im}^{j+1}(\xi_i)}{dz^5},
\end{cases} \tag{29}$$

Where $z_{i-1} < \xi < z_{i+1}$.
Substituting

$$\sigma\varepsilon\Xi_\beta = \varpi_\beta^{j+1}\frac{d\hat{\Im}_\beta^{j+1}}{dz} + r_\beta^{j+1}\Im_\beta^{j+1} - \varphi_\beta^{j+1}, \;\; \beta = i, i \pm 1$$

into Equation (18), we get

$$\begin{aligned}
\sigma\varepsilon&\left(\hat{\Im}_{i-1}^{j+1} - 2\hat{\Im}_i^{j+1} + \hat{\Im}_{i+1}^{j+1}\right)\\
&= \ell^2\omega_1\left(\varpi_{i-1}^{j+1}\frac{d\hat{\Im}_{i-1}^{j+1}}{dz} + r_{i-1}^{j+1}\hat{\Im}_{i-1}^{j+1} - \varphi_{i-1}^{j+1}\right)\\
&+ 2\ell^2\omega_2\left(\varpi_i^{j+1}\frac{d\hat{\Im}_i^{j+1}}{dx} + r_i^{j+1}\hat{\Im}_i^{j+1} - \varphi_i^{j+1}\right)\\
&+ \ell^2\omega_1\left(\varpi_{i+1}^{j+1}\frac{d\hat{\Im}_{i+1}^{j+1}}{dz} + r_{i+1}^{j+1}\hat{\Im}_{i+1}^{j+1} - \varphi_{i+1}^{j+1}\right).
\end{aligned} \tag{30}$$

Considering the corresponding exact solution to Equation (30), we have

$$\sigma\varepsilon\left(\hat{\Im}^{j+1}(z_{i-1})-2\hat{\Im}^{j+1}(z_i)+\hat{\Im}^{j+1}(z_{i+1})\right)$$
$$=\ell^2\omega_1\varpi^{j+1}(z_{i-1})\frac{d\hat{\Im}^{j+1}(z_{i-1})}{dz}+$$
$$\ell^2\omega_1\left(r^{j+1}(z_{i-1})\hat{\Im}^{j+1}(z_{i-1})-\varphi^{j+1}(z_{i-1})\right)$$
$$+2\ell^2\omega_2\left(\varpi^{j+1}(x_i)\frac{d\hat{\Im}^{j+1}(x_i)}{dz}+r^{j+1}(z_i)\hat{\Im}^{j+1}(z_i)\right)$$
$$-2\ell^2\omega_2 g^{j+1}(z_i)+\ell^2\omega_1\left(\varpi^{j+1}(\ell_{i+1})\frac{d\hat{\Im}^{j+1}(x_{i+1})}{dz}+\right.$$
$$\left.r^{j+1}(z_{i+1})\hat{\Im}^{j+1}(z_{i+1})-\varphi^{j+1}(z_{i+1})\right). \quad (31)$$

Subtracting Equation (30) from Equation (31) and denoting $e_\beta=\Im^{j+1}(z_\beta)-\hat{\Im}^{j+1}_\beta$, for $\beta=i,i\pm1$ we arrive at

$$\left(\sigma\varepsilon-\ell^2\omega_1 r^{j+1}_{i-1}\right)e_{i-1}+\left(-2\sigma\varepsilon-2\ell^2\omega_2 r^{j+1}_i\right)e_i$$
$$+\left(\sigma\varepsilon-\ell^2\omega_1 r^{j+1}_{i+1}\right)e_{i+1}$$
$$=\ell^2\left(\omega_1\varpi^{j+1}_{i-1}e'_{i-1}+2\omega_2\varpi^{j+1}_i e'_i+\omega_1\varpi^{j+1}_{i+1}e'_{i+1}\right). \quad (32)$$

Inserting Equation (29) in Equation (32), we obtain

$$\left(\sigma\varepsilon-\ell^2\omega_1 r^{j+1}_{i-1}\right)e_{i-1}+\left(-2\sigma\varepsilon-2\ell^2\omega_2 r^{j+1}_i\right)e_i$$
$$+\left(\sigma\varepsilon-\ell^2\omega_1 r^{j+1}_{i+1}\right)e_{i+1},$$
$$=\frac{\ell^4}{3}\left(\omega_1\varpi^{j+1}_{i-1}-\omega_2\varpi^{j+1}_i+\omega_1\varpi^{j+1}_{i+1}\right)\frac{d^3\Im^{j+1}(z_i)}{dz^3}$$
$$+\frac{\ell^5}{12}\left(-\omega_1\varpi^{j+1}_{i-1}+\omega_1\varpi^{j+1}_{i+1}\right)\frac{d^4\Im^{j+1}(z_i)}{dz^4}$$
$$+\frac{\ell^6}{60}\left(2\omega_1\varpi^{j+1}_{i-1}-\omega_2\varpi^{j+1}_i+2\omega_1\varpi^{j+1}_{i+1}\right)\frac{d^5\Im^{j+1}(\xi_i)}{dz^5}. \quad (33)$$

Using the expressions $\varpi_{i-1}=\varpi_i-\ell\varpi'_i+\frac{\ell^2}{2!}\varpi^{(2)}(\zeta_i)$ and $\varpi_{i+1}=\varpi_i+\ell\varpi'_i+\frac{\ell^2}{2!}\varpi^{(2)}(\zeta_i)$ in Equation (33), we have

$$\left(\sigma\varepsilon-\ell^2\omega_1 r^{j+1}_{i-1}\right)e_{i-1}$$
$$+\left(-2\sigma\varepsilon-2\ell^2\omega_2 r^{j+1}_i\right)e_i+\left(\sigma\varepsilon-\ell^2\omega_1 r^{j+1}_{i+1}\right)e_{i+1}=E_i(\ell), \quad (34)$$

Where $E_i(\ell)=\frac{\ell^4}{3}(2\omega_1-\omega_2)\varpi^{j+1}_i\frac{d^3\Im^{j+1}(z_i)}{dz^3}+O(\ell^6)$. Hence, for the choice of $\omega_1+\omega_2=1/2$, we obtain $E_i(\ell)=O(\ell^4)$.
The matrix representation of Equation (34) is

$$(\Gamma-\Lambda)\top=\hat{E}, \quad (35)$$

Where $\Gamma=trid(-\sigma\varepsilon,2\sigma\varepsilon,-\sigma\varepsilon)$, $\Lambda=trid\left(\ell^2\omega_1 r^{j+1}_{i-1},2\ell^2\omega_1 r^{j+1}_i,\ell^2\omega_1 r^{j+1}_{i+1}\right)$, $\top=[e_1,e_2,\cdots,e_{N-1}]^t$, and $\hat{E}=[-E_1(\ell),-E_2(\ell),\cdots,-E_{N-1}(\ell)]^t$.

Following Venkata et al. [30], it can be shown that

$$\|\top\|\le\frac{C}{\ell^2}\times O\left(\ell^4\right)=C\left(\ell^2\right), \quad (36)$$

Where C is a constant, independent of ℓ and ε.

Theorem 4.4. Let $\Im(z,t)$ and $\hat{\Im}^{j+1}_i$ be the solution of Equations (5) and (28) at each grid point (z_i,t_{j+1}), respectively. Then, the following uniform error bound holds

$$\max_{i,j}\left|\Im(z_i,t_{j+1})-\hat{\Im}^{j+1}_i\right|\le C\left(\tau+\ell^2\right).$$

Proof. By combining the result of Lemmas 3.2 and 4.3, the required bound is obtained.

4.1. Temporal Richardson extrapolation

In this section, we use the Richardson extrapolation technique to improve the accuracy and order of convergence of the discrete method (28) in the time direction. For that, we consider the tensor product meshes $\overline{\Upsilon}^{N,\tau}$ and $\overline{\Upsilon}^{N,\tau/2}$, where both the meshes $\overline{\Upsilon}^{N,\tau}$ and $\overline{\Upsilon}^{N,\tau/2}$ are uniform and identical in spatial direction and uniform in time with step sizes τ and $\tau/2$, respectively. Let $\overline{\Upsilon}_0^{N,\tau}=\overline{\Upsilon}^{N,\tau}\cap\overline{\Upsilon}^{N,\tau/2}$. It is clear that $\overline{\Upsilon}_0^{N,\tau}=\overline{\Upsilon}^{N,\tau}$. Furthermore, let $\Im^1(z_i,t_{j+1})$, for all $(z_i,t_{j+1})\in\overline{\Upsilon}^{N,\tau}$ and $\Im^2(z_i,t_{j+1})$, for all $(z_i,t_{j+1})\in\overline{\Upsilon}^{N,\tau/2}$ be the solutions of the discrete scheme (28). Then we define

$$\Im_{exp}(z_i,t_{j+1})=2\Im^2(z_i,t_{j+1})-\Im^1(z_i,t_{j+1}),(z_i,t_{j+1})\in\overline{\Upsilon}_0^{N,\tau}, \quad (37)$$

Where $\Im_{exp}$ is the Richardson extrapolation of $\Im$, which has an enhanced order of convergence by one in time. This technique is known as Richardson extrapolation technique.

Theorem 4.5 (Error After Extrapolation). Assume that $N\ge N_0$ satisfies the assumptions (28). Let $\Im$ be the solution of Equation (5) and $\Im_{exp}$ be the extrapolated solution of Equation (28) on the two nested meshes $\overline{\Upsilon}^{N,\tau}$ and $\overline{\Upsilon}^{2N,\tau/2}$. Then, the new error bound takes the form

$$\left\|(\Im-\Im_{exp})(z_i,t_{j+1})\right\|\le C\left((\tau)^2+\ell^2\right).$$

Proof. We consider the expansion of $\Im^n(z_i,t_{j+1}),(z_i,t_{j+1})\in\overline{\Upsilon}^{N,\tau/n}$ for $n=1,2$ as

$$\Im^n(z_i,t_{j+1})=\Im(z_i,t_{j+1})-2^{-(n-1)}\tau\xi^n(z_i,t_{j+1})+\Gamma^n(z_i,t_{j+1}), \quad (38)$$

TABLE 1 Comparison of $E_\varepsilon^{N,\tau}$, $E^{N,\tau}$, $r^{N,\tau}$, $(E_\varepsilon^{N,\tau})extp$, $E_{extp}^{N,\tau}$, and $r_{extp}^{N,\tau}$ for Example 5.1 with results in Liu et al. [22].

ε	$N = 2^5$	2^6	2^7	2^8	2^9	2^{10}
↓	M = 20	40	80	160	320	640
Before richardson extrapolation						
2^{-6}	2.8408e-03	1.4682e-03	7.4364e-04	3.7377e-04	1.8731e-04	9.3761e-05
2^{-8}	4.0156e-03	2.0020e-03	9.8557e-04	4.8685e-04	2.4170e-04	1.2040e-04
2^{-10}	4.8909e-03	2.4214e-03	1.1410e-03	5.4081e-04	2.6150e-04	1.2838e-04
2^{-12}	5.3650e-03	2.8201e-03	1.3852e-03	6.4340e-04	2.9416e-04	1.3782e-04
2^{-14}	5.4742e-03	2.9441e-03	1.5118e-03	7.4879e-04	3.5703e-04	1.6336e-04
2^{-16}	5.4827e-03	2.9641e-03	1.5399e-03	7.8120e-04	3.8923e-04	1.9002e-04
2^{-18}	5.4827e-03	2.9644e-03	1.5416e-03	7.8591e-04	3.9638e-04	1.9820e-04
$E^{N,\tau}$	5.4827e-03	2.9644e-03	1.5416e-03	7.8591e-04	3.9638e-04	1.9820e-04
$r^{N,\tau}$	0.8871	0.9433	0.9720	0.9875	0.9999	
After richardson extrapolation						
2^{-6}	1.1757e-04	3.3841e-05	9.0748e-06	2.3391e-06	6.0263e-07	1.4996e-07
2^{-8}	1.6763e-04	5.5737e-05	1.5609e-05	4.1335e-06	1.0628e-06	2.8261e-07
2^{-10}	2.5551e-04	6.9513e-05	1.8120e-05	4.4961e-06	1.1200e-06	2.8300e-07
2^{-12}	2.7297e-04	7.2989e-05	1.8782e-05	4.5990e-06	1.1281e-06	2.8341e-07
2^{-14}	2.7342e-04	7.3024e-05	1.8789e-05	4.6006e-06	1.1284e-06	2.8346e-07
2^{-16}	2.7377e-04	7.3049e-05	1.8794e-05	4.6016e-06	1.1287e-06	2.8350e-07
2^{-18}	2.7377e-04	7.3049e-05	1.8794e-05	4.6016e-06	1.1287e-06	2.8350e-07
$E_{extp}^{N,\tau}$	2.7377e-04	7.3049e-05	1.8794e-05	4.6016e-06	1.1287e-06	2.8350e-07
$r_{extp}^{N,\tau}$	1.9060	1.9586	2.0301	2.0275	1.9932	
Results in Liu et al. [22]						
2^{-6}	0.025289	0.017672	0.0090066	0.0048378	0.0025035	0.0012731
2^{-8}	0.038607	0.019497	0.011221	0.0062852	0.0033405	0.0017233
2^{-10}	0.093183	0.070120	0.044773	0.020546	0.010545	0.0051608
2^{-12}	0.17017	0.10083	0.062216	0.039526	0.020493	0.010027
2^{-14}	0.20410	0.16703	0.092110	0.052580	0.028766	0.016523
2^{-16}	0.20450	0.15975	0.12612	0.06.9772	0.038531	0.020526
2^{-18}	0.25614	0.21031	0.13406	0.085618	0.048834	0.026219
$E^{N,\tau}$	0.25614	0.21031	0.13406	0.085618	0.048834	0.026219
$r^{N,\tau}$	0.2844	0.6496	0.6469	0.8100	0.8973	

The symbol ↓ shows values of ε in the column.

Where $\Im^n$ is the approximate solution and Γ^n, for $n = 1, 2$, is the remainder term. The function ξ is the solution of the problem

$$\pounds_\varepsilon \xi(z_i, t_{j+1}) = -2^{-1} \frac{\partial^2}{\partial t^2} \Im(z_i, t_{j+1}), (z, t) \in \Upsilon \quad (39)$$
$$\xi(z, t) = 0, (z, t) \in \partial\Upsilon.$$

To show the convergence of the Richardson technique, we need to formulate the estimate for the remainder term Γ^n on $\overline{\Upsilon}^{N,\tau/n}$ for $n = 1, 2$. It is clear that $\Gamma^n(z_i, t_{j+1}) = 0$ for all $(z_i, t_{j+1}) \in \partial\Upsilon^{N,\tau/n}$, where $\partial\Upsilon^{N,\tau/n}$ denotes the boundary of $\overline{\Upsilon}^{N,\tau/n}$. In addition, for all $(z_i, t_{j+1}) \in \Upsilon^{N,\tau/n}$, where $n = 1, 2$, we have

$$\begin{aligned} |\pounds^{N,\tau/k}\Gamma^n(z_i, t_{j+1})| =& |\pounds^{N,\tau/n}\left(\Im^n - \Im\right)(z_i, t_{j+1}) \\ & -2^{-(n-1)}\tau\pounds^{N,\tau/n}\xi(z_i, t_{j+1})| \\ \leq & \, C\left((\tau)^2 + \ell^2\right). \end{aligned}$$

Then applying the discrete maximum principle, we get $|\Gamma^n\left(z_i, t_{j+1}\right)| \leq C\left((\tau)^2 + \ell^2\right)$.

TABLE 2 Comparison of $E_{\varepsilon}^{N,\tau}$, $E^{N,\tau}$, $r^{N,\tau}$, $(E_{\varepsilon}^{N,\tau})extp$, $E_{extp}^{N,\tau}$, and $r_{extp}^{N,\tau}$ for Example 5.2 with results in Liu et al. [22].

ε	$N = 2^5$	2^6	2^7	2^8	2^9	2^{10}
$\downarrow$	$M = 20$	40	80	160	320	640
Before richardson extrapolation						
2^{-6}	2.2225e-03	1.1579e-03	5.8916e-04	2.9661e-04	1.4877e-04	7.4503e-05
2^{-8}	3.4104e-03	1.7015e-03	8.3475e-04	4.1158e-04	2.0411e-04	1.0162e-04
2^{-10}	4.4223e-03	2.1536e-03	1.0013e-03	4.7090e-04	2.2677e-04	1.1112e-04
2^{-12}	5.0534e-03	2.6284e-03	1.2705e-03	5.7857e-04	2.6065e-04	1.2122e-04
2^{-14}	5.2401e-03	2.8042e-03	1.4305e-03	7.0176e-04	3.2939e-04	1.4772e-04
2^{-16}	5.2884e-03	2.8530e-03	1.4768e-03	7.4538e-04	3.6902e-04	1.7849e-04
2^{-18}	5.3005e-03	2.8654e-03	1.4887e-03	7.5704e-04	3.8034e-04	1.8926e-04
$E^{N,\tau}$	5.3005e-03	2.8654e-03	1.4887e-03	7.5704e-04	3.8034e-04	1.8926e-04
$r^{N,\tau}$	0.8874	0.9447	0.9756	0.9931	1.0069	
After richardson extrapolation						
2^{-6}	9.4527e-05	2.7189e-05	8.0891e-06	2.3427e-06	6.1922e-07	1.5941e-07
2^{-8}	1.4676e-04	4.7702e-05	1.3418e-05	3.5488e-06	9.9968e-07	2.8086e-07
2^{-10}	2.7265e-04	7.1303e-05	1.7154e-05	4.2520e-06	1.0546e-06	2.6286e-07
2^{-12}	2.7366e-04	7.2313e-05	1.8164e-05	4.3530e-06	1.0975e-06	2.7226e-07
2^{-14}	2.7381e-04	7.2463e-05	1.8371e-05	4.4612e-06	1.1010e-06	2.7378e-07
2^{-16}	2.7384e-04	7.2530e-05	1.8382e-05	4.4819e-06	1.1109e-06	2.7602e-07
2^{-18}	2.7388e-04	7.2533e-05	1.8384e-05	4.4820e-06	1.1110e-06	2.7603e-07
$E_{extp}^{N,\tau}$	2.7388e-04	7.2533e-05	1.8384e-05	4.4820e-06	1.1110e-06	2.7603e-07
$r_{extp}^{N,\tau}$	1.9168	1.9802	2.0362	2.0123	2.0090	
Results in Liu et al. [22]						
2^{-6}	0.038767	0.018983	0.0096122	0.0050867	0.0026211	0.0013306
2^{-8}	0.044450	0.020109	0.010519	0.059653	0.0031896	0.0016508
2^{-10}	0.083339	0.066120	0.040769	0.019284	0.0091775	0.0049998
2^{-12}	0.18762	0.084106	0.057234	0.03309	0.019041	0.0092260
2^{-14}	0.19755	0.15347	0.083864	0.046945	0.025508	0.014930
2^{-16}	0.21340	0.15016	0.11582	0.061958	0.033441	0.017672
2^{-18}	0.28299	0.17036	0.11805	0.074251	0.041879	0.022413
$E^{N,\tau}$	0.28299	0.17036	0.11805	0.074251	0.041879	0.022413
$r^{N,\tau}$	0.7322	0.5292	0.6689	0.8262	0.9019	

The symbol ↓ shows values of ε in the column.

Then this estimate for Γ^n, together with the Equations (37) and (38), yields

$$\begin{aligned}&\left(\Im-\Im_{\exp}\right)\left(z_i,t_{j+1}\right)=\Im\left(z_i,t_{j+1}\right)\\&-\left(2\Im^2(z_i,t_{j+1})-\Im^1(z_i,t_{j+1}),(z_i,t_{j+1})\right)=O\left((\tau)^2+\ell^2\right).\end{aligned} \quad (40)$$

Equation (40) gives the following error bound for the Richardson extrapolate scheme:

$$\left\|(\Im-\Im_{exp})(z_i,t_{j+1})\right\|\leq C\left((\tau)^2+\ell^2\right).$$

TABLE 3 Comparison of $E_\varepsilon^{N,\tau}$, $E^{N,\tau}$, $r^{N,\tau}$, $(E_\varepsilon^{N,\tau})extp$, $E_{extp}^{N,\tau}$, and $r_{extp}^{N,\tau}$ for Example 5.3 with results in Kaushik and Sharma [20] for $\tau = 0.001$.

$\varepsilon \downarrow N \rightarrow$	16	32	64	128	256
Before richardson extrapolation					
2^{-6}	1.1441e-02	3.3769e-03	1.0928e-03	5.2467e-04	2.5971e-04
2^{-8}	2.4264e-02	1.0286e-02	3.5936e-03	1.2628e-03	5.5544e-04
2^{-10}	2.9305e-02	1.4883e-02	7.0408e-03	3.0236e-03	1.2559e-03
2^{-12}	3.0111e-02	1.6059e-02	8.3119e-03	4.0215e-03	2.1264e-03
2^{-14}	3.0130e-02	1.6165e-02	8.5381e-03	4.3332e-03	1.8652e-03
2^{-16}	3.0130e-02	1.6165e-02	8.5447e-03	4.3660e-03	2.2016e-03
2^{-18}	3.0130e-02	1.6165e-02	8.5447e-03	4.3661e-03	2.2042e-03
2^{-20}	3.0130e-02	1.6165e-02	8.5447e-03	4.3661e-03	2.2042e-03
$E^{N,\tau}$	3.0130e-02	1.6165e-02	8.5447e-03	4.3661e-03	2.2042e-03
$r^{N,\tau}$	0.89833	0.91977	0.96868	0.98609	
After richardson extrapolation					
2^{-6}	1.1118e-03	4.0166e-04	1.3159e-04	4.1281e-05	1.0687e-05
2^{-8}	1.2086e-03	4.4528e-04	1.6502e-04	4.9481e-05	1.2714e-05
2^{-10}	1.3224e-03	5.3328e-04	1.9675e-04	5.8906e-05	1.5096e-05
2^{-12}	1.4246e-03	5.4330e-04	1.9900e-04	5.9563e-05	1.5227e-05
2^{-14}	1.4275e-03	5.4350e-04	1.9907e-04	5.9565e-05	1.5228e-05
2^{-16}	1.4280e-03	5.4357e-04	1.9909e-04	5.9571e-05	1.5241e-05
2^{-18}	1.4280e-03	5.4357e-04	1.9909e-04	5.9571e-05	1.5241e-05
2^{-20}	1.4280e-03	5.4357e-04	1.9909e-04	5.9571e-05	1.5241e-05
$E_{exp}^{N,\tau}$	1.4280e-03	5.4357e-04	1.9909e-04	5.9571e-05	1.5241e-05
$r_{exp}^{N,\tau}$	1.3935	1.4490	1.7407	1.9667	
Results in Kaushik and Sharma [20]					
2^{-6}	3.9974e-02	2.1987e-02	1.1611e-02	5.969e-03	3.028e-03
2^{-8}	4.0840e-02	2.2530e-02	1.1904e-02	6.126e-03	3.108e-03
2^{-10}	4.1057e02	2.2675e-02	1.1982e-02	6.166 e-03	3.129e-03
2^{-12}	4.1111e-02	2.2711e-02	1.2002e-02	6.176e-3	3.134e-03
2^{-14}	4.1111e-02	2.2711e-02	1.2002e-02	6.176e-03	3.134e-03
2^{-16}	4.1128e-02	2.2722e-02	1.2008e-02	6.179e-03	3.136e-03
2^{-18}	4.1129e-2	2.2723e-2	1.2008e-2	6.179e-3	3.136e-3
2^{-20}	4.1129e-2	2.2723e-2	1.2008e-2	6.179e-3	3.136e-3
$E^{N,\tau}$	4.1129e-2	2.2723e-2	1.2008e-2	6.179e-3	3.136e-3
$r^{N,\tau}$	0.8560	0.9202	0.9586	0.9784	

The symbol $\downarrow$ shows values of ε in the column and $\rightarrow$ shows number nodal points in space direction.

5. Numerical examples, results, and discussion

Four test examples are presented to illustrate the efficiency of the proposed method. The exact solutions for these problems are unknown for comparison. Therefore, we use the double mesh principle to estimate the error as given in Gupta et al. [31].

$$E_\varepsilon^{N,\tau} = \max_{(z_i,t_{j+1})\in\Upsilon^{N,M}} \left| \left(\Im^{N,\tau}(z_i,t_{j+1}) - \Im^{2N,\tau/2}(z_i,t_{j+1})\right)\right|,$$

(Before extrapolation),

$$(E_\varepsilon^{N,\tau})extp = \max_{(z_i,t_{j+1})\in\Upsilon^{N,M}} \left| \left(\Im_{extp}^{N,\tau}(z_i,t_{j+1}) - \Im_{extp}^{2N,\tau/2}(z_i,t_{j+1})\right)\right|,$$

(After extrapolation),

TABLE 4 Comparison of $E_\varepsilon^{N,\tau}$, $E^{N,\tau}$, $r^{N,\tau}$, $(E_\varepsilon^{N,\tau})extp$, $E_{extp}^{N,\tau}$, and $r_{extp}^{N,\tau}$ for Example 5.4 with results in Derzie et al. [29].

ε	N = 64	128	256	512
↓	M = 20	40	80	160
Before richardson extrapolation				
10^0	1.0681e-02	5.6600e-03	2.9086e-03	1.4738e-03
10^{-2}	1.2760e-02	6.7431e-03	3.4351e-03	1.7144e-03
10^{-4}	1.2850e-02	6.8269e-03	3.5117e-03	1.7800e-03
10^{-6}	1.2850e-02	6.8276e-03	3.5125e-03	1.7808e-03
$E^{N,\tau}$	1.2850e-02	6.8276e-03	3.5125e-03	1.7808e-03
$r^{N,\tau}$	0.91232	0.95888	0.97997	
After richardson extrapolation				
10^0	6.5119e-04	1.6178e-04	4.0204e-05	1.0013e-05
10^{-2}	8.1736e-04	1.6222e-04	5.3063e-05	1.3552e-05
10^{-4}	8.9593e-04	2.2944e-04	5.7591e-05	1.4263e-05
10^{-6}	8.9673e-04	2.3022e-04	5.8203e-05	1.4621e-05
$E_{extp}^{N,\tau}$	8.9673e-04	2.3022e-04	5.8203e-05	1.4621e-05
$r_{extp}^{N,\tau}$	1.9617	1.9838	1.9931	
Results in Derzie et al. [29]				
10^0	0.01070	0.002336	0.0005212	0.0001290
10^{-2}	0.02550	0.005796	0.001409	0.0003555
10^{-4}	0.09929	0.07841	0.06262	0.04828
10^{-6}	0.09929	0.07841	0.06260	0.04979
$E^{N,\tau}$	0.09929	0.07841	0.06260	0.04979
$r^{N,\tau}$	0.34061	0.32488	0.33031	

The symbol ↓ shows values of ε in the column.

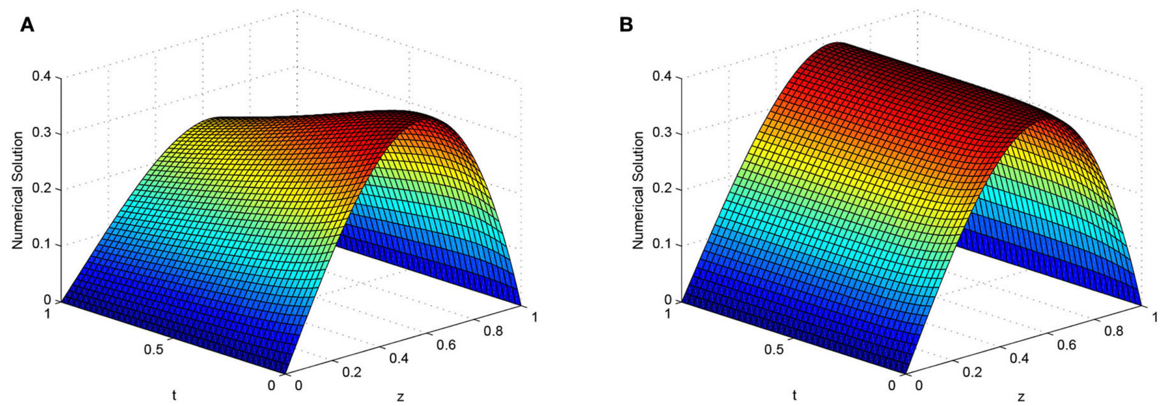

FIGURE 1
Numerical solution for $N = 64$, $M = 40$, and $\varepsilon = 2^{-18}$ **(A)** Example 5.1 and **(B)** Example 5.2.

Where $\Im(z_i, t_{j+1})$ and $\Im_{extp}(z_i, t_{j+1})$ denote the numerical solution obtained before and after extrapolation, respectively, in $\Upsilon^{N,M}$ with N mesh intervals in the spatial direction and M mesh intervals in the temporal direction. The rate of convergence is calculated as

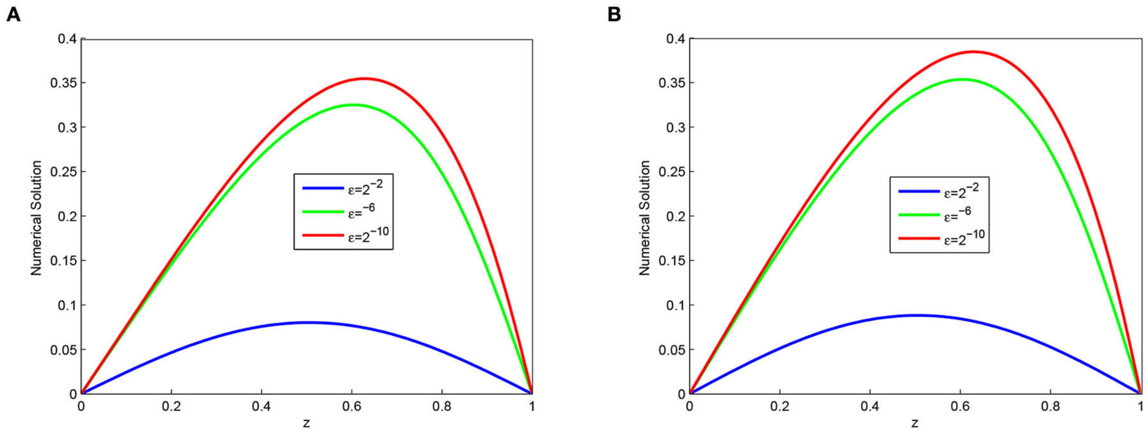

FIGURE 2
Effect of the ε on behavior of the solution with layer formation **(A)** Example 5.1 and **(B)** Example 5.2.

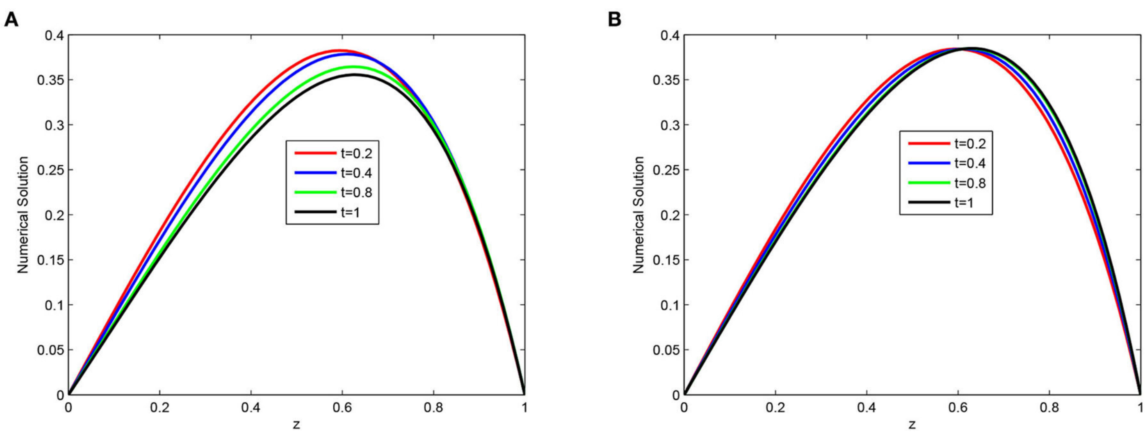

FIGURE 3
Numerical solution for $N = 64, M = 40$, and $\varepsilon = 2^{-16}$ and at different time levels **(A)** Example 5.1 and **(B)** Example 5.2.

$$r_{\varepsilon}^{N,\tau} = \log_2\left(E_{\varepsilon}^{N,\tau}/E_{\varepsilon}^{2N,\tau/2}\right), \text{(Before extrapolation)},$$
$$(r_{\varepsilon}^{N,\tau})extp = \log_2\left((E_{\varepsilon}^{N,\tau})extp/(E_{\varepsilon}^{2N,\tau/2})extp\right),$$
$$\text{(After extrapolation)}.$$

For each N and M, the parameter uniform maximum absolute error and uniform rate of convergence are computed using

$$E^{N,\tau} = \max_{\varepsilon} E_{\varepsilon}^{N,\tau}, \text{(Before extrapolation)},$$
$$E_{extp}^{N,\tau} = \max_{\varepsilon} E_{\varepsilon extp}^{N,\tau}, \text{(After extrapolation)},$$
$$r^{N,\tau} = \log_2\left(E^{N,\tau t}/E^{2N,\tau/2}\right), \text{(Before extrapolation)},$$
$$r_{extp}^{N,\tau} = \log_2\left(E_{extp}^{N,\tau}/E_{extp}^{2N,\tau/2}\right), \text{(After extrapolation)}.$$

Example 5.1. Consider the following SPBHE:

$$\begin{cases} \dfrac{\partial \Im}{\partial t} - \varepsilon \dfrac{\partial^2 \Im}{\partial z^2} + \Im \dfrac{\partial \Im}{\partial z} - (1 - \Im)(\Im - 0.5) = 0, (z,t) \in \Upsilon, \\ \Im(z,0) = z(1 - z^2), z \in \chi_z, \\ \Im(0,t) = 0 = \Im(1,t), t \in \chi_t. \end{cases}$$

Example 5.2. Consider the following SPBHE:

$$\begin{cases} \dfrac{\partial \Im}{\partial t} - \varepsilon \dfrac{\partial^2 \Im}{\partial z^2} + \Im \dfrac{\partial \Im}{\partial z} = 0, (z,t) \in \Upsilon, \\ \Im(z,0) = z(1 - z^2), z \in \chi_z, \\ \Im(0,t) = 0 = \Im(1,t), t \in \chi_t. \end{cases}$$

Example 5.3. Consider the following SPBHE:

$$\begin{cases} \dfrac{\partial \Im}{\partial t} - \varepsilon \dfrac{\partial^2 \Im}{\partial z^2} + \Im \dfrac{\partial \Im}{\partial z} = (1 - \Im)(\Im - 0.5), (z,t) \in \Upsilon, \\ \Im(z,0) = \sin(\pi z), z \in \chi_z, \\ \Im(0,t) = 0 = \Im(1,t), t \in \chi_t. \end{cases}$$

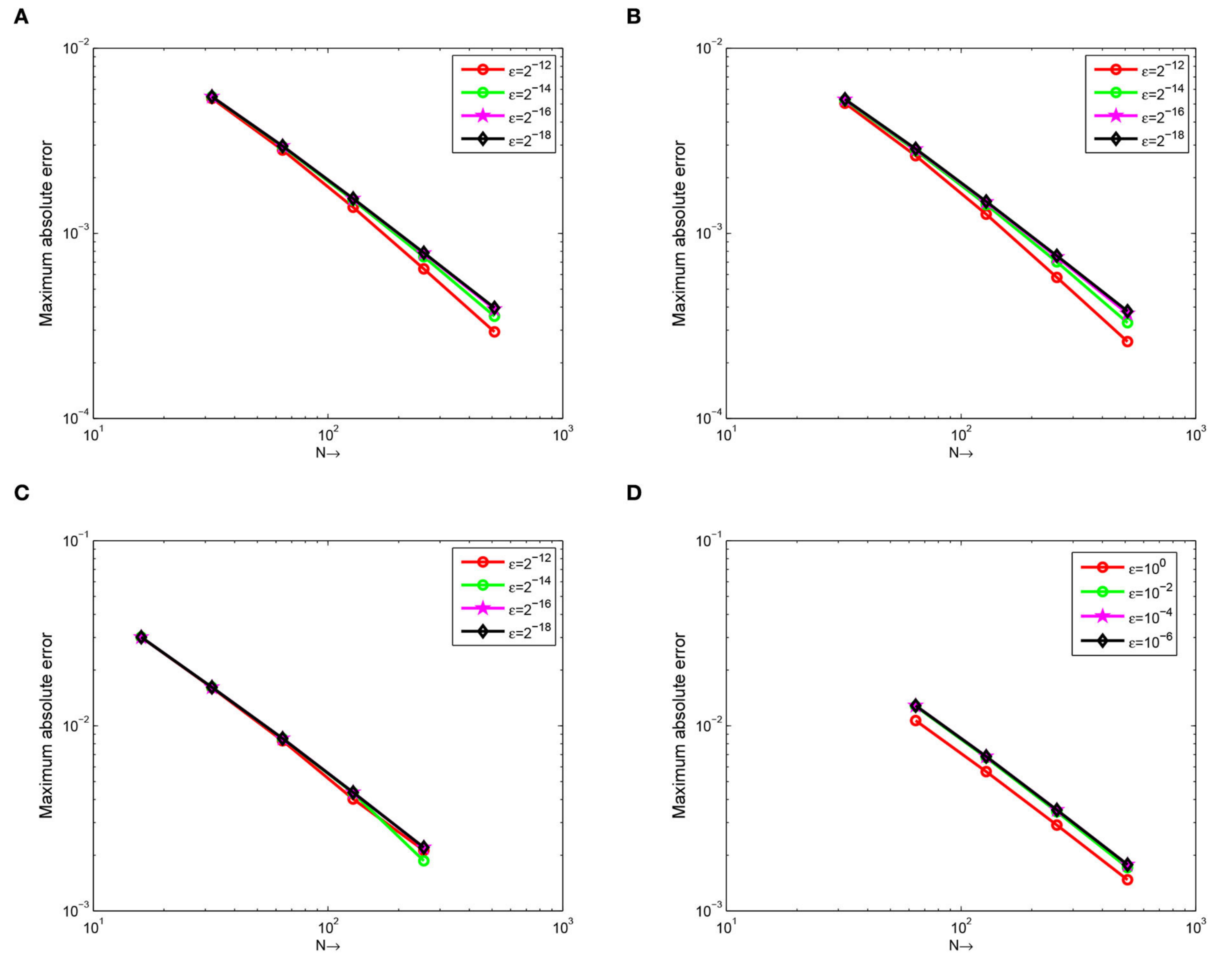

FIGURE 4
Example 5.1 on **(A)**, Example 5.2 on **(B)**, Example 5.3 on **(C)**, and Example 5.4 on **(D)**, the error plot for different values of ε.

Example 5.4. Consider the following SPBHE:

$$\begin{cases} \frac{\partial \Im}{\partial t} - \varepsilon \frac{\partial^2 \Im}{\partial z^2} + \Im \frac{\partial \Im}{\partial z} = 0, (z,t) \in \Upsilon, \\ \Im(z,0) = \sin(\pi z), z \in \chi_z, \\ \Im(0,t) = 0 = \Im(1,t) = 0, t \in \chi_t. \end{cases}$$

The computed $E_\varepsilon^{N,\tau}, E^{N,\tau}$, $r^{N,\tau}$, $(E_\varepsilon^{N,\tau})extp$, $E_{extp}^{N,\tau}$, and $r_{extp}^{N,\tau}$ of the considered problems using the proposed scheme are depicted in Tables 1–4. From these tables, one can observe that the developed scheme converges independent of the perturbation parameter with the order of convergence one before Richardson extrapolation and with the order of convergence two after Richardson extrapolation. In addition, from these Tables, one can observe the comparison of the scheme with the results in the articles [20, 22, 29]. The comparison confirms that the scheme provides a more accurate result than some schemes in Derzie et al. [29], Kaushik and Sharma [20], and Liu et al. [22]. In Figure 1, the numerical solution profile of Examples 5.1 and 5.2 when $N = 2^6, M = 40$, and $\varepsilon = 2^{-18}$ are given, which shows that a strong boundary layer is created near $z = 1$. The effect of ε on the solution profile is shown in Figure 2 and the figure shows as $\varepsilon \to 0$ strong boundary layer is created near $z = 1$. The effect of the time step on the solution profile is depicted in Figure 3. This figure depicts the existence of the boundary layer at $z = 1$ with time variable $t \to 1$. Figure 4 shows the uniform convergence of the proposed scheme in the error plot for Examples 5.1–5.4.

6. Conclusion

A higher order parameter uniform numerical scheme for singularly perturbed unsteady Burger-Huxley equation is presented. The developed scheme constitutes the implicit Euler in the time variable and the exponential spline method in the space variable. In addition, Richardson's extrapolation technique is implemented to improve the accuracy in the time direction. The stability and convergence analysis of the scheme are discussed and proved. It is found that the scheme proposed is uniformly convergent with linear and second order convergent before and after Richardson extrapolation. Four test examples are presented to illustrate the efficiency of the proposed scheme. It is found that the proposed scheme gives better results than the methods described in Derzie et al. [29], Kaushik and Sharma [20], and Liu et al. [22].

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

ITD and GFD: conceptualization, investigation and formal analysis, and visualization. ITD: software programming and writing—original draft. GFD: writing—review and editing. All authors contributed to the article and approved the submitted version.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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