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A New Tobit Ridge-Type Estimator of the Censored Regression Model With Multicollinearity Problem

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In the censored regression model, the Tobit maximum likelihood estimator is unstable and inefficient in the occurrence of the multicollinearity problem. To reduce this problem's effects, the Tobit ridge and the Tobit Liu estimators are proposed. Therefore, this study proposes a new kind of the Tobit estimation called the Tobit new ridge-type (TNRT) estimator. Also, the TNRT estimator was theoretically compared with the Tobit maximum likelihood, the Tobit ridge, and the Tobit Liu estimators *via* the mean squared error criterion. Moreover, we performed a Monte Carlo simulation to study the performance of the TNRT estimator compared with the previously defined estimators. Also, we used the Mroz dataset to confirm the theoretical and the simulation study results.

Keywords: censored regression model, multicollinearity, Tobit Liu estimator, Tobit ridge estimator, Tobit new ridge-type estimator

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INTRODUCTION

The limited dependent variables (LDVs) in the regression models are defined as the censored, the discrete, and the truncated outcomes. Tobin [1] introduced the Tobit model of the censored dependent variable, which is related to the LDVs, and Goldberger [2] gave its current name. The censored data appear when the dependent variable has a loss of information, while the truncated data appear when the dependent and the independent variables have a loss of information. In this study, we used the standard Tobit regression model, which is the Type 1 model of the Tobit models (Type 1–5) categorized by Amemiya [3] to deal with the censored dataset and their estimation. The censored normal regression model, which is called the Tobit model, is used to relieve the deficiency of biasedness and inconsistency of the results of using the least squares estimator (LSE). Therefore, to determine the estimates of the parameter and to find the estimates of statistical inference, the Tobit maximum likelihood estimator (TMLE) is used. When the explanatory (independent) variables are not independent, it becomes a problem called multicollinearity, which this problem often ignored in the censored regression models. Also, the multicollinearity makes the Tobit maximum likelihood estimates of the regression coefficients incorrect, unreliable, and unstable; because the mean squared error (MSE) values of these estimates are inflated. For this case, Khalaf et al. [4] examined the multicollinearity effects on the TMLE, and they introduced the Tobit ridge estimator (TRE). Then, Alhusseini and Odah [5] introduced a Tobit principal component estimator. Also, Toker et al. [6] introduced a Tobit Liu estimator (TLE).

In the linear regression model (LRM), several alternative estimators of the regression coefficients have been produced for the LSE when the multicollinearity problem happens because, in this case, the LSE gives large variances, wrong signs, and becomes unstable. The most popular estimators are the ridge estimator of Hoerl and Kennard [7] and the Liu estimator of Liu [8]. Recently, Kibria and Lukman [9] proposed a new ridge-type estimator (NRTE). The NRTE has been extended in different regression models in different studies, such as Lukman et al. [10], Lukman et al. [11], Akram et al. [12], Dawoud and Abonazel [13], Awwad et al. [14], and Abonazel et al. [15]. The multicollinearity is known to be a terrible problem in the Tobit model like in the LRM. For handling multicollinearity, some studies gave and investigated some biased estimators in the LRM for a long time, but there is little investigation of these estimators in the Tobit model. However, studies of the biased estimators instead of TMLE in deleting multicollinearity effects on regression coefficients in the Tobit model are needed. In this context, the TRE was introduced by Khalaf et al. [4] and the TLE by Toker et al. [6] were the biased estimation beginning points in the Tobit model. Then, we defined the Tobit NRTE (TNRTE) in this study. Also, we focus on the theoretical properties of the TNRTE by the MSE criterion and to compare them to the TMLE, the TRE, and the TLE.

The next content of this study is given as follows: Methodology Section defines the Tobit regression model and provides the TNRTE and the theoretical properties. A Monte Carlo Simulation Section deals with the Monte Carlo simulation study. A Real Life Data Section deals with the Mroz dataset. Conclusion Section includes the concluding remarks.

METHODOLOGY

Tobit Regression Model

The model of the Tobit regression is

$$y_i^* = x_i\beta + u_i;\ i = 1, 2, ..., n, \tag{1}$$

where y_i^* is called the dependent latent variable, x_i is an i-th row of the known matrix X with the dimension $n \times (p+1)$; where p is the number of the explanatory variables. β is the unknown $(p+1) \times 1$ coefficient vector (when the model contains the intercept β_0), and u_i is called an error term that is independent, follows a normal distribution by mean, and equals 0 and variance equals σ^2. We considered the left censoring, where y_i is defined as follows:

$$y_i = \begin{cases} y_i^*, & if\ y_i^* > 0 \\ 0, & otherwise \end{cases} \tag{2}$$

On the basis of n observations on y_i and x_i, the β and σ^2 estimation issues are noted. For the defined model in Equation (1), assuming that n_a is the observation number for $y_i = 0$ and is the observation number for $y_i > 0$, that is, non-zero for y_i occur first, then the log-likelihood function of the censored data is given as

$$\log S(X;\ \beta, \sigma^2) = \sum_{n_a} \log(1 - G_i) + \sum_{n_b} \log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \sum_{n_b} \frac{1}{2\sigma^2}(y_i - x_i\beta)^2, \tag{3}$$

where $G_i = \int\limits_{-\infty}^{x_i\beta} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}q^2} dq$.

The TMLE of β is identified after solving the derivate of Equation (3), but it is not a linear function of β, so it can be solved iteratively by Fisher's scoring method that comprises using the second derivative. The Fisher's scoring method is given as

$$\hat{\beta}^{(r)} = \hat{\beta}^{(r-1)} - L^{-1}(\hat{\beta}^{(r-1)})\ Q(\hat{\beta}^{(r-1)}). \qquad r = 1, 2, ..., \tag{4}$$

where $L(\hat{\beta}^{(r-1)}) = E\left(\frac{\partial^2 \log S(X;\beta,\sigma^2)}{\partial\beta\,\partial\beta'}\right)_{\beta=\hat{\beta}^{(r-1)}} = \frac{1}{\sigma^2}\bar{D}$ is the matrix of the Fisher information which is given at $\hat{\beta}^{(r-1)}$ where $\hat{\beta}^{(r)}$ is β estimate at iteration(r), $\hat{\beta}^{(r-1)}$ is β estimate at iteration $(r-1)$, $\bar{D} = X'DX$, D is called as the diagonal matrix and $Q(\hat{\beta}^{(r-1)}) = E\left(\frac{\partial \log S(X;\beta,\sigma^2)}{\partial\beta}\right)_{\beta=\hat{\beta}^{(r-1)}}$. So, the TMLE is written as:

$$\begin{aligned}\hat{\beta}^{(r)} &= \hat{\beta}^{(r-1)} - \sigma^2(\bar{D})^{-1}\ Q(\hat{\beta}^{(r-1)}) \\ &= (\bar{D})^{-1}(\bar{D}\hat{\beta}^{(r-1)} - \sigma^2\, Q(\hat{\beta}^{(r-1)})).\end{aligned} \tag{5}$$

Then, $Q(\hat{\beta}^{(r-1)})$is given as:

$$Q(\hat{\beta}^{(r-1)}) = \frac{1}{\sigma^2}(\bar{D})(\hat{\beta}^{(r-1)} - \hat{\beta}^{(r)}). \tag{6}$$

Since the TMLE becomes inefficient and unstable when the multicollinearity problem occurs, Khalaf et al. [4] proposed the TRE and Toker et al. [6] proposed the TLE to eliminate the effects of this problem.

The TRE is given iteratively as

$$\hat{\beta}_k^{(r)} = \hat{\beta}_k^{(r-1)} - \left[\ \hat{\beta}^{(r-1)} - (\bar{D} + kI)^{-1}(\bar{D})\hat{\beta}^{(r)}\right]_{\beta=\hat{\beta}^{(r-1)}} \tag{7}$$

and the first step of the TRE is

$$\begin{aligned}\hat{\beta}_k^{(1)} &= \hat{\beta}_k^{(0)} - \left[\hat{\beta}^{(0)} - (\bar{D} + kI)^{-1}(\bar{D})\hat{\beta}^{(1)}\right]_{\beta=\hat{\beta}^{(0)}} \\ &= (\hat{\bar{D}}^{(0)} + kI)^{-1}(\hat{\bar{D}}^{(0)})\hat{\beta}^{(1)},\end{aligned} \tag{8}$$

such that $\hat{\beta}^{(0)}$ is the first estimate of β, $\hat{\bar{D}}^{(0)} = X'\hat{D}^{(0)}X$, $\hat{D}^{(0)}$ is given at $\beta^{(0)}$, the TMLE first step values are as same as that of the TRE, and $\hat{\beta}^{(1)}$is the first step of the TMLE. When $k = 0$, $\hat{\beta}_k^{(1)} = \hat{\beta}^{(1)}$.

The TLE is given iteratively as

$$\hat{\beta}_d^{(r)} = \hat{\beta}_d^{(r-1)} - \left[\hat{\beta}^{(r-1)} - (\bar{D} + I)^{-1}(\bar{D} + dI)\hat{\beta}^{(r)}\right]_{\beta=\hat{\beta}^{(r-1)}} \tag{9}$$

and the first step of the TLE is

$$\hat{\beta}_d^{(1)} = \hat{\beta}_d^{(0)} - \left[\hat{\beta}^{(0)} - (\bar{D}+I)^{-1}(\bar{D}+dI)\hat{\beta}^{(1)}\right]_{\beta=\hat{\beta}^{(0)}} = (\hat{\bar{D}}^{(0)}+I)^{-1}(\hat{\bar{D}}^{(0)}+dI)\hat{\beta}^{(1)}, \quad (10)$$

where the TMLE first step values are as same as that of the TLE if $d = 1$, $\hat{\beta}_d^{(1)} = \hat{\beta}^{(1)}$ [see Amemiya [16], Fair [17], and Toker et al. [6] for more details].

New Ridge-Type Estimator

The usefulness of the NRTE among the one-parameter estimators (RE and LE) in many different regression models and the extension of the one-parameter estimators to the area of the Tobit regression model encouraged us to derive the NRTE in this model as follows:

By extending Equation (3), which is the censored data log-likelihood function with the term of penalization, as

$$J = \log S(X;\beta,\sigma^2) + \frac{k}{2\sigma^2}[(\beta+\hat{\beta})'(\beta+\hat{\beta}) - c], \quad (11)$$

where $\frac{k}{2\sigma^2}$ is called a Lagrangian multiplier and c is a constant, and by differentiating J due to β, we got

$$\frac{\partial J}{\partial \beta} = Q(\beta) + \frac{k}{\sigma^2}(\beta+\hat{\beta}), \quad (12)$$

where $Q(\beta) = \frac{\partial \log S(X;\beta,\sigma^2)}{\partial\beta}$.

By finding the J second derivative due to β and then taking the expectation, we got the following form for the matrix:

$$E\left(\frac{\partial^2 J}{\partial\beta\,\partial\beta'}\right)_{\beta=\hat{\beta}^{(r-1)}} = \frac{1}{\sigma^2}(\bar{D}+kI)_{\beta=\hat{\beta}^{(r-1)}}. \quad (13)$$

Then, we employed the scoring of Fisher's method in order to introduce the TNRTE as:

$$\hat{\beta}_{TNRTE}^{(r)} = \hat{\beta}_{TNRTE}^{(r-1)} - L^{-1}(\hat{\beta}_{TNRTE}^{(r-1)})\,Q(\hat{\beta}_{TNRTE}^{(r-1)}) = \hat{\beta}_{TNRTE}^{(r-1)} - \left[\sigma^2(\bar{D}+kI)^{-1}\left(Q(\hat{\beta}^{(r-1)}) + \frac{k}{\sigma^2}(\beta+\hat{\beta})\right)\right]_{\beta=\hat{\beta}^{(r-1)}} \quad (14)$$

By using Equation (4), we have the TNRTE in its final form as:

$$\begin{aligned}\hat{\beta}_{TNRTE}^{(r)} &= \hat{\beta}_{TNRTE}^{(r-1)} - \left[\sigma^2(\bar{D}+kI)^{-1}\left(\tfrac{1}{\sigma^2}(\bar{D})(\hat{\beta}^{(r-1)}-\hat{\beta}^{(r)}) + \tfrac{k}{\sigma^2}(\beta+\hat{\beta})\right)\right]_{\beta=\hat{\beta}^{(r-1)}} \\ &= \hat{\beta}_{TNRTE}^{(r-1)} - \left[(\bar{D}+kI)^{-1}\left((\bar{D})(\hat{\beta}^{(r-1)}-\hat{\beta}^{(r)}) + k(\hat{\beta}^{(r-1)}+\hat{\beta}^{(r)})\right)\right]_{\beta=\hat{\beta}^{(r-1)}} \\ &= \hat{\beta}_{TNRTE}^{(r-1)} - \left[(\bar{D}+kI)^{-1}\left((\bar{D}+kI)\,\hat{\beta}^{(r-1)} - (\bar{D}-kI)\,\hat{\beta}^{(r)}\right)\right]_{\beta=\hat{\beta}^{(r-1)}} \\ &= \hat{\beta}_{TNRTE}^{(r-1)} - \left[\hat{\beta}^{(r-1)} - (\bar{D}+kI)^{-1}(\bar{D}-kI)\hat{\beta}^{(r)}\right]_{\beta=\hat{\beta}^{(r-1)}}\end{aligned} \quad (15)$$

The TNRTE of Equation (15) was obtained iteratively. The first step of the TNRTE is given as follows:

$$\hat{\beta}_{TNRTE}^{(1)} = \hat{\beta}_{TNRTE}^{(0)} - \left[\hat{\beta}^{(0)} - (\bar{D}+kI)^{-1}(\bar{D}-kI)\hat{\beta}^{(1)}\right]_{\beta=\hat{\beta}^{(0)}}, \quad (16)$$

TABLE 1 | Values of factors that are considered in the simulation.

Factor	Symbol	Design
Censoring level	CL	5, 25, 50%
Sample size	n	100, 400, 800
Variance	σ	0.5, 1, 5
Degree of correlation	τ	0.85, 0.9, 0.95, 0.99
Number of explanatory variables	p	4, 8
Number of replicates	MCN	1,000

and the first step of the TNRTE is

$$\hat{\beta}_{TNRTE}^{(1)} = (\hat{\bar{D}}^{(0)}+kI)^{-1}(\hat{\bar{D}}^{(0)}-kI)\hat{\beta}^{(1)}, \quad (17)$$

where the first step values of the TNRTE are same as that of the Tobit LE and $\hat{D}^{(0)}$ is evaluated at $\beta^{(0)}$ if $k=0$, $\hat{\beta}_{TNRTE}^{(1)} = \hat{\beta}^{(1)}$.

Asymptotic MSE Comparisons

To observe the estimators' characteristics, the MSE criterion was preferred. When $\bar{B}$ is an estimator of B, then the matrix form of the MSE criterion is given as

$$MSE(\bar{B}) = Var(\bar{B}) + Bias(\bar{B})\,Bias(\bar{B})', \quad (18)$$

where var($\bar{B}$)is the matrix form of the variance-covariance and $Bias(\bar{B})$ is the bias vector of $\bar{B}$ estimator. Then, the scalar MSE is given by

$$mse(\bar{B}) = trace(MSE(\bar{B})), \quad (19)$$

Since the TMLE for the first step is known as an asymptotically unbiased estimator, it means that the asymptotic matrix form of the MSE equals the asymptotic matrix form of the variance-covariance as follows:

$$MSE(\hat{\beta}^{(1)}) = Var(\hat{\beta}^{(1)}) = \sigma^2(\hat{\bar{D}}^{(0)})^{-1} \quad (20)$$

The asymptotic MSE matrix form of $\hat{\beta}_k^{(1)}$ is given as

$$MSE(\hat{\beta}_k^{(1)}) = \sigma^2(\hat{\bar{D}}^{(0)}+kI)^{-1}\hat{\bar{D}}^{(0)}(\hat{\bar{D}}^{(0)}+kI)^{-1} + k^2(\hat{\bar{D}}^{(0)}+kI)^{-1}\beta\beta'(\hat{\bar{D}}^{(0)}+kI)^{-1}. \quad (21)$$

The asymptotic MSE matrix form of $\hat{\beta}_d^{(1)}$ is given as

$$MSE(\hat{\beta}_d^{(1)}) = \sigma^2[(\hat{\bar{D}}^{(0)}+I)^{-1}(\hat{\bar{D}}^{(0)}+dI)(\hat{\bar{D}}^{(0)})^{-1}(\hat{\bar{D}}^{(0)}+dI)(\hat{\bar{D}}^{(0)}+I)^{-1}] + (d-1)^2(\hat{\bar{D}}^{(0)}+I)^{-1}\beta\beta'(\hat{\bar{D}}^{(0)}+I)^{-1}. \quad (22)$$

The first step TNRTE asymptotic bias and its asymptotic variance-covariance forms are given as follows:

$$Bias(\hat{\beta}_{TNRTE}^{(1)}) = [\,(\hat{D}^{(0)}+kI)^{-1}(\hat{D}^{(0)}-kI) - I\,]\,\beta. \quad (23)$$

TABLE 2 | Simulation results in case of $p = 4$ and $\sigma = 0.5$.

CL	n	τ	$\hat{\alpha}^{(1)}$	$\hat{\alpha}_k^{(1)}(\hat{k}_M)$	$\hat{\alpha}_d^{(1)}(\hat{d}_{alt})$	$\hat{\alpha}_{TNRTE}^{(1)}(\hat{k}_H)$	$\hat{\alpha}_{TNRTE}^{(1)}(\hat{k}_{\min})$
0.05	100	0.85	0.08439	0.08177	0.08024	0.07580	**0.04701**
		0.90	0.14477	0.14108	0.13687	0.13340	**0.09588**
		0.95	0.37866	0.30013	0.28805	0.21957	**0.14546**
		0.99	2.12123	1.69281	1.17409	1.36538	**1.07352**
	400	0.85	0.06948	0.06873	0.06862	0.06682	**0.04976**
		0.90	0.03736	0.03698	0.03664	0.03595	**0.02226**
		0.95	0.15997	0.15486	0.15322	0.14427	**0.10026**
		0.99	0.51461	0.44249	0.41698	0.36556	**0.26895**
	800	0.85	0.01500	0.01495	0.01492	0.01482	**0.01131**
		0.90	0.02420	0.02403	0.02393	0.02356	**0.01506**
		0.95	0.05889	0.05823	0.05779	0.05653	**0.03707**
		0.99	0.26189	0.24239	0.23242	0.21132	**0.13378**
0.25	100	0.85	0.15183	0.13607	0.13891	0.11350	**0.09149**
		0.90	0.34139	0.23299	0.27936	0.16125	**0.13711**
		0.95	0.81721	0.33112	0.47175	0.18070	**0.15651**
		0.99	3.07911	1.94488	1.30970	1.30760	**0.94765**
	400	0.85	0.09315	0.09179	0.09245	**0.08912**	0.09653
		0.90	0.14761	0.14084	0.14458	0.13033	**0.12393**
		0.95	0.28787	0.21218	0.25017	0.14216	**0.10127**
		0.99	1.16488	0.51121	0.64053	0.28221	**0.23849**
	800	0.85	0.05474	0.05454	0.05462	**0.05407**	0.06394
		0.90	0.14315	0.13700	0.14052	0.12498	**0.09854**
		0.95	0.11686	0.10747	0.11202	0.09119	**0.06602**
		0.99	1.23499	0.81775	0.89744	0.53421	**0.35129**
0.50	100	0.85	0.94398	0.49023	0.72343	0.34857	**0.32451**
		0.90	0.89523	0.37400	0.61122	0.29265	**0.28724**
		0.95	0.45112	0.15818	0.25473	**0.12841**	0.12884
		0.99	4.57183	0.72390	0.72960	**0.39133**	0.45197
	400	0.85	0.41065	0.30470	0.38918	0.25460	**0.24267**
		0.90	0.36824	0.27886	0.34462	0.22898	**0.20956**
		0.95	0.78458	0.39098	0.66041	0.29614	**0.27413**
		0.99	6.15895	3.91878	2.45550	1.67715	**0.37830**
	800	0.85	0.28978	0.27517	0.28713	0.25792	**0.24808**
		0.90	0.38991	0.31360	0.37730	0.26372	**0.24913**
		0.95	0.32927	0.29935	0.32278	0.28405	**0.28284**
		0.99	0.64817	0.29716	0.46906	0.25371	**0.24986**

and

$$Var(\hat{\beta}_{TNRTE}^{(1)}) = \sigma^2[(\hat{\hat{D}}^{(0)} + kI)^{-1}(\hat{\hat{D}}^{(0)} - kI)(\hat{\hat{D}}^{(0)})^{-1} (\hat{\hat{D}}^{(0)} - kI)(\hat{\hat{D}}^{(0)} + kI)^{-1}]. \quad (24)$$

Then, the asymptotic MSE matrix form of $\hat{\beta}_{TNRTE}^{(1)}$ is given as

$$MSE(\hat{\beta}_{TNRTE}^{(1)}) = \sigma^2[(\hat{\hat{D}}^{(0)} + kI)^{-1}(\hat{\hat{D}}^{(0)} - kI)(\hat{\hat{D}}^{(0)})^{-1} (\hat{\hat{D}}^{(0)} - kI)(\hat{\hat{D}}^{(0)} + kI)^{-1}] + [(\hat{\hat{D}}^{(0)} + kI)^{-1}(\hat{\hat{D}}^{(0)} - kI) - I] \beta\beta'[(\hat{\hat{D}}^{(0)} + kI)^{-1}(\hat{\hat{D}}^{(0)} - kI) - I]'. \quad (25)$$

Model (1) is written in the canonical form using the orthogonal transformation and the spectral decomposition such that the Fisher matrix form of the first step is given as $\hat{\hat{D}}^{(0)} = C\bar{W}C'$, where $C = [C_0, C_1, ..., C_p]$ is called a $(p+1)\times(p+1)$ orthogonal matrix form and $\hat{\hat{D}}^{(0)} = C\bar{W}C'$ refers to the eigenvectors columns, $C'\hat{\hat{D}}^{(0)}C = M'\hat{D}^{(0)}M = \bar{W} = diag(\bar{w}_j)$ is called a $(p+1)\times(p+1)$ diagonal matrix form with the $\hat{\hat{D}}^{(0)}$ ($\bar{w}_0 \geq \bar{w}_1 \geq ... \geq \bar{w}_p > 0$) eigenvalues on the diagonal, such that $M = XC$. The canonical form formula of the asymptotic matrix form and the scalar MSE for $\hat{\alpha}^{(1)}$, $\hat{\alpha}_k^{(1)}$, $\hat{\alpha}_d^{(1)}$ and $\hat{\alpha}_{TNRTE}^{(1)}$ are written as follows:

$$MSE(\hat{\alpha}^{(1)}) = \sigma^2(\bar{W})^{-1}, \quad (26)$$

TABLE 3 | Simulation results in case of $p = 4$ and $\sigma = 1$.

CL	n	τ	$\hat{\alpha}^{(1)}$	$\hat{\alpha}_k^{(1)}(\hat{k}_M)$	$\hat{\alpha}_d^{(1)}(\hat{d}_{alt})$	$\hat{\alpha}_{TNRTE}^{(1)}(\hat{k}_H)$	$\hat{\alpha}_{TNRTE}^{(1)}(\hat{k}_{\min})$
0.05	100	0.85	0.27994	0.22248	0.23948	0.15818	**0.09746**
		0.90	0.48548	0.36945	0.38882	0.26648	**0.18523**
		0.95	1.06487	0.53619	0.59676	0.29376	**0.21853**
		0.99	6.09446	3.40860	1.50770	1.88323	**0.79975**
	400	0.85	0.12066	0.11220	0.11558	0.09571	**0.05239**
		0.90	0.12223	0.11110	0.11400	0.09042	**0.04230**
		0.95	0.29436	0.24058	0.25852	0.17596	**0.10436**
		0.99	1.21836	0.66503	0.71062	0.40058	**0.30620**
	800	0.85	0.03940	0.03824	0.03857	0.03533	**0.01791**
		0.90	0.06245	0.05920	0.06033	0.05186	**0.02352**
		0.95	0.14561	0.13252	0.13633	0.10842	**0.05392**
		0.99	0.66375	0.42483	0.47623	0.26713	**0.19121**
0.25	100	0.85	0.33660	0.21404	0.26973	0.14257	**0.12302**
		0.90	0.65097	0.29570	0.45169	0.17768	**0.15457**
		0.95	1.35205	0.43767	0.61171	0.19088	**0.13035**
		0.99	6.27692	2.95777	1.26949	1.38631	**0.60606**
	400	0.85	0.12446	0.11477	0.12118	0.10195	**0.10170**
		0.90	0.23884	0.19092	0.22362	0.15172	**0.13461**
		0.95	0.37085	0.19966	0.30011	0.11732	**0.09775**
		0.99	1.80126	0.61418	0.76889	0.25384	**0.13237**
	800	0.85	0.07414	0.07139	0.07309	0.06594	**0.06361**
		0.90	0.17009	0.15060	0.16434	0.12400	**0.09976**
		0.95	0.21158	0.16012	0.19310	0.11050	**0.08175**
		0.99	1.74511	0.87871	1.07884	0.45816	**0.25214**
0.50	100	0.85	1.13850	0.51946	0.81290	0.35420	**0.33801**
		0.90	1.13729	0.40701	0.68195	0.30323	**0.30289**
		0.95	0.85386	0.20759	0.36866	**0.13717**	0.13773
		0.99	5.99679	0.90344	0.65449	**0.41361**	0.43331
	400	0.85	0.43862	0.28981	0.40717	0.24688	**0.24386**
		0.90	0.42703	0.27502	0.38620	0.22024	**0.20849**
		0.95	1.02445	0.43173	0.79452	0.29342	**0.25947**
		0.99	7.23887	3.73973	2.27123	1.30613	**0.39580**
	800	0.85	0.31729	0.28246	0.31201	0.25644	**0.24980**
		0.90	0.43023	0.30451	0.40978	0.25171	**0.24425**
		0.95	0.38920	0.30205	0.37000	0.27884	**0.27794**
		0.99	0.88767	0.31974	0.54905	0.25139	**0.24847**

$$MSE(\hat{\alpha}_k^{(1)}) = \sigma^2 (\bar{W}+kI)^{-1} \bar{W} (\bar{W}+kI)^{-1} + k^2 (\bar{W}+kI)^{-1} \alpha\alpha' (\bar{W}+kI)^{-1} \tag{27}$$

$$MSE(\hat{\alpha}_d^{(1)}) = \sigma^2 [(\bar{W}+I)^{-1} (\bar{W}+dI)(\bar{W})^{-1}(\bar{W}+dI)(\bar{W}+I)^{-1}] + (d-1)^2 (\bar{W}+I)^{-1} \alpha\alpha' (\bar{W}+I)^{-1} \tag{28}$$

$$MSE(\hat{\alpha}_{TNRTE}^{(1)}) = \sigma^2 [(\bar{W}+kI)^{-1} (\bar{W}-kI)(\bar{W})^{-1}(\bar{W}-kI)(\bar{W}+kI)^{-1}] + 4k^2 (\bar{W}+kI)^{-1} \alpha\alpha' (\bar{W}+kI)^{-1} \tag{29}$$

$$mse(\hat{\alpha}^{(1)}) = \sigma^2 \sum_{j=0}^{p} \frac{1}{\bar{w}_j}, \tag{30}$$

$$mse(\hat{\alpha}_k^{(1)}) = \sum_{j=0}^{p} \frac{\sigma^2 \bar{w}_j + k^2 \alpha_j^2}{(\bar{w}_j + k)^2}, \tag{31}$$

$$mse(\hat{\alpha}_d^{(1)}) = \sum_{j=0}^{p} \frac{\sigma^2 (\bar{w}_j + d)^2 + (d-1)^2 \bar{w}_j \alpha_j^2}{\bar{w}_j (\bar{w}_j + 1)^2}, \tag{32}$$

$$mse(\hat{\alpha}_{TNRTE}^{(1)}) = \sum_{j=0}^{p} \frac{\sigma^2 (\bar{w}_j - k)^2 + 4k^2 \bar{w}_j \alpha_j^2}{\bar{w}_j (\bar{w}_j + k)^2}, \tag{33}$$

where $\alpha = C'\beta$, $\hat{\alpha}^{(1)} = C'\hat{\beta}^{(1)}$, $\hat{\alpha}_k^{(1)} = C'\hat{\beta}_k^{(1)}$, $\hat{\alpha}_d^{(1)} = C'\hat{\beta}_d^{(1)}$ and $\hat{\alpha}_{TNRTE}^{(1)} = C'\hat{\beta}_{TNRTE}^{(1)}$.

TABLE 4 | Simulation results in case of $p = 4$ and $\sigma = 5$.

CL	n	τ	$\hat{\alpha}^{(1)}$	$\hat{\alpha}_k^{(1)}(\hat{k}_M)$	$\hat{\alpha}_d^{(1)}(\hat{d}_{alt})$	$\hat{\alpha}_{TNRTE}^{(1)}(\hat{k}_H)$	$\hat{\alpha}_{TNRTE}^{(1)}(\hat{k}_{\min})$
0.05	100	0.85	5.85274	0.39991	0.80830	**0.18888**	0.19297
		0.90	9.23801	0.43827	0.75055	0.20275	**0.17161**
		0.95	20.26041	0.63066	**0.54674**	0.75467	0.93491
		0.99	107.51962	3.14511	**0.33799**	1.95772	1.76879
	400	0.85	1.70770	0.16823	0.75322	0.04656	**0.03788**
		0.90	2.53743	0.16743	0.78535	0.04929	**0.04044**
		0.95	4.58955	0.17121	0.79950	0.06594	**0.04300**
		0.99	21.62036	0.42102	0.40384	0.25861	**0.13540**
	800	0.85	0.73929	0.08473	0.47144	0.02380	**0.02045**
		0.90	1.19120	0.09931	0.61501	0.02961	**0.02455**
		0.95	2.38632	0.08583	0.77279	0.02664	**0.01796**
		0.99	11.96986	0.24125	0.53698	0.13878	**0.06955**
0.25	100	0.85	5.77461	0.46378	0.71224	**0.29294**	0.30568
		0.90	11.21582	0.48987	0.74074	0.22880	**0.18155**
		0.95	18.42567	0.63905	0.54834	0.56494	**0.50903**
		0.99	89.60013	2.05524	**0.26170**	0.98369	0.41537
	400	0.85	1.38431	0.42819	0.68425	0.18593	**0.17036**
		0.90	3.08322	0.35974	0.97073	0.15886	**0.13434**
		0.95	4.08642	0.25720	0.67809	0.12900	**0.09643**
		0.99	22.17859	0.40916	0.36989	0.31214	**0.13774**
	800	0.85	0.83619	0.17091	0.53177	0.06465	**0.05761**
		0.90	1.18690	0.23904	0.63327	0.10522	**0.09467**
		0.95	2.81468	0.17317	0.85394	0.07262	**0.06217**
		0.99	15.32023	0.41075	0.72909	0.15574	**0.12420**
0.50	100	0.85	9.33358	0.75539	1.21896	0.63396	**0.61351**
		0.90	10.71809	0.76879	0.97905	0.71838	**0.71674**
		0.95	17.16682	0.42785	0.44256	0.42213	**0.32800**
		0.99	64.19689	0.91622	**0.55578**	1.46511	0.66360
	400	0.85	1.84858	0.58827	0.86659	0.36689	**0.34593**
		0.90	2.42662	0.44374	0.82334	0.25360	**0.22669**
		0.95	6.36484	0.40697	0.98929	0.29202	**0.24581**
		0.99	42.22717	1.41102	0.73127	0.44336	**0.27481**
	800	0.85	1.24810	0.45470	0.82589	0.26673	**0.25520**
		0.90	1.79880	0.40571	0.95688	0.25133	**0.23683**
		0.95	2.05814	0.41260	0.74084	0.26399	**0.24359**
		0.99	7.95036	0.33278	0.52525	0.32700	**0.23893**

The lemmas below are useful to be used in the theoretical comparisons among the above estimators.

Lemma 1: Suppose for the matrices $n \times n$, if $F > 0$ and $I > 0$ (or $I \geq 0$), then $F > I$ iff $\lambda_{\max}(IF^{-1}) < 1$ such that $\lambda_{\max}(IF^{-1})$ is the matrix IF^{-1} maximum eigenvalue [18].

Lemma 2: If the matrix F is defined as an $n \times n$ positive definite, i.e., $F > 0$, as well as α is a vector, then, $F - \alpha\,\alpha' > 0$ iff $\alpha' F^{-1}\alpha < 1$ [19].

Lemma 3: Suppose $\alpha_i = K_i\, m$, $i = 1, 2$ are two α linear estimators and suppose $Diff = Cov\,(\hat{\alpha}_1) - Cov\,(\hat{\alpha}_2) > 0$, where $Cov\,(\hat{\alpha}_i)$; $i = 1, 2$ refers to $\hat{\alpha}_i$ covariance matrix and $b_i = Bias\,(\hat{\alpha}_i) = (K_i X - I)\alpha$, $i = 1, 2$ [20], then consequently,

$$\Delta(\hat{\alpha}_1 - \hat{\alpha}_2) = MSE(\hat{\alpha}_1) - MSE(\hat{\alpha}_2) = \sigma^2 Diff + b_1 b'_1 - b_2 b'_2 > 0 \tag{34}$$

iff $b'_2[\sigma^2 Diff + b'_1 b_1]^{-1} b_2 < 1$, where $MSE\,(\hat{\alpha}_i) = Cov\,(\hat{\alpha}_i) + b_i b'_i$.

Comparisons Among the Estimators

Theorem 1: $\hat{\alpha}_{TNRTE}^{(1)}$ is superior to $\hat{\alpha}^{(1)}$ iff

$$\begin{gathered}\alpha' [\,(\bar{W} + kI)^{-1} (\bar{W} - kI)\, - I_p]' \times [\sigma^2\,((\bar{W})^{-1} \\ -(\bar{W} + kI)^{-1} (\bar{W} - kI)\,(\bar{W})^{-1} (\bar{W} - kI)(\bar{W} + kI)^{-1})] \times \\ [\,(\bar{W} + kI)^{-1} (\bar{W} - kI)\, - I_p]\,\alpha\; < 1\end{gathered} \tag{35}$$

Proof: The dispersion difference is:

$$Cov\;(\hat{\alpha}^{(1)}) - Cov\,(\hat{\alpha}_{TNRTE}^{(1)}) = \sigma^2\,((\bar{W})^{-1} - (\bar{W} + kI)^{-1} (\bar{W} - kI)(\bar{W})^{-1} (\bar{W} - kI)(\bar{W} + kI)^{-1}).$$

TABLE 5 | Simulation results in case of $p = 8$ and $\sigma = 0.5$.

CL	*n*	τ	$\hat{\alpha}^{(1)}$	$\hat{\alpha}_k^{(1)}(\hat{k}_M)$	$\hat{\alpha}_d^{(1)}(\hat{d}_{alt})$	$\hat{\alpha}_{TNRTE}^{(1)}(\hat{k}_H)$	$\hat{\alpha}_{TNRTE}^{(1)}(\hat{k}_{\min})$
0.05	100	0.85	0.21299	0.21003	0.20261	0.19563	**0.12179**
		0.90	0.56969	0.51409	0.48277	0.36650	**0.18440**
		0.95	0.83207	0.72366	0.63084	0.50335	**0.28609**
		0.99	16.52158	10.44695	**2.35048**	4.92202	2.37078
	400	0.85	0.27154	0.26411	0.26211	0.23113	**0.13296**
		0.90	0.13371	0.12809	0.12646	0.10156	**0.02604**
		0.95	0.31037	0.29873	0.29093	0.25102	**0.13099**
		0.99	1.18991	1.01467	0.86321	0.71893	**0.45445**
	800	0.85	0.05024	0.04994	0.04966	0.04810	**0.02107**
		0.90	0.05721	0.05666	0.05614	0.05330	**0.01864**
		0.95	0.18054	0.17206	0.17089	0.13469	**0.04490**
		0.99	1.35089	1.07159	1.04126	0.69134	**0.43935**
0.25	100	0.85	1.56275	0.58823	1.03474	0.22315	**0.19675**
		0.90	1.89425	0.74730	1.13449	0.31928	**0.26897**
		0.95	3.92006	2.28015	2.15826	1.18369	**0.83884**
		0.99	15.48598	3.40925	0.68261	0.49189	**0.45279**
	400	0.85	0.21475	0.17820	0.19909	0.09342	**0.05105**
		0.90	0.51782	0.36158	0.45851	0.16621	**0.10651**
		0.95	1.89760	1.16085	1.37948	0.40165	**0.11108**
		0.99	3.85082	1.25270	1.42291	0.43649	**0.29770**
	800	0.85	0.36483	0.31813	0.35016	0.20515	**0.13247**
		0.90	0.12170	0.10803	0.11754	0.08141	**0.07711**
		0.95	0.55161	0.37153	0.48398	0.16621	**0.08142**
		0.99	3.11611	0.91577	1.55483	0.29862	**0.18130**
0.50	100	0.85	4.46921	1.76414	2.69672	0.83017	**0.64431**
		0.90	4.40601	1.93398	2.24012	0.70013	**0.41793**
		0.95	7.70547	2.09302	1.91177	0.47669	**0.38454**
		0.99	16.11585	1.44523	**0.69649**	0.80656	0.86553
	400	0.85	0.58769	0.25876	0.50664	**0.19675**	0.19763
		0.90	0.86237	0.34608	0.72974	0.22902	**0.22895**
		0.95	1.80472	0.60017	1.26100	0.35283	**0.33833**
		0.99	21.83544	5.59858	3.62775	1.13593	**0.75091**
	800	0.85	0.47881	0.34735	0.46012	0.28121	**0.27713**
		0.90	0.83702	0.49046	0.76243	0.29857	**0.27414**
		0.95	1.49459	0.57785	1.22709	0.31494	**0.27184**
		0.99	3.12040	0.66685	1.30715	0.27054	**0.24975**

We observed that $(\bar{W})^{-1} - (\bar{W}+kI)^{-1}(\bar{W}-kI)(\bar{W})^{-1}(\bar{W}-kI)(\bar{W}+kI)^{-1}$ is positive definite since$(\bar{w}_j+k)^2 - (\bar{w}_j-k)^2 = 4\,\bar{w}_j k > 0$, for $k > 0$. By Lemma 3, the proof is completed.

Theorem 2: When $\lambda_{\max}(IF^{-1}) < 1$, $\hat{\alpha}_{TNRTE}^{(1)}$ is superior to $\hat{\alpha}_k^{(1)}$ iff

$$\begin{gathered}\alpha'[(\bar{W}+kI)^{-1}(\bar{W}-kI)-I_p]'\times\\ [V_1+((\bar{W}+kI)^{-1}(\bar{W})^{-1}-I_p)\alpha\alpha'((\bar{W}+kI)^{-1}(\bar{W})^{-1}-I_p)']\\ [(\bar{W}+kI)^{-1}(\bar{W}-kI)-I_p]\,\alpha<1\end{gathered} \tag{36}$$

$$\lambda_{\max}(IF^{-1}) < 1, \tag{37}$$

where

$$\begin{aligned}V_1 &= \sigma^2\,((\bar{W}+kI)^{-1}\,\bar{W}(\bar{W}+kI)^{-1}-(\bar{W}+kI)^{-1}(\bar{W}-kI)\\ &\quad(\bar{W})^{-1}(\bar{W}-kI)(\bar{W}+kI)^{-1}),\\ I &= k\,(\bar{W}+kI)^{-1}(\bar{W})^{-1}(\bar{W}+kI)^{-1},\\ F &= 2\,(\bar{W}+kI)^{-1}(\bar{W}+kI)^{-1}.\end{aligned}$$

Proof:

$$\begin{aligned}&V_1=\sigma^2\,((\bar{W}+kI)^{-1}\,\bar{W}(\bar{W}+kI)^{-1}-(\bar{W}+kI)^{-1}(\bar{W}-kI)\\ &(\bar{W})^{-1}(\bar{W}-kI)(\bar{W}+kI)^{-1})\\ &\quad=\sigma^2 k\,(2\,(\bar{W}+kI)^{-1}(\bar{W}+kI)^{-1}-k\,(\bar{W}+kI)^{-1}\\ &\quad(\bar{W})^{-1}(\bar{W}+kI)^{-1})\\ &\quad=\sigma^2\,k\,(F-I)\end{aligned}$$

TABLE 6 | Simulation results in case of $p = 8$ and $\sigma = 1$.

CL	n	τ	$\hat{\alpha}^{(1)}$	$\hat{\alpha}_k^{(1)}(\hat{k}_M)$	$\hat{\alpha}_d^{(1)}(\hat{d}_{alt})$	$\hat{\alpha}_{TNRTE}^{(1)}(\hat{k}_H)$	$\hat{\alpha}_{TNRTE}^{(1)}(\hat{k}_{\min})$
0.05	100	0.85	1.15846	0.87888	0.91697	0.51322	**0.27689**
		0.90	1.53553	1.15383	1.07084	0.68093	**0.38196**
		0.95	4.01636	2.37195	1.80330	1.01617	**0.57745**
		0.99	12.27522	8.05584	2.46922	4.06623	**2.06109**
	400	0.85	0.19309	0.17901	0.18189	0.12447	**0.03658**
		0.90	0.43058	0.37714	0.39135	0.23424	**0.09405**
		0.95	0.73881	0.55514	0.60638	0.27658	**0.12419**
		0.99	4.29109	2.14779	1.75127	0.85536	**0.46653**
	800	0.85	0.11483	0.10893	0.11095	0.08191	**0.02236**
		0.90	0.16419	0.15237	0.15634	0.10503	**0.02677**
		0.95	0.32358	0.28838	0.29549	0.18040	**0.05814**
		0.99	1.60044	0.77540	0.97348	0.29564	**0.18525**
0.25	100	0.85	2.32574	1.08986	1.36734	0.27468	**0.13717**
		0.90	5.22360	2.78669	2.68889	1.03991	**0.49823**
		0.95	5.40588	3.28294	2.54175	1.61957	**0.97862**
		0.99	27.30491	8.10032	**1.35084**	1.91940	1.40683
	400	0.85	0.44521	0.26832	0.39387	0.11254	**0.08142**
		0.90	0.76937	0.45170	0.65382	0.18471	**0.11600**
		0.95	2.29500	1.21621	1.50964	0.27296	**0.10602**
		0.99	5.17561	1.69015	1.39349	0.33433	**0.13233**
	800	0.85	0.29510	0.24088	0.27898	0.12772	**0.06503**
		0.90	0.36689	0.24308	0.33794	0.12569	**0.10288**
		0.95	0.37569	0.20298	0.31807	0.08142	**0.06467**
		0.99	3.96692	1.07023	1.72913	0.30098	**0.20096**
0.50	100	0.85	3.37604	1.06933	1.78404	0.57155	**0.55877**
		0.90	1.37876	0.32910	0.68637	**0.19041**	0.19209
		0.95	5.84301	1.00650	1.43225	**0.41051**	0.42935
		0.99	73.10146	29.99253	**1.11649**	1.68054	2.97445
	400	0.85	1.12473	0.44371	0.94728	0.25330	**0.23185**
		0.90	1.83650	0.65919	1.39858	0.25178	**0.18772**
		0.95	1.39164	0.41154	0.91057	**0.30256**	0.30279
		0.99	12.10185	1.84634	1.58779	0.38332	**0.36864**
	800	0.85	0.48052	0.34534	0.46263	0.29511	**0.29213**
		0.90	0.53634	0.30901	0.49602	0.26758	**0.26706**
		0.95	1.81295	0.66496	1.39854	0.30232	**0.25480**
		0.99	4.02489	0.89612	1.47110	0.29179	**0.26844**

where $I = k(\bar{W}+kI)^{-1}(\bar{W})^{-1}(\bar{W}+kI)^{-1}$ and $F = 2(\bar{W}+kI)^{-1}(\bar{W}+kI)^{-1}$

It is clear that, for $k > 0$ and $0 < d < 1$, $F > 0$ and $I > 0$. It is obvious that $F - I > 0$ if and only if $\lambda_{\max}(IF^{-1}) < 1$, where $\lambda_{\max}(IF^{-1})$ is the maximum eigenvalue of the matrix IF^{-1}. By Lemma 1, the proof is completed.

Theorem 3: $\hat{\alpha}_{TNRTE}^{(1)}$ is superior to $\hat{\alpha}_d^{(1)}$ if and only if

$$\begin{gathered}\alpha'[(\bar{W}+kI)^{-1}(\bar{W}-kI)-I_p]'\times\\ [V_2+(1-d)^2(\bar{W}+I_p)^{-1}\alpha\,\alpha'(\bar{W}+I_p)^{-1}]\times\\ [(\bar{W}+kI)^{-1}(\bar{W}-kI)-I_p]\,\alpha<1\end{gathered} \tag{38}$$

where

$$\begin{aligned}V_2=\sigma^2\,&((\bar{W}+I)^{-1}(\bar{W}+dI)(\bar{W})^{-1}(\bar{W}+dI)(\bar{W}+I)^{-1}-\\ &(\bar{W}+kI)^{-1}(\bar{W}-kI)(\bar{W})^{-1}(\bar{W}-kI)(\bar{W}+kI)^{-1})\end{aligned}$$

Proof: The dispersion difference is

$$\begin{aligned}V_2&=\sigma^2((\bar{W}+I)^{-1}(\bar{W}+dI)(\bar{W})^{-1}(\bar{W}+dI)(\bar{W}+I)^{-1}-\\ &(\bar{W}+kI)^{-1}(\bar{W}-kI)(\bar{W})^{-1}(\bar{W}-kI)(\bar{W}+kI)^{-1})\\ &=\sigma^2 diag\left\{\frac{(\bar{w}_j+d)^2}{\bar{w}_j(\bar{w}_j+1)^2}-\frac{(\bar{w}_j-k)^2}{\bar{w}_j(\bar{w}_j+k)^2}\right\}_{j=0}^{p}.\end{aligned}$$

We observed that $((\bar{W}+I)^{-1}(\bar{W}+dI)(\bar{W})^{-1}(\bar{W}+dI)(\bar{W}+I)^{-1} - (\bar{W}+kI)^{-1}(\bar{W}-kI)(\bar{W})^{-1}(\bar{W}-$

TABLE 7 | Simulation results in case of $p = 8$ and $\sigma = 5$.

CL	n	τ	$\hat{\alpha}^{(1)}$	$\hat{\alpha}_k^{(1)}(\hat{k}_M)$	$\hat{\alpha}_d^{(1)}(\hat{d}_{alt})$	$\hat{\alpha}_{TNRTE}^{(1)}(\hat{k}_H)$	$\hat{\alpha}_{TNRTE}^{(1)}(\hat{k}_{\min})$
0.05	100	0.85	16.48429	0.76265	1.79934	**0.47392**	0.58901
		0.90	24.14235	1.02268	1.45280	0.77891	**1.03974**
		0.95	53.51362	2.15790	**1.02047**	1.79561	2.33215
		0.99	239.12441	9.28487	**0.34627**	4.55732	5.75055
	400	0.85	3.49016	0.13836	1.51275	**0.03610**	0.03901
		0.90	6.20417	0.20077	1.81764	**0.05704**	0.06378
		0.95	11.94495	0.35280	1.73986	**0.09322**	0.10407
		0.99	59.06904	1.59133	0.78171	**0.32741**	0.37704
	800	0.85	1.89563	0.08764	1.17527	**0.02067**	0.02213
		0.90	2.73088	0.09386	1.39951	**0.01762**	0.01943
		0.95	5.56415	0.13841	1.71080	**0.02300**	0.02574
		0.99	27.36199	0.61708	1.10996	**0.13521**	0.14900
0.25	100	0.85	18.87774	0.82976	1.55346	**0.63487**	0.74055
		0.90	34.43712	2.11859	1.93708	**0.82710**	0.99253
		0.95	56.67286	2.54356	1.20241	**1.20350**	1.40572
		0.99	234.54677	6.24261	**0.46401**	6.35297	7.65207
	400	0.85	3.64835	0.19408	1.46753	0.08255	**0.08183**
		0.90	6.64063	0.26842	1.94108	**0.10631**	0.11199
		0.95	16.55755	0.74765	2.08509	**0.21553**	0.21790
		0.99	53.92402	1.12368	0.64415	**0.37636**	0.40181
	800	0.85	2.28659	0.16307	1.35829	**0.06684**	0.06702
		0.90	2.58707	0.19128	1.28871	0.10171	**0.10106**
		0.95	5.41396	0.18338	1.54659	0.06600	**0.06500**
		0.99	32.35663	0.77203	1.22296	**0.19496**	0.20403
0.50	100	0.85	20.54085	1.08011	2.06415	**0.91851**	0.96065
		0.90	21.67315	0.57351	1.04308	**0.54817**	0.55866
		0.95	65.76152	2.32354	**1.07240**	2.24306	2.64051
		0.99	299.40422	12.27341	**0.42142**	11.04547	11.98258
	400	0.85	5.14950	0.39194	1.96635	0.25346	**0.24852**
		0.90	7.85514	0.34316	2.15083	0.14905	**0.14598**
		0.95	14.82821	0.65250	1.89624	0.47094	**0.45788**
		0.99	69.45204	1.45136	**0.85636**	0.94116	0.88360
	800	0.85	1.88897	0.37811	1.18269	0.25336	**0.25122**
		0.90	3.04633	0.40105	1.51743	0.30356	**0.30175**
		0.95	8.83255	0.40226	2.39202	**0.20527**	0.20621
		0.99	26.02972	0.54693	0.96760	0.27806	**0.25399**

$kI)\,(\bar{W}+kI)^{-1})$ is applicable if and only if $(\bar{w}_j+k)^2\,(\bar{w}_j+d)^2 - (\bar{w}_j-k)^2(\bar{w}_j+1)^2 > 0$. For $k > 0$, it was observed that $(\bar{w}_j+k)^2\,(\bar{w}_j+d)^2 - (\bar{w}_j-k)^2(\bar{w}_j+1)^2 > 0$. By Lemma 3, the proof is completed.

The Selection of *k* Parameter of the TNRTE

Using the Kibria and Lukman [9] method, the optimal biasing parameter k of the TNRTE is given as:

$$k_j = \frac{\sigma^2}{2\alpha_j^2 + (\sigma^2/\bar{w}_j)}, \tag{39}$$

and using the unbiased estimates of σ^2 and α^2, the optimal estimated k of the TNRTE is given as:

$$\hat{k}_j = \frac{\hat{\sigma}^2}{2\hat{\alpha}_j^2 + (\hat{\sigma}^2/\bar{w}_j)}. \tag{40}$$

A MONTE CARLO SIMULATION

To explain the performance of the proposed TNRTE compared with other mentioned estimators, we conducted the simulation experiments using some different factor levels. The design is constructed by following the techniques of Kibria [21], Yenilmez et al. [22], Khalaf et al. [4], Yenilmez and Kantar [23], Toker et al. [6], and Yenilmez et al. [24]. The correlation degree (τ)

TABLE 8 | The regression coefficients and the MSE results.

Estimator	α_0	α_1	α_2	α_3	α_4	α_5	MSE	*k/d*
$\hat{\alpha}^{(1)}$	−2.5349	−0.1648	0.6825	−2.6877	0.0781	0.2214	56.6834	NA
$\hat{\alpha}_k^{(1)}(\hat{k}_M)$	−0.4558	−0.1806	0.5691	−2.0635	−0.0077	0.2243	7.7517	2.270
$\hat{\alpha}_d^{(1)}(\hat{d}_{alt})$	−0.8312	−0.1811	0.6051	−2.4213	0.0083	0.2230	9.9986	0.029
$\hat{\alpha}_{TNRTE}^{(1)}(\hat{k}_{\min})$	1.2387	−0.1987	0.5007	−1.9321	−0.0770	0.2254	34.3714	1.342
$\hat{\alpha}_{TNRTE}^{(1)}(\hat{k}_H)$	−0.3934	−0.1883	0.5992	−2.5793	−0.0090	0.2226	8.0258	0.296
$\hat{\alpha}_k^{(1)}(0.3)$	−1.4565	−0.1766	0.6405	−2.6324	0.0343	0.2220	20.1089	0.300
$\hat{\alpha}_d^{(1)}(0.3)$	−1.3068	−0.1766	0.6267	−2.4957	0.0278	0.2225	16.9422	0.300
$\hat{\alpha}_{TNRTE}^{(1)}(0.3)$	−0.3780	−0.1884	0.5986	−2.5771	−0.0096	0.2226	8.0235	0.300

among the explanatory variables is one of the essential factors in the simulation. For providing the correlation changing range, the data were also generated using the next model:

$$x_{ij} = \sqrt{(1-\tau^2)}\, z_{ij} + \tau\, z_{ip}, \qquad i = 1, .2, .., n;\ j = 1, 2, ..., p, \tag{41}$$

where z_{ij} is given and follows a standard normal. The dependent variable is given using the next equation:

$$y_i^* = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + u_i \qquad i = 1, .2, .., n, \tag{42}$$

where u_i's are considered as pseudo-random numbers, which are independent and identical and have $N(0, \sigma^2)$, and the parameter vector is considered as $\beta'\beta = 1$ as in the studies of Dawoud and Abonazel [25], Awwad et al. [26], Awwad et al. [14], Abonazel and Dawoud [27], Algamal and Abonazel [28], Abonazel et al. [15], and Abonazel et al. [29]. So, the dependent variable has been censored using Equation (2). Also, all factors used in this simulation are stated in **Table 1**.

The TRE, the TLE, and the proposed TNRTE estimated biasing parameters used in this simulation study are given as follows:

1. The estimated parameter of k for the TRE is considered according to Hoerl and Kennard [7], as

$$\hat{k}_M = \min\left\{\frac{\hat{\sigma}^2}{\left(\hat{\alpha}_j^{(1)}\right)^2}\right\}_{j=0}^{p} \tag{43}$$

2. The estimated parameter d for the TLE is considered, according to Liu [8] as follows

$$\hat{d}_{opt} = 1 - \hat{\sigma}^2\left[\frac{\sum\limits_{j=0}^{p}\left(1/(\bar{w}_j(\bar{w}_j+1))\right)}{\sum\limits_{j=0}^{p}\left(\left(\hat{\alpha}_j^{(1)}\right)^2/(\bar{w}_j+1)^2\right)}\right], \tag{44}$$

when $\hat{d}_{opt}$ has negative value, Ozkale and Kaciranlar [30] considered the alternative parameter of d as:

$$\hat{d}_{alt} = \min\left[\frac{\left(\hat{\alpha}_j^{(1)}\right)^2}{(\hat{\sigma}^2/\bar{w}_j) + \left(\hat{\alpha}_j^{(1)}\right)^2}\right]_{j=0}^{p}. \tag{45}$$

3. Following the study of Kibria and Lukman [9], the estimated biasing parameter minimum value and the harmonic-mean of k for the proposed TNRTE are considered as follows:

$$\hat{k}_{\min} = \min\left\{\frac{\hat{\sigma}^2}{2\left(\hat{\alpha}_j^{(1)}\right)^2 + (\hat{\sigma}^2/\bar{w}_j)}\right\}_{j=0}^{p} \tag{46}$$

$$\hat{k}_H = \frac{\hat{\sigma}^2\,(p+1)}{\sum\limits_{j=0}^{p}\left(2\left(\hat{\alpha}_j^{(1)}\right)^2 + (\hat{\sigma}^2/w_j)\right)}. \tag{47}$$

To examine the performances of the TMLE, TRE, TLE, and the proposed TNRTE, we computed the estimated MSE (EMSE) as:

$$EMSE(\alpha^*) = \frac{1}{MCN}\sum_{l=1}^{MCN}(\alpha_l^* - \alpha)'(\alpha_l^* - \alpha), \tag{48}$$

where α_l^* is called an estimator as well as α is called a true parameter. The simulation results (EMSE values) are stated in **Tables 2–7**, the smallest value of the EMSE is highlighted in bold.

Based on the simulation results, we conclude the following:

1. The EMSE increases as n decreases.
2. The EMSE increases as p increases.
3. The EMSE increases as τ increases.
4. The EMSE increases as σ increases.
5. The EMSE increases as the CL increases.
6. The TMLE exhibited the least performance at all levels of multicollinearity and censoring.
7. The TNRTE and the TLE outperform the TRE for all cases.
8. The proposed TNRTE has few EMSE values near to that of TLE in case of large σ and p values.
9. The proposed TNRTE with the biasing parameters $\hat{k}_{\min}$ performs the best of all other mentioned estimators in terms of the EMSE, followed by the proposed TNRTE with the biasing parameters $\hat{k}_H$ in most cases.
10. The proposed TNRTE performance and others almost depend on the determination of their biasing parameter estimators.
11. Finally, the proposed TNRTE performs the best of all other mentioned estimators in terms of the EMSE in most cases.

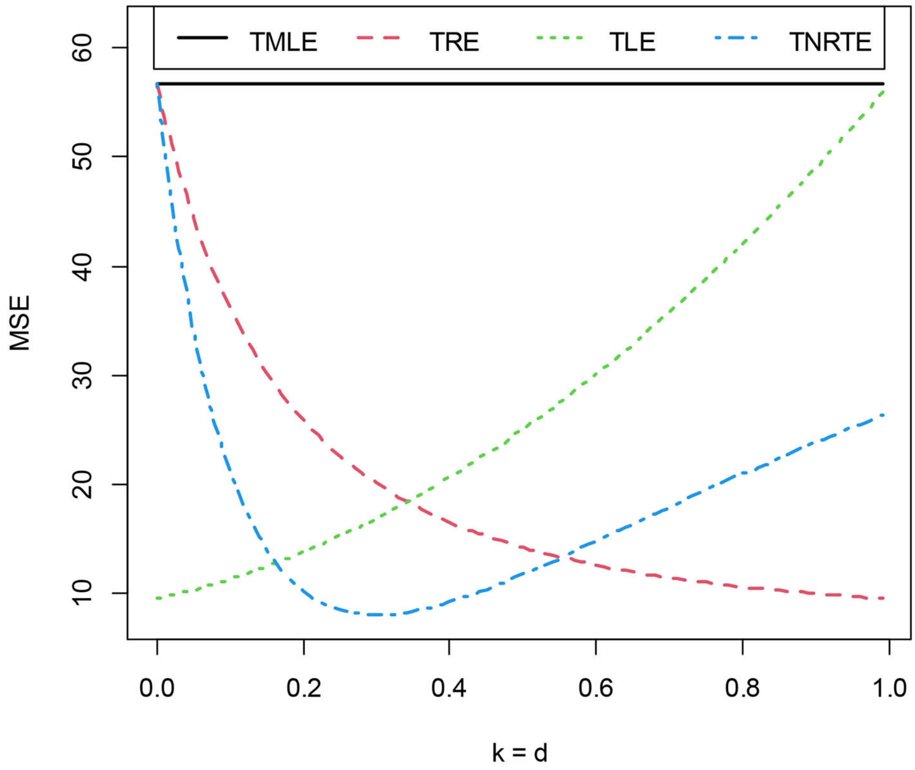

FIGURE 1 | MSE of TMLE, TRE, TLE, and TNRTE for diffrent *k, d*.

A REAL-LIFE DATA

In this section, we have the Mroz dataset that was originally adopted by Mroz [31] to clarify the performance of the proposed TNRTE and other mentioned estimators. The Mroz data contains 753 cases of married women with 21 variables, and the ages of these women range from 30 to 60 years. Three hundred twenty-five of the 753 cases from these women have an average wage of zero in an hour. Then, Barros et al. [32] considered the average hourly wage of the women as a dependent variable (y), while the independent variables are as follows: age of the women (x_1), education of the women (x_2), number of children <6 years (x_3), number of children between the ages 6 and 18 (x_4), and previous labor market experience of the women (x_5). With the method of Toker et al. [6], to examine the existence of multicollinearity, or not, the $\bar{W}$ matrix eigenvalues are given as 69,601.81, 1,723.52, 334.22, 54.43, 6.22, and 0.36, and the condition number is calculated as 441.09, and these results connote that there is high multicollinearity. The parameters and MSE are estimated and presented in **Table 8**.

Table 8 shows that the TMLE performs worse as expected. Also, the TRE has a near MSE value with the biasing parameter estimator $\hat{k}_M$ to that of the proposed TNRTE with biasing parameter estimator $\hat{k}_H$. Moreover, the proposed TNRTE has the lowest MSE value among the mentioned estimators (TRE and TLE), followed by TLE and then the TRE, when $k = d = 0.3$; this means that the proposed TNRTE is the best in this case.

Figure 1 shows that the proposed TNRTE with biasing parameter k from 0.18 to 0.58 performing better than other mentioned estimators, and when k equals 0.36, the proposed TNRTE has the least MSE; which means it is the best of all given estimators, while the TMLE performs the worst as expected.

CONCLUSIONS

In this study, we proposed the Tobit new ridge-type estimator (TNRTE) for overcoming the multicollinearity problem of the censored model. Theoretically, we compared the proposed TNRTE with some given estimators: the Tobit maximum likelihood estimator (TMLE), the Tobit ridge estimator (TRE), and the Tobit Liu estimator (TLE), and gave biasing parameter estimators of the proposed TNRTE. Then, a simulation study was performed to know the performance of the TMLE, the TRE, and the TLE with the proposed TNRTE. The results of the simulation indicate that the proposed TNRTE is better than other existing estimators in most cases. Moreover, real-life Mroz data were used to clarify the study results.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

ID, MA, and FA contributed to conception and structural design of the manuscript. MA performed the simulation and application. All authors contributed to manuscript revision, read, and approved the submitted version.

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