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Controllability of Hilfer fractional Langevin evolution equations

Haihua Wang* and Junhua Ku

College of Science, Qiongtai Normal University, Haikou, China

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EDITED BY
Zhouchao Wei,
China University of Geosciences Wuhan, China

REVIEWED BY
Savin Treanta,
Polytechnic University of Bucharest, Romania
Omar Abu Arqub,
Al-Balqa Applied University, Jordan

*CORRESPONDENCE
Haihua Wang
✉ wanghoiwan@163.com

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The existence of fractional evolution equations has attracted a growing interest in recent years. The mild solution of fractional evolution equations constructed by a probability density function was first introduced by El-Borai. Inspired by El-Borai, Zhou and Jiao gave a definition of mild solution for fractional evolution equations with Caputo fractional derivative. Exact controllability is one of the fundamental issues in control theory: under some admissible control input, a system can be steered from an arbitrary given initial state to an arbitrary desired final state. In this article, using the (α, β) resolvent operator and three different fixed point theorems, we discuss the control problem for a class of Hilfer fractional Langevin evolution equations. The exact controllability of Hilfer fractional Langevin systems is established. An example is also discussed to illustrate the results.

KEYWORDS

control, mild solution, existence, non-compactness, evolution

1. Introduction

The application of fractional differential equations to many engineering and scientific disciplines is very important, as numerous fractional-order derivatives are used in the mathematical modeling in the fields of physics, chemistry, electrodynamics of complex media, and polymer rheology, see [1–10]. Currently, fractional differential equations are used extensively in every branch of science, for example, the electrical closed loops can be expressed as fractional equations by Kirchhoff's law [11]. In 2000, Hilfer introduced the definition of Hilfer fractional derivative $D_{0+}^{\alpha,\beta}$. Especially, $D_{0+}^{\alpha,0}$ became the famous Riemman–Liouville fractional derivative whereas $D_{0+}^{\alpha,1}$ coincided with another fractional derivative, namely, the Caputo fractional derivative.

The study of fractional differential equations in infinite dimensional spaces includes the theoretical aspects, such as the existence and uniqueness of solutions, the numerical solutions, and so on. In general, it is interesting to find the existence of mild solutions, to arrive at the fact that, some technical tools, such as the method of lower and upper solutions and various fixed point theorems, are usually applied to the proof of existence.

The exact or approximate controllability is important in control theory. With some control input, a system can be guided from an initial state to any desired ultimate state. There are various articles with respect to the exact or approximate controllability of fractional differential equations [12–16]. However, a few articles have been written about the exact controllability of Hilfer fractional evolution equations.

Langevin first proposed a Brownian motion equation in 1908 and Langevin's equation was named so from then on. There have been a remarkably large number of frequently used theories to explain how physical phenomena evolve in fluctuating environments with respect to the Langevin equation. For example, if white noise is taken to be the random fluctuation force, Brownian motion can be well-described by the Langevin equation. More generally, if white noise is not taken to be the random fluctuation force, the generalized Langevin equation can be used to describe the particle's motion [17]. The formulation of Langevin equation is not unique. Currently, several versions of the conventional Langevin equation have been used in complex media to describe dynamical processes in a fractal medium, the reader can consult [18–21].

In 2012, Ahmad et al. [18] investigated the following fractional Langevin equation:

$$\begin{cases} {}^cD^{\beta}({}^cD^{\alpha}+\lambda)x(t)=f(t,x(t)), 0<t<1, 0<\alpha\leq 1, 1<\beta\leq 2,\\ x(0)=0, x(\eta)=0, x(1)=0,\end{cases}$$

where ${}^cD^{\alpha}$ denotes the Caputo fractional derivative, and the authors obtained the existence of solutions by Krasnoselskii's fixed point theorem and the Banach contraction mapping theory, respectively.

In 2018, Lv et al. [22] considered approximative controllability of Hilfer fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha,\beta}x(t)+Ax(t)=f(t,x(t))+(Bu)(t), t\in(0,b],\\ \lim\limits_{t\to 0+}\left(I_{0+}^{(1-\beta)(2-\alpha)}x\right)(t)=0, \lim\limits_{t\to 0+}\frac{\mathrm{d}}{\mathrm{d}t}\left(I_{0+}^{(1-\beta)(2-\alpha)}x\right)(t)=b_1,\end{cases}$$

where $D_{0+}^{\alpha,\beta}$ denotes the Hilfer fractional derivative, $A\in Sect(\theta)$, where $\theta\in\rho(A)\cap[0,(1-\frac{\alpha}{2})\pi)]$, and b_1 is an element in Banach space X. The control term $u\in L^p(J,U)$, the approximate controllability of the above system, was discussed.

Recently, Gou et al. [23] discussed the controllability of an impulsive evolution equation. They proved that the system is controllable on J under the Mönch fixed point theorem.

However, controllability of the Hilfer Langevin evolution equation has received little attention. For the above-mentioned aspects, we discuss the controllability for a class of Hilfer Langevin evolution equations of the form:

$$\begin{cases} D_{0+}^{\alpha_1,\beta_1}(D_{0+}^{\alpha_2,\beta_2}+A)x(t)=f(t,x(t))+Bu(t),\ t\in J=[0,b], b>0,\\ \lim\limits_{t\to 0+}\left(I_{0+}^{(1-\alpha_2)(1-\beta_2)}x\right)(t)=0, \lim\limits_{t\to 0+}\frac{\mathrm{d}}{\mathrm{d}t}\left(I_{0+}^{(1-\alpha_1)(1-\beta_1)+(1-\alpha_2)}x\right)\\ (t)+h(x)=x_0,\end{cases} \tag{1.1}$$

where $D_{0+}^{\alpha_i,\beta_i}$, $i=1,2$ denotes the Hilfer fractional derivative, respectively. $0<\alpha_i\leq 1$, $0\leq\beta_i\leq 1$, satisfies $1<\alpha_1+\alpha_2\leq 2$. A generates a strongly continuous (α_2,δ)-resolvent family $S_{\alpha_2,\delta}(t)$ $(t\geq 0)$, where $0<\delta\leq\alpha_1+\alpha_2$. The function $f:J\times E\to E$, let U be a Banach space, the control term $u\in L^2(J,U)$, $B:U\to E$ is linear and bounded.

This article aimed to study the controllability of system 1.1. The main approach is based on three different fixed point theorems and the properties of (α_2,δ)-resolvent operators. The structure of this article is given as follows: In Section 2, we list some notations, definitions, and preliminaries, which will be used in the next section. In Section 3, Theorem 3.1 is obtained without the compactness of the resolvent family, and Theorems 3.2 and 3.3 are obtained *via* compactness. Section 4 is devoted to illustrating the application of the results by an example.

2. Preliminaries and Lemmas

Throughout we let E be a Banach space with norm $\|\cdot\|$. The space $C(J,E)$ denotes the space of continuous functions on J and taking values in E, with the norm $\|x\|_C=\max_{t\in J}\|x(t)\|$, for $x\in C(J,E)$. We consider the $L^p(J,R^+)$ of Lebesgue p-integrable functions with $1<p<\infty$ on J, and let $\|f\|_{L^p}$ denote the norm of $L^p(J,R^+)$. Let $B(Y,X)$ denote the space of bounded linear operators from Y to X, $B(X)=B(X,X)$ for short. Let $A\in B(E)$, $\rho(A)$ is defined by the set of $\{\lambda:(\lambda I-A)^{-1}$ exists in $B(E)\}$.

Let $g_\gamma(\gamma>0)$ denote the function

$$g_\gamma(t)=\begin{cases}\frac{t^{\gamma-1}}{\Gamma(\gamma)}, & t>0,\\ 0, & t\leq 0.\end{cases}$$

For two given functions f_1 and f_2, the convolution of them is expressed in the form $(f_1*f_2)(t)=\int_0^t f_1(t-s)f_2(s)\,\mathrm{d}s$.

Definition 2.1. Li et al. [24] $\{S(t)\}_{t\geq 0}\subset B(E)$ is called exponentially bounded (EB) if there are constants $\omega\in R$ and $M>0$, such that

$$\|S(t)\|\leq M\mathrm{e}^{\omega t}, \text{for all } t>0.$$

ω or more precisely (M,ω) is called a type of $S(t)$.

Definition 2.2. Kilbas et al. [8] Let $\gamma>0$, the γ-order Riemann–Liouville fractional integral of function $f:[0,\infty)\to R$ is given by $I_{0+}^{\gamma}f(t)=(g_\gamma*f)(t)$, $t>0$.

Definition 2.3. Hilfer et al. [25] The Hilfer fractional derivative $D_{0+}^{\alpha_1,\beta_1}f(t)$ of order $\alpha_1\in(n-1,n]$ and type $\beta_1\in[0,1]$ is defined by

$$D_{0+}^{\alpha_1,\beta_1}f(t)=\left(I_{0+}^{\beta_1(n-\alpha_1)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(I_{0+}^{(1-\beta_1)(n-\alpha_1)}f\right)\right)(t).$$

If f is taking values in E, then the corresponding integrals of the above two definitions are given in the sense of Bochner.

Lemma 2.1. Hilfer [6] Let $f\in L(0,b)$, $n-1<\alpha_1\leq n$, $0\leq\beta_1\leq 1$, and $I_{0+}^{(1-\beta_1)(n-\alpha_1)}f\in AC^k[0,b]$, then

$$\begin{aligned}&\left(I_{0+}^{\alpha_1}D_{0+}^{\alpha_1,\beta_1}f\right)(t)=f(t)\\ &-\sum_{k=0}^{n-1}\frac{(t-s)^{k-(n-\alpha_1)(1-\beta_1)}}{\Gamma(k-(n-\alpha_1)(1-\beta_1)+1)}\lim_{t\to 0+}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(I_{0+}^{(1-\beta_1)(n-\alpha_1)}f\right)(t).\end{aligned}$$

Definition 2.4. Chang et al. [26] Let A be a closed linear operator in Banach space E with domain $D(A)\subset E$. Assume that $\alpha,\beta>0$, A is called the generator of the resolvent family (α,β), if there exists an $\omega\geq 0$ and $S_{\alpha,\beta}$ is strongly continuous from $[0,\infty)$ to $B(E)$, such that $S_{\alpha,\beta}(t)$ is EB, $\{\lambda^\alpha:(\lambda^\alpha I-A)^{-1}$ exists in $B(E)$, $\mathrm{Re}\lambda>\omega\}$,

$$\lambda^{\alpha-\beta}(\lambda^\alpha I-A)^{-1}x=\int_0^\infty \mathrm{e}^{-\lambda t}S_{\alpha,\beta}(t)x\,\mathrm{d}t, \mathrm{Re}\lambda>\omega, x\in E. \tag{2.1}$$

Then $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is called the resolvent family (α,β) generated by operator A. It is simply said that $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is generated by operator A.

Lemma 2.2. Li et al. [24] Let $\alpha,\beta>0$ and $\{S_{\alpha,\beta}(t)\}_{t\geq 0}\subset B(E)$ is generated by operator A. Then, the main properties of $S_{\alpha,\beta}(t)$ are as per the following:

(i) For $t\geq 0$ and $x\in D(A)$, we have $S_{\alpha,\beta}(t)x\in D(A)$. Moreover, $S_{\alpha,\beta}(t)Ax=AS_{\alpha,\beta}(t)x$;

(ii) For $x \in E$, $t \geq 0$, we have $\int_0^t g_\alpha(t-s)S_{\alpha,\beta}(s)x\,\mathrm{d}s \in D(A)$, and

$$S_{\alpha,\beta}(t)x = g_\beta(t)x + A\int_0^t g_\alpha(t-s)S_{\alpha,\beta}(s)x\,\mathrm{d}s;$$

moreover, if $x \in D(A)$, then the second term on the right-hand side of the above equality can be replaced by

$$\int_0^t g_\alpha(t-s)AS_{\alpha,\beta}(s)x\,\mathrm{d}s.$$

Theorem 2.1. Ponce [27] Let $\alpha > 0$, $1 < \beta \leq 2$. Assume that $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is generated by operator A. Then for $t > 0$, $S_{\alpha,\beta}(t)$ is continuous in $B(E)$.

Lemma 2.3. Ponce [27] $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is generated by operator A and (M,ω) is a type of $S_{\alpha,\beta}(t)$. Then for $\gamma > 0$, $\{S_{\alpha,\beta+\gamma}(t)\}_{t\geq 0}$ is generated by operator A and $(M/\omega^\gamma, \omega)$ is a type of $S_{\alpha,\beta+\gamma}(t)$.

Definition 2.5. Ponce [27] If $S_{\alpha,\beta}(t)$ is a compact operator for all $t > 0$, then we call the resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ as compact.

Theorem 2.2. Ponce [27] Let $\alpha > 0$, $1 < \beta \leq 2$, $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is generated by operator A and (M,ω) is a type of $S_{\alpha,\beta}(t)$, and the following two conclusions are equivalent:

(i) For $t > 0$, $S_{\alpha,\beta}(t)$ is compact.
(ii) For $\mu > \omega^{1/\alpha}$, $(\mu I - A)^{-1}$ is compact.

Lemma 2.4. Let $\alpha > 0$, $0 < \beta \leq 1$, $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is generated by operator A and (M,ω) is a type of $S_{\alpha,\beta}(t)$. For $t > 0$, $S_{\alpha,\beta}(t)$ is uniform continuous. Then the following two conclusions are equivalent:

(i) For $t > 0$, $S_{\alpha,\beta}(t)$ is a compact operator.
(ii) For $\mu > \omega^{1/\alpha}$, $(\mu I - A)^{-1}$ is compact.

Proof. If (i) is true, for $\lambda > \omega$. Then we obtain

$$\lambda^{\alpha-\beta}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty \mathrm{e}^{-\lambda t}S_{\alpha,\beta}(t)x\,\mathrm{d}t,$$

from Definition 2.4. However, note that $\{S_{\alpha,\beta}(t)\}_{t>0}$ is uniform continuous by our hypothesis, where we can see that $(\lambda^\alpha I - A)^{-1}$ is compact using Lemma 2.1 in Chang et al. [26].

On the contrary, for every fixed $t > 0$, let $0 \leq \beta \leq 1$. For $g_{\frac{\beta}{2}} \in L^1_{loc}[0,\infty)$ and therefore, by proposition in Haase [28], we obtain

$$\lim_{N\to\infty}\frac{1}{2\pi i}\int_{\omega-iN}^{\omega+iN} e^{\lambda t}(L(g_{\frac{\beta}{2}} * S_{\alpha,\frac{\beta}{2}}))(\lambda)d\lambda = g_{\frac{\beta}{2}} * S_{\alpha,\frac{\beta}{2}} = S_{\alpha,\beta}(t),$$

in $B(E)$. Hence, for $t > 0$,

$$\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-\beta}(\lambda^\alpha I - A)^{-1}d\lambda = S_{\alpha,\beta}(t),$$

where Γ is a vertical path lying in $Re(z) = \omega$. By Lemma 2.4 and hypothesis, we observe for $t > 0$, $S_{\alpha,\beta}(t)$ is compact.

The definition and some Lemmas of Hausdorff measure of non-compactness can be found in Banas and Goebel [29], Deimling [30], Guo and Sun [31], and Lakshmikantham and Leela [32], so we omit their details here.

Lemma 2.5. Let $\alpha > 0$, $\beta > 1$, $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is generated by operator A and $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is strongly continuous. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}S_{\alpha,\beta}(t)x = S_{\alpha,\beta-1}(t)x, \text{for } t \in J, x \in E. \tag{2.2}$$

Proof. Using (2.1), we have for $t \geq 0$,

$$\lambda^{\alpha-\beta}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty \mathrm{e}^{-\lambda t}S_{\alpha,\beta}(t)x\,\mathrm{d}t, \mathrm{Re}\lambda > \omega, \tag{2.3}$$

$$\lambda^{\alpha-\beta+1}(\lambda^\alpha I - A)^{-1}x\int_0^\infty \mathrm{e}^{-\lambda t}S_{\alpha,\beta-1}(t)x\,\mathrm{d}t, \mathrm{Re}\lambda > \omega. \tag{2.4}$$

(2.3) and (2.4) together imply

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\lambda t}S_{\alpha,\beta}(t)x\,\mathrm{d}t &= \lambda^{\alpha-\beta}(\lambda^\alpha I - A)^{-1}x\\ &= \frac{1}{\lambda}\lambda^{\alpha-\beta+1}(\lambda^\alpha I - A)^{-1}x\\ &= \frac{1}{\lambda}\int_0^\infty \mathrm{e}^{-\lambda t}S_{\alpha,\beta-1}(t)x\,\mathrm{d}t\\ &= \int_0^\infty \mathrm{e}^{-\lambda t}(g_1 * S_{\alpha,\beta-1})(t)x\,\mathrm{d}t\\ &= \int_0^\infty \mathrm{e}^{-\lambda t}\left(\int_0^t S_{\alpha,\beta-1}(s)ds\right)x\,\mathrm{d}t,\end{aligned}$$

It is easy to see that $S_{\alpha,\beta}(t) = \int_0^t S_{\alpha,\beta-1}(s)\,\mathrm{d}s$, then we obtain (2.2) is true.

Remark 2.1. If $\beta = 2$ or $\beta = \alpha$, the corresponding results can be found in Gou and Li [23].

Lemma 2.6. Let $0 < \delta \leq \alpha_1 + \alpha_2$, $\{S_{\alpha_2,\delta}\}_{t\geq 0}$ is generated by operator $-A$. Suppose that $x \in C(J,E)$, if for $t \in J$, $x(t) \in D(-A)$ satisfies problem (1.1) and $Ax \in L^1((0,b),E)$, then we have

$$x(t) = ((g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta}) * (\bar{f} + Bu))(t) + (g_{\gamma_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t)(x_0 - h(x)), \tag{2.5}$$

where $\bar{f}(t) = f(t,x(t))$.

Proof. Using Liouville operators with $I^{\alpha_1}_{0+}$ on both sides of the equation

$$D^{\alpha_1,\beta_1}_{0+}(D^{\alpha_2,\beta_2}_{0+} + A)x(t) = f(t,x(t)) + Bu(t),$$

in view of Lemma 2.1, we obtain

$$(D^{\alpha_2,\beta_2}_{0+} + A)x(t) = I^{\alpha_1}_{0+}(\bar{f} + Bu)(t) + \frac{c_0}{\Gamma(\gamma_1)}t^{\gamma_1-1}, \tag{2.6}$$

where $\gamma_1 = \alpha_1 + \beta_1 - \alpha_1\beta_1$. Using Liouville operators with $I^{\alpha_2}_{0+}$ on both sides of equation (2.6) again, we obtain

$$\begin{aligned}x(t) &= I^{\alpha_1+\alpha_2}_{0+}(\bar{f} + Bu)(t) - I^{\alpha_2}_{0+}(Ax)(t) + \tfrac{c_0}{\Gamma(\gamma_1+\alpha_2)}t^{\gamma_1+\alpha_2-1}\\ &\quad + \tfrac{c_1}{\Gamma(\gamma_2)}t^{\gamma_2-1},\end{aligned} \tag{2.7}$$

where $\gamma_2 = \alpha_2 + \beta_2 - \alpha_2\beta_2$. In view of the condition, we obtain $c_0 = x_0 - h(x)$ and $c_1 = 0$. Then we rewrite the representation of (2.7) as

$$\begin{aligned}x(t) &= (g_{\alpha_1+\alpha_2} * \bar{f})(t) + (g_{\alpha_1+\alpha_2} * Bu)(t) - (g_{\alpha_2} * Ax)(t)\\ &\quad + \tfrac{x_0-h(x)}{\Gamma(\gamma_1+\alpha_2)}t^{\gamma_1+\alpha_2-1}.\end{aligned} \tag{2.8}$$

Applying the Laplace transform to (2.8), we obtain

$$(Lx)(\lambda) = \frac{1}{\lambda^{\alpha_1+\alpha_2}}(L\bar{f})(\lambda) + \frac{1}{\lambda^{\alpha_1+\alpha_2}}(LBu)(\lambda) - \frac{1}{\lambda^{\alpha_2}}A(Lx)(\lambda) + (x_0 - h(x))\frac{1}{\lambda^{\gamma_1+\alpha_2}}.$$

Thus, we obtain

$$(Lx)(\lambda) = \frac{1}{\lambda^{\alpha_1}}(\lambda^{\alpha_2}I + A)^{-1}(L\bar{f})(\lambda) + \frac{1}{\lambda^{\alpha_1}}(\lambda^{\alpha_2}I + A)^{-1}(LBu)(\lambda) + (x_0 - h(x))\frac{1}{\lambda^{\gamma_1}}(\lambda^{\alpha_2}I + A)^{-1}.$$

Currently, by Definition 2.4, we can apply the inverse Laplace transform to the above equation, therefore

$$\begin{aligned} x(t) &= ((S_{\alpha_2,\alpha_1+\alpha_2}) * (\bar{f} + Bu))(t) + S_{\alpha_2,\gamma_1+\alpha_2}(t)(x_0 - h(x)) \\ &= ((g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta}) * (\bar{f} + Bu))(t) \\ &+ (g_{\gamma_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t)(x_0 - h(x)). \end{aligned}$$

Definition 2.6. Let $0 < \delta \le \alpha_1 + \alpha_2$, $\{S_{\alpha_2,\delta}(t)\}_{t\ge0}$ is generated by $-A$. We say that $x(t)$ is a mild solution of (1.1) if $\lim_{t\to0+}\frac{\mathrm{d}}{\mathrm{d}t}(I_{0+}^{(1-\alpha_1)(1-\beta_1)+(1-\alpha_2)}x)(t) + h(x) = x_0$, $\lim_{t\to0+}(I_{0+}^{(1-\alpha_2)(1-\beta_2)}x)(t) = 0$, $x(\cdot) \in C(J,E)$ satisfies the equation

$$x(t) = ((g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta}) * (\bar{f} + Bu))(t) + (g_{\gamma_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t)(x_0 - h(x)).$$

3. Main results

Let x be an arbitrary function in $C(J,E)$, which we denote by $x_b = x(b)$ during the final stages at time b in E.

Definition 3.1. Let the initial condition $x_0 \in E$ and final stages $x_b \in E$, if there exists a control term $u \in L^2(J,U)$, such that $x(t)$ is the mild solution of (1.1) with respect to u, which satisfies

$$\lim_{t\to0+}\left(I_{0+}^{(1-\alpha_2)(1-\beta_2)}x\right)(t) = 0,\ \lim_{t\to0+}\frac{\mathrm{d}}{\mathrm{d}t}\left(I_{0+}^{(1-\alpha_1)(1-\beta_1)+(1-\alpha_2)}x\right)(t) + h(x) = x_0$$

and $x(b) = x_b$, then we say that system (1.1) can be controlled on J.

Theorem 3.1. Let $0 < \delta \le \alpha_1 + \alpha_2$, $\{S_{\alpha_2,\delta}(t)\}_{t\ge0}$ is generated by operator $-A$ and (M,ω) is a type of $S_{\alpha_2,\delta}(t)$. Assume that the following conditions are satisfied:

(H1) $f: J\times E \to E$ satisfies the Carathéodory conditions.

(H2) There exist $q_1 \in [0,1)$ and two functions $m \in L^{\frac{1}{q_1}}(J,R^+)$, $\Phi \in C(R^+,R^+)$ which are non-decreasing that satisfy

$$\|f(t,x)\| \le m(t)\Phi(\|x\|), \text{for } x \in E, \text{a.e. } t \in J.$$

(H3) There exist $q_2 \in [0,1)$ and a function $n \in L^{\frac{1}{q_2}}(J,R^+)$, such that for every bounded set D in E,

$$\alpha\left(f(t,D)\right) \le n(t)\alpha(D), \text{for a.e. } t \in J.$$

(H4) (i) The function $h: C(J,E) \to E$ and there exist $c_1, c_2 \ge 0$, such that

$$\|h(x)\| \le c_1, \|h(x) - h(y)\| \le c_2\|x - y\|, x \in C(J,E);$$

(ii) There exists $l > 0$, such that for every bounded subset D in E,

$$\alpha(h(D)) \le l\alpha(D),$$

(H5) $W: L^2(J,U) \to E$ is a linear operator, which is given by

$$Wu = \int_0^b S_{\alpha_2,\alpha_1+\alpha_2}(b-s)Bu(s)ds, u = u^x,$$

where u^x is defined in (3.4).

(i) The inverse operator $W^{-1}: E \to L^2(J,U)\backslash\ker W$ exists, if there exist $M_1 > 0$, $M_2 > 0$, such that $\|B\| \le M_1$, $\|W^{-1}\| \le M_2$;

(ii) There exist $q_3 \in [0,1)$ and a function $K \in L^{\frac{1}{q_3}}(J,R^+)$, such that for every bounded subset D in E,

$$\alpha\left(W^{-1}(D)(t)\right) \le K(t)\alpha(D), t \in J.$$

Assume that $\max\{\Lambda_1,\Lambda_2\} < 1$, where

$$\Lambda_1 = \frac{M}{\omega^{\alpha_1+\alpha_2-\delta}}\left(1 + \frac{MM_1M_2}{\omega^{\alpha_1+\alpha_2-\delta}}\frac{1}{\sqrt{2\omega}}e^{\omega b}\right)\left(\frac{1-q_1}{\omega}\right)^{1-q_1} e^{\omega b}\|m\|_{L^{\frac{1}{q_1}}}\liminf_{r\to+\infty}\frac{\Phi(r)}{r}, \tag{3.1}$$

$$\Lambda_2 = Me^{\omega b}\left[1 + \frac{2MM_1}{\omega^{\alpha_1+\alpha_2-\delta}}\left(\frac{1-q_3}{\omega}\right)^{1-q_3}e^{\omega b}\|K\|_{L^{\frac{1}{q_3}}}\right] \times \left[\frac{1}{\omega^{\gamma_1+\alpha_2-\delta}}l + \frac{2}{\omega^{\alpha_1+\alpha_2-\delta}}\left(\frac{1-q_2}{\omega}\right)^{1-q_2}\|n\|_{L^{\frac{1}{q_2}}}\right], \tag{3.2}$$

then system (1.1) can be controlled on J.

Proof. Let us consider operator T in $C(J,E)$ as follows:

$$\begin{aligned}(Tx)(t) &= ((g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta}) * (\bar{f} + Bu))(t) \\ &+ (g_{\alpha_2+\gamma_2-\delta} * S_{\alpha_2,\delta})(t)(x_0 - h(x)), t \in J,\end{aligned} \tag{3.3}$$

where the control term u is given by $u(t) = u^x(t)$, $x \in C(J,E)$ is given by

$$u^x(t) = W^{-1}\Big[x_b - (g_{\alpha_2+\gamma_2-\delta} * S_{\alpha_2,\delta})(b)(x_0 - h(x)) - ((g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta}) * \bar{f})(b)\Big](t). \tag{3.4}$$

Taking the control (3.4) in (3.3), we obtain $(Tx)(b) = x_b$. Next, we illustrate that the non-linear operator T has a fixed point.

Step 1: $T(B_r) \subset B_r$ for some positive number r

If not, then for every $r > 0$, there exist $x_r \in B_r$ and $t_r \in J$, such that $\|(Tx_r)(t_r)\| > r$. First, we observe that

$$\begin{aligned}\|(Tx)(t)\| \le& \int_0^t \|(g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t-s)\|_B \|f(s,x(s))\|\,ds\\ &+ \|(g_{\gamma_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t)\|_B \|(x_0-h(x))\|\\ &+ \int_0^t \|(g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t-s)\|_B \|Bu^x(s)\|\,ds\\ \le& \int_0^t \frac{Me^{\omega(t-s)}}{\omega^{\alpha_1+\alpha_2-\delta}} m(s)\Phi(\|x(s)\|)\,ds\\ &+ \frac{Me^{\omega t}}{\omega^{\gamma_1+\alpha_2-\delta}}\|(x_0-h(x))\|\\ &+ \int_0^t \frac{Me^{\omega(t-s)}}{\omega^{\alpha_1+\alpha_2-\delta}} M_1\|u^x(s)\|\,ds\\ \le& \frac{M}{\omega^{\alpha_1+\alpha_2-\delta}}\int_0^t e^{\omega(t-s)} m(s)\Phi(\|x(s)\|)\,ds\\ &+ \frac{Me^{\omega t}}{\omega^{\gamma_1+\alpha_2-\delta}}(\|x_0\|+c_1)\\ &+ \frac{MM_1}{\omega^{\alpha_1+\alpha_2-\delta}}\left[\frac{1}{2\omega}\left(e^{2\omega t}-1\right)\right]^{\frac{1}{2}}\|u^x\|_{L^2},\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}\|u^x\|_{L^2} \le M_2\Bigg[&\|x_b\| + \frac{Me^{\omega b}}{\omega^{\gamma_1+\alpha_2-\delta}}(\|x_0\|+c_1)\\ &+ \frac{M}{\omega^{\alpha_1+\alpha_2-\delta}}\int_0^b e^{\omega(b-s)} m(s)\Phi(\|x(s)\|)\,ds\Bigg].\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we conclude that

$$\begin{aligned}\|(Tx)(t)\| \le& M\left\{1+\frac{MM_1M_2}{\omega^{\alpha_1+\alpha_2-\delta}}\left[\frac{1}{2\omega}\left(e^{2\omega b}-1\right)\right]^{\frac{1}{2}}\right\}\\ &\left[\frac{e^{\omega b}}{\omega^{\gamma_1+\alpha_2-\delta}}(\|x_0\|+c_1)\right.\\ &\left.+\frac{1}{\omega^{\alpha_1+\alpha_2-\delta}}\int_0^b e^{\omega(b-s)} m(s)\Phi(\|x(s)\|)\,ds\right]\\ &+\frac{MM_1M_2}{\omega^{\alpha_1+\alpha_2-\delta}}\left[\frac{1}{2\omega}\left(e^{2\omega b}-1\right)\right]^{\frac{1}{2}}\|x_b\|.\end{aligned}$$

Consequently,

$$\begin{aligned}r <& \|(Tx_r)(t_r)\| \le M\left\{1+\frac{MM_1M_2}{\omega^{\alpha_1+\alpha_2-\delta}}\right.\\ &\left.\left[\frac{1}{2\omega}\left(e^{2\omega b}-1\right)\right]^{\frac{1}{2}}\right\}\left\{\frac{e^{\omega b}}{\omega^{\gamma_1+\alpha_2-\delta}}(\|x_0\|+c_1)\right.\\ &\left.+\frac{\Phi(r)}{\omega^{\alpha_1+\alpha_2-\delta}}\left[\frac{1-q_1}{\omega}\left(e^{\frac{\omega}{1-q_1}b}-1\right)\right]^{1-q_1}\|m\|_{L^{\frac{1}{q_1}}}\right\}\\ &+\frac{MM_1M_2}{\omega^{\alpha_1+\alpha_2-\delta}}\left[\frac{1}{2\omega}\left(e^{2\omega b}-1\right)\right]^{\frac{1}{2}}\|x_b\|\\ \le& M\left(1+\frac{MM_1M_2}{\omega^{\alpha_1+\alpha_2-\delta}}\frac{1}{\sqrt{2\omega}}e^{\omega b}\right)\left[\frac{e^{\omega b}}{\omega^{\gamma_1+\alpha_2-\delta}}(\|x_0\|+c_1)\right.\\ &\left.+\frac{\Phi(r)}{\omega^{\alpha_1+\alpha_2-\delta}}\left(\frac{1-q_1}{\omega}\right)^{1-q_1}e^{\omega b}\|m\|_{L^{\frac{1}{q_1}}}\right]\\ &+\frac{MM_1M_2}{\omega^{\alpha_1+\alpha_2-\delta}}\frac{1}{\sqrt{2\omega}}e^{\omega b}\|x_b\|.\end{aligned} \tag{3.7}$$

Dividing (3.7) by r and passing to the lower limit as $r \to +\infty$ yield

$$\begin{aligned}&\frac{M}{\omega^{\alpha_1+\alpha_2-\delta}}\left(1+\frac{MM_1M_2}{\omega^{\alpha_1+\alpha_2-\delta}}\frac{1}{\sqrt{2\omega}}e^{\omega b}\right)\left(\frac{1-q_1}{\omega}\right)^{1-q_1}\\ &e^{\omega b}\|m\|_{L^{\frac{1}{q_1}}}\liminf_{r\to+\infty}\frac{\Phi(r)}{r} \ge 1,\end{aligned}$$

which contradicts $\Lambda_1 < 1$. Hence, $T(B_r) \subset B_r$ for some $r > 0$.

Step 2: $T : B_r \to B_r$ is continuous.

Assume that $\{x_n\} \subset B_r$ satisfying $x_n \to x$. Let us show that $\|Tx_n - Tx\|_C \to 0$. For this, we consider the inequality

$$\begin{aligned}\|(Tx_n)(t)-(Tx)(t)\| \le& \int_0^t \|(g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t-s)\|_B\|\bar{f}_n(s)\\ &-\bar{f}(s)\|\,ds + \int_0^t \|(g_{\alpha_1+\alpha_2-\delta}\\ &* S_{\alpha_2,\delta})(t-s)\|_B\|Bu^{x_n}(s)-Bu^x(s)\|\,ds\\ &+\|(g_{\gamma_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t)\|_B\|h(x_n)-h(x)\|\\ \le& \frac{M}{\omega^{\alpha_1+\alpha_2-\delta}}\int_0^t e^{\omega(t-s)}\|\bar{f}_n(s)-\bar{f}(s)\|\,ds\\ &+\frac{MM_1}{\omega^{\alpha_1+\alpha_2-\delta}}\left[\frac{1}{2\omega}\left(e^{2\omega t}-1\right)\right]^{\frac{1}{2}}\|u^{x_n}-u^x\|_{L^2}\\ &+\frac{Me^{\omega t}}{\omega^{\gamma_1+\alpha_2-\delta}}c_2\|x_n-x\|,\end{aligned} \tag{3.8}$$

where $\bar{f}_n(t) = f(t, x_n(t))$ and

$$\begin{aligned}\|u^{x_n}-u^x\|_{L^2} \le M_2\Bigg[&\frac{Me^{\omega b}}{\omega^{\gamma_1+\alpha_2-\delta}}c_2\|x_n-x\|_C\\ &+\frac{M}{\omega^{\alpha_1+\alpha_2-\delta}}\int_0^b e^{\omega(b-s)}\|\bar{f}_n(s)-\bar{f}(s)\|\,ds\Bigg].\end{aligned} \tag{3.9}$$

By means of the Lebesgue dominated convergence theorem and condition (H1), together with (3.8) and (3.9), proves that $\|Tx_n - Tx\|_C \to 0$ as $n \to \infty$.

Step 3: T satisfies conditions of the Mönch fixed point theorem.

Let D be a countable subset in B_r satisfying D is a subset in the closed convex hull of $\{0\} \cup T(D)$, and we will later prove $\alpha(D) = 0$. Assume, without loss of generality, that $D = \{x_n\}_{n=1}^{\infty} \subset B_r$, let $0 \le t_1 < t_2 \le b$, then

$$\begin{aligned}&\|(Tx_n)(t_1)-(Tx_n)(t_2)\|\\ &\le \int_0^{t_1}\|(g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t_2-s)-(g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t_1-s)\|_B\\ &\times\|\bar{f}(s)+Bu^{x_n}(s)\|\,ds\\ &+\int_{t_1}^{t_2}\|(g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t_2-s)\|_B\|\bar{f}(s)+Bu^{x_n}(s)\|\,ds\\ &+\|(g_{\gamma_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t_2)-(g_{\gamma_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t_1)\|_B\|(x_0-h(x_n))\|.\end{aligned}$$

By Lemma 2.5, $(g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t) = S_{\alpha_2,\alpha_1+\alpha_2}(t)$ and $(g_{\gamma_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t) = S_{\alpha_2,\gamma_1+\alpha_2}(t)$ for $t \ge 0$. Furthermore, by Theorem 2.1, we obtain $S_{\alpha_2,\alpha_2+\alpha_1}(t)$ and $S_{\alpha_2,\gamma_1+\alpha_2}(t)$ which are norm continuous. Since the right-hand side of the inequality approaches zero as $t_2 \to t_1$, $T(D)$ is equicontinuous on J.

Using the properties of the measure of non-compactness in Deimling [30], Lakshmikantham and Leela [32],

$$\begin{aligned}\alpha\left(Tx_n(t)\right) \leq{}& \alpha\left(\left\{((g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta}) * \bar{f}(s))(t)\right\}\right)\\ &+\alpha\left(\left\{((g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta}) * Bu^{x_n})(t)\right\}\right)\\ &+\alpha\left(\left\{(g_{\gamma_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t)(h(x_n))\right\}\right)\\ \leq{}& \frac{2M}{\omega^{\alpha_1+\alpha_2-\delta}}\int_0^t e^{\omega(t-s)}n(s)\,\mathrm{d}s\alpha\left(\{x_n\}\right)\\ &+\frac{2MM_1}{\omega^{\alpha_1+\alpha_2-\delta}}\int_0^t e^{\omega(t-s)}\alpha\left(\{u^{x_n}(s)\}\right)\,\mathrm{d}s\\ &+\frac{Me^{\omega t}}{\omega^{\gamma_1+\alpha_2-\delta}}l\alpha\left(\{x_n\}\right),\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}\alpha\left(\left\{u^{x_n}(s)\right\}_{n=1}^{\infty}\right) \leq{}& K(s)\Bigg[\alpha\left(\left\{(g_{\gamma_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(b)h(x_n)\right\}\right)\\ &+\alpha\left(\left\{((g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta}) * \bar{f}_n)(b)\right\}\right)\Bigg]\\ \leq{}& K(s)\alpha\left(\{x_n\}_{n=1}^{\infty}\right)\Bigg(\frac{Me^{\omega b}}{\omega^{\gamma_1+\alpha_2-\delta}}l\\ &+\frac{2M}{\omega^{\alpha_1+\alpha_2-\delta}}\int_0^b e^{\omega(b-s)}n(s)\,\mathrm{d}s.\end{aligned} \tag{3.11}$$

By (3.10) and (3.11), we obtain

$$\begin{aligned}\alpha\left(Tx_n(t)\right) \leq{}& \frac{2M}{\omega^{\alpha_1+\alpha_2-\delta}}\alpha(D)\left[\frac{1-q_2}{\omega}\left(e^{\frac{\omega}{1-q_2}t}-1\right)\right]^{1-q_2}\|n\|_{L^{\frac{1}{q_2}}}\\ &+\frac{2MM_1}{\omega^{\alpha_1+\alpha_2-\delta}}\alpha(D)\Bigg(\frac{Me^{\omega b}}{\omega^{\gamma_1+\alpha_2-\delta}}l\\ &+\frac{2M}{\omega^{\alpha_1+\alpha_2-\delta}}\int_0^b e^{\omega(b-s)}n(s)ds\\ &\times\int_0^t e^{\omega(t-s)}K(s)ds+\frac{Me^{\omega t}}{\omega^{\gamma_1+\alpha_2-\delta}}l\alpha(D)\\ \leq{}& \frac{2M}{\omega^{\alpha_1+\alpha_2-\delta}}\alpha(D)\left(\frac{1-q_2}{\omega}\right)^{1-q_2}e^{\omega t}\|n\|_{L^{\frac{1}{q_2}}}\\ &+\frac{2M^2M_1}{\omega^{\alpha_1+\alpha_2-\delta}}\alpha(D)\Bigg[\frac{e^{\omega b}}{\omega^{\gamma_1+\alpha_2-\delta}}l\\ &+\frac{2}{\omega^{\alpha_1+\alpha_2-\delta}}\left(\frac{1-q_2}{\omega}\right)^{1-q_2}e^{\omega b}\|n\|_{L^{\frac{1}{q_2}}}\\ &\times\left(\frac{1-q_3}{\omega}\right)^{1-q_3}e^{\omega t}\|K\|_{L^{\frac{1}{q_3}}}+\frac{Me^{\omega t}}{\omega^{\gamma_1+\alpha_2-\delta}}l\alpha(D)\\ \leq{}& Me^{\omega b}\left[1+\frac{2MM_1}{\omega^{\alpha_1+\alpha_2-\delta}}\left(\frac{1-q_3}{\omega}\right)^{1-q_3}e^{\omega b}\|K\|_{L^{\frac{1}{q_3}}}\right]\\ &\times\left[\frac{1}{\omega^{\gamma_1+\alpha_2-\delta}}l+\frac{2}{\omega^{\alpha_1+\alpha_2-\delta}}\left(\frac{1-q_2}{\omega}\right)^{1-q_2}\|n\|_{L^{\frac{1}{q_2}}}\right]\\ &\alpha(D),\end{aligned}$$

we have

$$\alpha(TD) \leq \Lambda_2\alpha(D).$$

Thus, by condition of the Mönch fixed point theorem, we obtain

$$\alpha(D) \leq \alpha\left(\overline{co}(\{0\}\cup T(D))\right).$$

We obtain $\alpha(D) = 0$ for $\Lambda_2 < 1$. Applying the Mönch fixed point theorem, we know that there exists a fixed point $x \in B_r$ of T, which, of course, is a mild solution of 1.1 and satisfies $x(b) = x_b$. Hence, system 1.1 can be controlled on J.

Theorem 3.2. Let $0 < \delta \leq \alpha_1 + \alpha_2$, $\left\{S_{\alpha_2,\delta}(t)\right\}_{t\geq 0}$ is generated by operator $-A$ and (M,ω) is a type of $S_{\alpha_2,\delta}(t)$. In addition to assumptions (H1), (H2), (H4)(i), and (H5)(i) of Theorem 3.1, we suppose that the following assumptions hold:

(H6) $(\lambda^{\alpha_2}I + A)^{-1}$ is compact for all $\lambda > \omega^{1/\alpha_2}$.

If $\max\{\Lambda_1, \Lambda_3\} < 1$, where $\Lambda_3 = \frac{Me^{\omega b}}{\omega^{\gamma_1+\alpha_1-\delta}}c_2$, then (1.1) can be controlled on J.

Proof. We define two operators T_1, T_2 in $C(J,E)$ as follows:

$$(T_1x)(t) = \int_0^t ((g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta}) * (\bar{f} + Bu)(t), t \in J, \tag{3.12}$$

$$(T_2x)(t) = (g_{\gamma_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t)(x_0 - h(x)), t \in J. \tag{3.13}$$

As in Step 1 of Theorem 3.1, we can find $r > 0$, such that $T_1x + T_2y \in B_r$ for $x, y \in B_r$. Moreover, with a similar method used in Step 2 of Theorem 3.1, it follows that T_1 is continuous on B_r and T_2 is a contraction on B_r. Currently, we are going to illustrate that $\{T_1x : x \in B_r\}$ is precompact. The uniformly bounded nature of $\{T_1x : x \in B_r\}$ is obvious.

Step 1: $\{T_1x : x \in B_r\}$ is an equicontinuous family.

For $x \in B_r$, without loss of generality, we assume that $0 \leq t_1 < t_2 \leq b$, then

$$\begin{aligned}&\|(T_1x)(t_1) - (T_1x)(t_2)\|\\ &\leq \int_0^{t_1}\|(g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t_2-s) - (g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t_1-s)\|_B\\ &\times\|\bar{f}(s) + Bu^x(s)\|\,\mathrm{d}s\\ &+\int_{t_1}^{t_2}\|(g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t_2-s)\|_B\|\bar{f}(s) + Bu^x(s)\|\,\mathrm{d}s := I_1 + I_2.\end{aligned}$$

For I_1, we have

$$\begin{aligned}I_1 \leq{}& \left(\Phi(r)\|m\|_{L^{\frac{1}{q_1}}} + +M_1\|u^x\|_{L^2}\right)\Bigg(\int_0^{t_1}\|(g_{\alpha_1+\alpha_2-\delta}\\ &* S_{\alpha_2,\delta})(t_2-s) - (g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t_1-s)\|_B^{\frac{1}{1-q_1}}\,\mathrm{d}s\Bigg)^{1-q_1}.\end{aligned} \tag{3.14}$$

By Theorem 2.1, we have the norm continuity of $S_{\alpha_2,\alpha_2+\alpha_1}(t)$ and therefore if $t_2 \to t_1$, then $S_{\alpha_2,\alpha_2+\alpha_1}(t_2-s) - S_{\alpha_2,\alpha_2+\alpha_1}(t_1-s) \to 0$ in $B(E)$. We can have that $\lim_{t_2\to t_1} I_1 = 0$ using Lebesgue's theorem.

For I_2, we have

$$\begin{aligned} I_2 &\leq \frac{M\Phi(r)}{\omega^{\alpha_1+\alpha_2-\delta}} \int_{t_1}^{t_2} e^{\omega(t_2-s)} m(s) ds \\ &+ \frac{MM_1}{\omega^{\alpha_1+\alpha_2-\delta}} \int_{t_1}^{t_2} e^{\omega(t_2-s)} u^x(s)\, ds \\ &\leq \frac{M\Phi(r)}{\omega^{\alpha_1+\alpha_2-\delta}} \|m\|_{L^{\frac{1}{q_1}}} \left[\frac{1-q_1}{\omega}\left(e^{\frac{\omega(t_2-t_1)}{1-q_1}}-1\right)\right]^{1-q_1} \\ &+ \frac{MM_1}{\omega^{\alpha_1+\alpha_2-\delta}} \|u^x\|_{L^2} \left[\frac{1}{2\omega}\left(e^{2\omega(t_2-t_1)}-1\right)\right]^{\frac{1}{2}}, \end{aligned} \tag{3.15}$$

and therefore $\lim_{t_2\to t_1} I_2 = 0$. From the above two inequalities, we find that $\{T_1 x : x \in B_r\}$ is an equicontinuous family.

Step 2: For every $t \in [0, b]$, it remains to show that $H(t) = \{(T_1x)(t) : x \in B_r\}$ is precompact.

First, it is obvious that $H(0)$ is precompact. Finally, let $0 < t \leq b$ be a fixed number. For $\forall\epsilon \in (0, t)$, we consider the operator T_1^ϵ on B_r by the formula

$$(T_1^\epsilon x)(t) = ((g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta}) * (\bar{f} + Bu))(t-\epsilon), x \in B_r.$$

From (H6) and Theorem 2.2, we know that the compactness of $\{(g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t-s)(\bar{f}(s)+Bu) : 0 \leq s \leq t-\epsilon\}$ for $\epsilon > 0$. Using the Mazur theorem and the mean-value theorem with respect to the Bochner integral, we have that for $\epsilon > 0$, $H_\epsilon(t) = \{(T_1^\epsilon x)(t) : x \in B_r\}$ is precompact in E. In addition, for every $x \in B_r$, we obtain

$$\begin{aligned} \|(T_1x)(t) - (T_1^\epsilon x)(t)\| &\leq \int_{t-\epsilon}^{t} \|(g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t-s)[\bar{f}(s) \\ &+ Bu(s)]\| ds \leq \frac{M\Phi(r)}{\omega^{\alpha_1+\alpha_2-\delta}} \|m\|_{L^{\frac{1}{q_1}}} \\ &\left[\frac{1-q_1}{\omega}\left(e^{\frac{\omega\epsilon}{1-q_1}}-1\right)\right]^{1-q_1} \\ &+ \frac{MM_1}{\omega^{\alpha_1+\alpha_2-\delta}} \|u^x\|_{L^2} \left[\frac{1}{2\omega}\left(e^{2\omega\epsilon}-1\right)\right]^{\frac{1}{2}}. \end{aligned}$$

Therefore, $H(t) = \{(T_1x)(t) : x \in B_r\}$ is precompact in E.

According to Ascoli–Arzela's Theorem and above, we conclude that $\{T_1x : x \in B_r\}$ is precompact. Thus, T_1 is a completely continuous operator by the continuity of T_1 and the relative compactness of $\{T_1x : x \in B_r\}$. According to Krasnoselskii's fixed point theorem, it is natural to obtain that $T_1 + T_2$ has a fixed point on B_r. Hence, (1.1) can be controlled on J, and the proof is complete.

Theorem 3.3. Let $0 < \delta \leq \alpha_1 + \alpha_2$, $\{S_{\alpha_2,\delta}(t)\}_{t\geq 0}$ is generated by operator $-A$ and (M, ω) is a type of $S_{\alpha_2,\delta}(t)$. In addition to assumptions (H1), (H2), (H4)(i), (H5)(i), and (H6) of Theorem 3.1, suppose that

(H7) For $0 < \delta \leq 1$, $\{S_{\alpha_2,\delta}(t)\}_{t>0}$ is uniform continuous.

Then (1.1) can be controlled on J for $\Lambda_1 < 1$.

Proof. We consider the operator T in $C(J, E)$, which is the same as (3.3). Similarly, there exists $r > 0$, such that $T : B_r \to B_r$ is continuous. We shall now examine the precompact nature of $\{Tx : x \in B_r\}$. Furthermore, we can see that $\{Tx : x \in B_r\}$ is not only uniformly bounded, but also equicontinuous.

Next, we verify that for all $t \in [0, b]$, $\{Tx(t) : x \in B_r\}$ is precompact. Obviously, $\{(Tx)(0) : x \in B_r\}$ is precompact. Let $0 < t \leq b$ be a number, $\forall\epsilon \in (0, t)$, we consider operator T^ϵ on B_r as follows:

$$(T^\epsilon x)(t) = S_{\alpha_2,\delta}(\epsilon)((g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta}) * (\bar{f} + Bu))(t-\epsilon), x \in B_r.$$

If $0 < \delta \leq 1$, then (H6), (H7), and Lemma 2.4 show that for $t > 0$, $S_{\alpha_2,\delta}(t)$ is compact, if $1 < \delta \leq \alpha_1 + \alpha_2$, then (H6) and Theorem 2.2 also illustrate that $S_{\alpha_2,\delta}(t)$ is compact for $t > 0$, Finally, we obtain that $\{(T^\epsilon x)(t) : x \in B_r\}$ is precompact in E for $\forall\epsilon \in (0, t)$. Furthermore, for every $x \in B_r$, we have

$$\begin{aligned} &\Big\| S_{\alpha_2,\delta}(\epsilon)((g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta}) * (\bar{f} + Bu))(t-\epsilon) \\ &- \int_0^{t-\epsilon} (g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t-s)[\bar{f}(s) + Bu(s)]\, ds \Big\| \\ &\leq \left(\Phi(r)\|m\|_{L^{\frac{1}{q_1}}} + M_1\|u^x\|_{L^2}\right) \left(\int_0^{t-\epsilon} \|S_{\alpha_2,\delta}(\epsilon)(g_{\alpha_1+\alpha_2-\delta} \right. \\ &* S_{\alpha_2,\delta})(t-s-\epsilon) \\ &\left. - (g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t-s)\|_B^{\frac{1}{1-q_1}}\, ds\right)^{1-q_1} \end{aligned}$$

By Theorem 2.1, $(g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta})(t)$ is norm continuous for all $t > 0$, using Lebesgue's theorem, we have

$$\begin{aligned} &\Big\| S_{\alpha_2,\delta}(\epsilon)((g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta}) * (\bar{f} + Bu))(t-\epsilon) - \int_0^{t-\epsilon} (g_{\alpha_1+\alpha_2-\delta} \\ &* S_{\alpha_2,\delta})(t-s)[\bar{f}(s) + Bu(s)]\, ds \Big\| \to 0, \epsilon \to 0. \end{aligned}$$

Hence, the set $((g_{\alpha_1+\alpha_2-\delta} * S_{\alpha_2,\delta}) * (\bar{f} + Bu))(t) : x \in B_r\}$, $t > 0$ is precompact. The compactness of $(g_{\alpha_2+\gamma_2-\delta} * S_{\alpha_2,\delta})(t)$ is obtained by Theorem 2.2. Hence, we have proved that for $t \in (0, b]$, $\{Tx(t) : x \in B_r\}$ is relatively compact in E. Consequently, by Ascoli–Arzela's Theorem, the set $\{Tx : x \in B_r\}$ is precompact. This further leads to T being compact on B_r. We therefore have, by applying Schauder's fixed point theorem, a fixed point on B_r of T, which implies that 1.1 can be controlled on J.

4. An example

Example 4.1. Set $E = U = L^2([0, \pi], R)$, $\alpha_i \in (0, 1]$, $\beta_i \in [0, 1]$, and $i = 0, 1$. We consider the fractional control system

$$\begin{cases} D_{0+}^{\alpha_1,\beta_1}\left(D_{0+}^{\alpha_2,\beta_2} + A\right) x(t,\xi) = f(t, x(t,\xi)) + Bu(t,\xi), t \in (0,1), \\ \xi \in [0,\pi], \\ \lim\limits_{t\to 0+} \left(I_{0+}^{(1-\alpha_2)(1-\beta_2)} x\right)(t,\xi) = 0, \\ \lim\limits_{t\to 0+} \dfrac{d}{dt}\left(I_{0+}^{(1-\alpha_1)(1-\beta_1)+(1-\alpha_2)} x\right)(t,\xi) = x_0(\xi), \end{cases} \tag{4.1}$$

where $t \in (0, 1)$, $\xi \in [0, \pi]$, and $D_{0+}^{\alpha_i,\beta_i}$ are Hilfer fractional derivatives. Operator A is given by $Ax = x - \frac{\partial^2 x}{\partial \xi^2}$, let $D(A) = \{x \in E : x, x'$ absolutely continuous, $x'' \in E, x(t, 0) = x(t, \pi) = 0\}$ and E be the domain and the range of A, respectively. We can see that $(1+n^2)$ and $x_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi)$ are the eigenvalues and the normalized eigenvectors of A, respectively.

For $x \in E$ and $1 \le \delta \le \alpha_1 + \alpha_2$, we have

$$\begin{aligned}\lambda^{\alpha_2-\delta}(\lambda^{\alpha_2}I+A)^{-1}x &= \sum_{n=1}^{\infty}\langle x, x_n\rangle x_n \frac{\lambda^{\alpha_2-\delta}}{\lambda^{\alpha_2}+(1+n^2)} \\ &= \sum_{n=1}^{\infty}\langle x, x_n\rangle x_n \int_0^{\infty} \mathrm{e}^{-\lambda t} t^{\delta-1} E_{\alpha_2,\delta} \\ &\quad (-(1+n^2)t^{\alpha_2})\,\mathrm{d}t \\ &= \int_0^{\infty}\langle x, x_n\rangle x_n \mathrm{e}^{-\lambda t}\sum_{n=1}^{\infty} t^{\delta-1}E_{\alpha_2,\delta} \\ &\quad (-(1+n^2)t^{\alpha_2})\,\mathrm{d}t.\end{aligned}$$

Hence, $\left\{S_{\alpha_2,\delta}(t)\right\}_{t\ge 0}$ is generated by operator $-A$,

$$S_{\alpha_2,\delta}(t)x = \sum_{n=1}^{\infty}\langle x, x_n\rangle x_n t^{\delta-1}E_{\alpha_2,\delta}(-(1+n^2)t^{\alpha_2}), x \in E,$$

which is norm continuous by the continuity of $E_{\alpha_2,\delta}(\cdot)$. Moreover, for $\lambda > 0$, we have $\lim_{n\to\infty}\frac{\lambda^{\alpha_2-\delta}}{\lambda^{\alpha_2}+(1+n^2)} = 0$, which implies that $\lambda^{\alpha_2-\delta}(\lambda^{\alpha_2}I + A)^{-1}$ is compact on the Hilbert space E, then $(\lambda^{\alpha_2}I + A)^{-1}$ is compact for $\lambda > 0$.

Otherwise, for each $x \in E$, we obtain $\|S_{\alpha_2,\delta}(t)x\| \le \frac{b^{\delta-1}}{\Gamma(\delta)}\|x\|$. Therefore, $S_{\alpha_2,\delta}(t)$ is of type $(b^{\delta-1}/\Gamma(\delta), 1)$.

Let $f(t,x) = \frac{e^{-t}}{1+t}x$, then we can choose $m(t) = \frac{e^{-t}}{1+t}$ and $\Phi = I$.

$$\|f(t,x)\| \le \frac{e^{-t}}{1+t}$$

Assume that $Bu(t) = \sum_{n=1}^{\infty}\hat{u}_n(t)x_n$, where

$$\hat{u}_n(t) = \begin{cases} 0, t \in \left[0, 1-\frac{1}{n}\right), \\ u_n(t), t \in \left[1-\frac{1}{n}, 1\right]. \end{cases}$$

Similar to Lv and Yang [22], we see that B is a bounded linear operator and W satisfies (H5). Then (4.1) can be controlled on J by Theorem 3.3.

5. Conclusion

In this article, we consider the exact controllability of a Hilfer fractional Langevin equation and the corresponding results are obtained using three fixed point theorems, respectively. One result is obtained without the compactness of proper $\left\{S_{\alpha_2,\delta}(t)\right\}$, whereas the other two results rely on the compactness of $\left\{S_{\alpha_2,\delta}(t)\right\}$.

Data availability statement

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

Author contributions

HW wrote and revised this manuscript. JK discussed with HW and gave some valuable suggestions in this paper. Both authors have participated in this research and approved the final manuscript.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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