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EDITED BY
Kateryna Buryachenko,
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Tuncer Acar,
Selçuk University, Türkiye
Bogdan Szal,
University of Zielona Góra, Poland

*CORRESPONDENCE
Olga Rovenska
rovenskaya.olga.math@gmail.com

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# Approximation of classes of Poisson integrals by rectangular Fejér means

Olga Rovenska*

Department of Theory of Functions, Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

The article is devoted to the problem of approximation of classes of periodic functions by rectangular linear means of Fourier series. Asymptotic equalities are found for upper bounds of deviations in the uniform metric of rectangular Fejér means on classes of periodic functions of several variables generated by sequences that tend to zero at the rate of geometric progression. In one-dimensional cases, these classes consist of Poisson integrals, namely functions that can be regularly extended in the fixed strip of a complex plane.

KEYWORDS

**linear method of approximation, extremal problem of approximation theory, Poisson integral, Fejér mean, exact asymptotic**

## 1 Introduction

Let $\mathbb{R}^d$ be the Euclidean space of vectors $\bar{x} = (x_1; x_2; \ldots; x_d)$. Let $f(\bar{x})$ be a function $2\pi$-periodic in each variable $x_i$, $i \in \overline{1,d}$ and summable on the set $\mathbb{T}^d = [-\pi;\pi]^d$, i.e., $f \in \mathrm{L}\left(\mathbb{T}^d\right)$, let

$$S[f](\bar{x}) = \sum_{\bar{k}\in\mathbb{Z}_+^d} 2^{-\gamma(\bar{k})} \sum_{\bar{s}\in\{0;1\}^d} a_{\bar{k}}^{\bar{s}}[f] \prod_{i=1}^{d} \cos\left(k_i x_i - \frac{s_i\pi}{2}\right)$$

be the complete Fourier series of function $f$, where

$$a_{\bar{k}}^{\bar{s}}[f] = \pi^{-d} \int\limits_{\mathbb{T}^d} f(\bar{x}) \prod_{i=1}^{d} \cos\left(k_i x_i - \frac{s_i\pi}{2}\right) dx_i,$$

are the Fourier coefficients of the function $f$, corresponding to the vectors $\bar{k} \in \mathbb{Z}_+^d$, $\bar{s} \in \{0;1\}^d$, and $\gamma(\bar{k})$ is the number of zero coordinates of the vector $\bar{k}$.

Let $\bar{\Lambda} = (\Lambda_1; \Lambda_2; \ldots; \Lambda_d)$ be the fixed set of infinite triangular matrices of numbers $\Lambda_i = \left\{\lambda_{k_i}^{(n_i)}\right\}$, $i \in \overline{1,d}$ such that $\lambda_0^{(n_i)} = 1$, $\lambda_{k_i}^{(n_i)} = 0$, $k_i \geq n_i$. Denote $\lambda_{\bar{k}}^{(\bar{n})} = \prod_{i=1}^{d} \lambda_{k_i}^{(n_i)}$, and $\mathbb{G}_{\bar{n}} = \prod_{i=1}^{d} [0; n_i - 1]$. If $\bar{k} \notin \mathbb{G}_{\bar{n}}$, then $\lambda_{\bar{k}}^{(\bar{n})} = 0$. For function $f \in \mathrm{L}\left(\mathbb{T}^d\right)$ the set $\bar{\Lambda}$ defines a family of trigonometric polynomials

$$U_{\bar{n}}[f;\bar{\Lambda}](\bar{x}) = \sum_{\bar{k}\in\mathbb{G}_{\bar{n}}} 2^{-\gamma(\bar{k})} \lambda_{\bar{k}}^{(\bar{n})} \sum_{\bar{s}\in\{0;1\}^d} a_{\bar{k}}^{\bar{s}}[f] \prod_{i=1}^{d} \cos\left(k_i x_i - \frac{s_i\pi}{2}\right).$$

The polynomials $U_{\bar{n}}[f;\bar{\Lambda}](\bar{x})$ are called rectangular linear means for $S[f](\bar{x})$. In particular, if $\lambda_{k_i}^{(n_i)}=1$, $\bar{k}\in\mathbb{G}_{\bar{n}}$, then $U_{\bar{n}}[f;\bar{\Lambda}](\bar{x})=S_{\bar{n}-1}[f](\bar{x})$ are the rectangular partial sums of $S[f](\bar{x})$, and if $\lambda_{k_i}^{(n_i)}=1-\frac{k_i}{n_i}$, $\bar{k}\in\mathbb{G}_{\bar{n}}$, then

$$U_{\bar{n}}[f;\bar{\Lambda}](\bar{x})=\sigma_{\bar{n}}[f](\bar{x})=\prod_{i=1}^{d}n_i^{-1}\sum_{\bar{k}\in\mathbb{G}_{\bar{n}}}S_{\bar{n}}[f](\bar{x})$$

are the rectangular Fejér means of $S[f](\bar{x})$.

Basic results relating to the approximation of functional classes by linear methods of summation of Fourier series can be found in books Timan [1], Lorentz [2], and Dyachenko [3]. Linear summation methods are widely used both for the solution of practical problems and for development of more advanced approximation methods. This chapter of approximation theory has been intensively developed over the past decades [4–9]. Here it is difficult to mention all the relevant published research papers in this area. Recently, we have seen the publication of several important works [10–15].

Let $C\left(\mathbb{T}^d\right)$ be the space of continuous $2\pi$-periodic in each variable's functions $f(\bar{x})$ with the norm

$$\|f\|:=\|f\|_C=\max_{\bar{x}\in\mathbb{T}^d}|f(\bar{x})|.$$

Let $\mathcal{J}(r)$ be the arbitrary subset of the set $\overline{\{1;d\}}$, where $r$ is the number of elements of the set $\mathcal{J}(r)$. Denote by $C^{\bar{q}}\left(\mathbb{T}^d\right)$, $\bar{q}\in(0;1)^d$ the set of functions $f\in C\left(\mathbb{T}^d\right)$ such that $\forall\mathcal{J}:=\mathcal{J}(r)\subseteq\overline{\{1;d\}}$, the series

$$\sum_{\substack{\bar{k}\in\mathbb{Z}_+^d,\\ k_j\neq 0,j\in\mathcal{J}}}2^{-\gamma(\bar{k})}\prod_{j\in\mathcal{J}}q_j^{-k_j}\sum_{\bar{s}\in\{0;1\}^d}a_{\bar{k}}^{\bar{s}}[f]\prod_{i=1}^{d}\cos\left(k_ix_i-\frac{s_i\pi}{2}\right)\qquad(1)$$

are the Fourier series of certain functions $\varphi_{\bar{q}}^{(\mathcal{J})}(\bar{x})\in L\left(\mathbb{T}^d\right)$, which are almost everywhere bounded by a unity, and the Fourier series of functions $\varphi_{\bar{q}}^{(\mathcal{J})}(\bar{x})$ do not contain terms independent of the variables $x_i$, $i\in\mathcal{J}(r)$.

For example, in the case $d=2$, the series (Equation 1) is as follows:

$$S\left[\varphi_{\bar{q}}^{(1)}\right](\bar{x})=\sum_{\bar{k}\in\mathbb{N}\times\mathbb{Z}_+}2^{-\gamma(\bar{k})}q_1^{-k_1}\sum_{\bar{s}\in\{0;1\}^2}a_{\bar{k}}^{\bar{s}}[f]$$
$$\cos\left(k_1x_1-\tfrac{s_1\pi}{2}\right)\cos\left(k_2x_2-\tfrac{s_2\pi}{2}\right),$$
$$S\left[\varphi_{\bar{q}}^{(2)}\right](\bar{x})=\sum_{\bar{k}\in\mathbb{Z}_+\times\mathbb{N}}2^{-\gamma(\bar{k})}q_2^{-k_2}\sum_{\bar{s}\in\{0;1\}^2}a_{\bar{k}}^{\bar{s}}[f]$$
$$\cos\left(k_1x_1-\tfrac{s_1\pi}{2}\right)\cos\left(k_2x_2-\tfrac{s_2\pi}{2}\right),$$
$$S\left[\varphi_{\bar{q}}^{(\mathcal{J})}\right](\bar{x})=\sum_{\bar{k}\in\mathbb{N}^2}2^{-\gamma(\bar{k})}q_1^{-k_1}q_2^{-k_2}\sum_{\bar{s}\in\{0;1\}^2}a_{\bar{k}}^{\bar{s}}[f]$$
$$\cos\left(k_1x_1-\tfrac{s_1\pi}{2}\right)\cos\left(k_2x_2-\tfrac{s_2\pi}{2}\right).$$

In the one-dimensional case, the classes $C^q\left(\mathbb{T}^1\right)$, $q\in(0;1)$ consist of continuous $2\pi$-periodic functions, given by the convolution

$$f(x)=A_0+\pi^{-1}\int_{\mathbb{T}^1}\varphi_q^{(1)}(x+t)\mathscr{P}_q(t)\,dt,\quad A_0-\text{const},$$

where

$$\mathscr{P}(q;t)=\sum_{k=0}^{\infty}q^k\cos kt=\frac{1-q\cos t}{1-2q\cos t+q^2},\quad q\in(0;1)$$

is the well-known Poisson kernel, the function $\varphi_q^{(1)}\in L(\mathbb{T}^1)$ $(\mathcal{J}(1)=i,\ i=1)$ satisfies almost everywhere the conditions $|\varphi_q^{(1)}(t)|\leq 1$, $\varphi_q^{(1)}\perp 1$.

In this work, we consider the problem of the exact upper bound for the approximation of periodic functions by linear means of the Fourier series. We employed methods for studying integral representations of deviations of polynomials, generated by linear summation methods of Fourier series of continuous periodic functions, developed in the works of Nikolskii [16], Telyakovskii [17], Stepanets [18], and others. This topic is currently being developed in the works of many authors [19–21].

Nikolskii [22] established the asymptotic equality as $n\to\infty$

$$\sup\left\{\|f-S_n[f]\|:f\in C^q\left(\mathbb{T}^1\right)\right\}=$$
$$\sup\left\{\left\|\frac{1}{\pi}\int_{\mathbb{T}^1}\varphi_q^{(1)}(x+t)\sum_{k=n+1}^{\infty}q^k\cos kt\,dt\right\|:|\varphi_q^{(1)}(t)|\leq 1,\ \varphi_q^{(1)}\perp 1\right\}$$
$$=\frac{8q^{n+1}}{\pi^2}K(q)+O(1)\frac{q^n}{n},$$

where $K(q)=\int_0^{\frac{\pi}{2}}(1-q^2\sin^2u)^{-\frac{1}{2}}\,du$ is the complete elliptic integral of the first kind and $O(1)$ is a quantity uniformly bounded with respect to $n$. Regarding the summability of Fourier series by Fejér means $\sigma_n[f]$, we proved the following two theorems [23–25].

**Theorem 1.** *Let $q_0$ be the only root of the equation $q^4-2q^3-2q^2-2q+1=0$, that belongs to the interval $(0;1)$, $q_0=\left(2+\sqrt{5}-2\sqrt{2+\sqrt{5}}\right)^{1/2}=0.346\ldots$. If $q\in(0;q_0]$, then the equality hold as $n\to\infty$*

$$\sup\left\{\|f-\sigma_n[f]\|:f\in C^q\left(\mathbb{T}^1\right)\right\}=\frac{4q}{\pi n(1+q^2)}+O(1)\frac{q^n}{n},$$

*where $O(1)$ is a quantity uniformly bounded with respect to $n$.*

**Theorem 2.** *If $q\in[q_0;1)$, then the equality hold as $n\to\infty$*

$$\sup\left\{\|f-\sigma_n[f]\|:f\in C^q\left(\mathbb{T}^1\right)\right\}$$
$$=\frac{2}{\pi n}\frac{(1+q^2)^2}{(1-q^2)\left(1-q^2+\sqrt{2(1+q^4)}\right)}+O(1)\frac{q^n}{n(1-q)^3},$$

*where $O(1)$ is uniformly bounded with respect to $n$, $q$.*

The purpose of this paper is to present the asymptotic equalities for upper bounds of deviations of rectangular Fejér means taken over multidimensional analogs of classes $C^q\left(\mathbb{T}^1\right)$. Similar asymptotic expansions for other rectangular linear methods can be found in Rukasov et al. [26] and Rovenska [27].

## 2 Result

The main result is the following.

**Theorem 3.** *Let $\bar{q} \in (0;1)^d$. Then*

$$\sup\left\{\|f-\sigma_{\bar{n}}[f]\| : f \in C^{\bar{q}}\left(\mathbb{T}^d\right)\right\} = \frac{4}{\pi}\sum_{i=1}^{d}\frac{A(q_i)}{n_i} \tag{2}$$
$$+O(1)\left(\sum_{i=1}^{d}\frac{q_i^{n_i}}{n_i(1-q_i)^3}+\sum_{r=2}^{d}\sum_{\mathcal{J}(r)\subset\overline{\{1,d\}}}\prod_{j\in\mathcal{J}(r)}\frac{1}{n_j(1-q_j)^3}\right),$$

*where*

$$A(q) = \begin{cases} \frac{q}{1+q^2}, & q\in(0;q_0] \\ \frac{(1+q^2)^2}{2(1-q^2)\left(1-q^2+\sqrt{2(1+q^4)}\right)}, & q\in[q_0;1), \end{cases}$$

*$q_0$ is the only root of the equation $q^4-2q^3-2q^2-2q+1=0$, that belongs to the interval $(0;1)$, $q_0 = 0.346\ldots$, $O(1)$ is a quantity, uniformly bounded with respect to $q_i$, $n_i$, $i\in\overline{\{1,d\}}$.*

*Proof*

First we find the upper estimate for the quantity

$$\sup\left\{\|f-\sigma_{\bar{n}}[f]\| : f\in C^{\bar{q}}\left(\mathbb{T}^d\right)\right\}. \tag{3}$$

Based on Theorem 1 in Rukasov et al. [26], $\forall f\in C^{\bar{q}}\left(\mathbb{T}^d\right)$, the equality holds

$$f(\bar{x})-\sigma_{\bar{n}}[f](\bar{x}) = \sum_{\bar{k}\in\mathbb{Z}_+^d}2^{-\gamma(\bar{k})}\sum_{\bar{s}\in\{0;1\}^d}a_{\bar{k}}^{\bar{s}}[f]\prod_{i=1}^{d}\cos\left(k_ix_i-\frac{s_i\pi}{2}\right)-$$
$$\sum_{\bar{k}\in\mathbb{G}_{\bar{n}}}2^{-\gamma(\bar{k})}\prod_{i=1}^{d}\left(1-\frac{k_i}{n_i}\right)\sum_{\bar{s}\in\{0;1\}^d}a_{\bar{k}}^{\bar{s}}[f]\prod_{i=1}^{d}\cos\left(k_ix_i-\frac{s_i\pi}{2}\right)=$$
$$\frac{1}{\pi}\sum_{i=1}^{d}\frac{1}{n_i}\int_{\mathbb{T}^1}\varphi_{q_i}^{(i)}\left(\bar{x}+t_i\bar{e}_i\right)\sum_{k_i=0}^{n_i-1}\sum_{\nu_i=k_i+1}^{\infty}q_i^{\nu_i}\cos\nu_it_i\,dt_i+$$
$$\sum_{r=2}^{d}(-1)^{r+1}\frac{1}{\pi^r}\sum_{\mathcal{J}(r)\subset\overline{\{1,d\}}}\int_{\mathbb{T}^r}\varphi_{\bar{q}}^{(\mathcal{J})}\left(\bar{x}+\sum_{j\in\mathcal{J}(r)}t_j\bar{e}_j\right)$$
$$\prod_{j\in\mathcal{J}(r)}\frac{1}{n_j}\sum_{k_j=0}^{n_j-1}\sum_{\nu_j=k_j+1}^{\infty}q_j^{\nu_j}\cos\nu_jt_j\,dt_j. \tag{4}$$

In Novikov et al. [24] and Rovenska [25] it was shown that

$$\sup\left\{\left\|\frac{1}{n}\int_{\mathbb{T}^1}\varphi_q^{(1)}(x+t)\sum_{k=0}^{n-1}\sum_{\nu=k+1}^{\infty}q^\nu\cos\nu t\,dt\right\| : |\varphi_q^{(1)}(t)|\le 1,\ \varphi_q^{(1)}\perp 1\right\}=$$
$$\frac{1}{n}\int_{\mathbb{T}^1}\varphi_q^{*(1)}(t)\sum_{k=0}^{n-1}\sum_{\nu=k+1}^{\infty}q^\nu\cos\nu t\,dt$$
$$=\frac{A(q)}{n}+O(1)\frac{q^n}{n(1-q)^3}, \tag{5}$$

where

$$\varphi_q^{*(1)}(t) = \begin{cases} \operatorname{sign}\left(\frac{\partial\mathscr{P}(q;t)}{\partial q}-\frac{\partial\mathscr{P}(q;t)}{\partial q}\Big|_{t=\frac{\pi}{2}}\right), & q\in(0;q_0], \\ \operatorname{sign}\left(\frac{\partial\mathscr{P}(q;t)}{\partial q}-\frac{\partial\mathscr{P}(q;t)}{\partial q}\Big|_{t=t_q}\right), & q\in[q_0;1), \end{cases} \tag{6}$$

and $t_q$ is determined by the condition

$$\frac{\partial\mathscr{P}(q;t)}{\partial q}\Big|_{t=t_q} = \frac{\partial\mathscr{P}(q;t)}{\partial q}\Big|_{t=t_q+\frac{\pi}{2}},\quad 0\le t_q\le\frac{\pi}{2}.$$

Combining Equations 4, 5, and 6, we obtain

$$\sup\left\{\|f-\sigma_{\bar{n}}[f]\| : f\in C^{\bar{q}}\left(\mathbb{T}^d\right)\right\}\le\frac{4}{\pi}\sum_{i=1}^{d}\frac{A(q_i)}{n_i}$$
$$+O(1)\left(\sum_{i=1}^{d}\frac{q_i^{n_i}}{n_i(1-q_i)^3}+\sum_{r=2}^{d}\sum_{\mathcal{J}(r)\subset\overline{\{1,d\}}}\prod_{j\in\mathcal{J}(r)}\frac{1}{n_j(1-q_j)^3}\right). \tag{7}$$

Next, we find the lower estimate of Equation 3. We construct the function $f^*(\bar{x})\in C^{\bar{q}}\left(\mathbb{T}^d\right)$ for which estimate Equation 7 cannot be improved. Based on equality Equation 3 we have

$$f(\bar{0})-\sigma_{\bar{n}}[f](\bar{0})$$
$$=\frac{1}{\pi}\sum_{i=1}^{d}\frac{1}{n_i}\int_{\mathbb{T}^1}\varphi_{q_i}^{(i)}\left(\bar{0}+t_i\bar{e}_i\right)\sum_{k_i=0}^{n_i-1}\sum_{\nu_i=k_i+1}^{\infty}q_i^{\nu_i}\cos\nu_it_i\,dt_i+$$
$$\sum_{r=2}^{d}(-1)^{r+1}\frac{1}{\pi^r}\sum_{\mathcal{J}(r)\subset\overline{\{1,d\}}}\int_{\mathbb{T}^r}\varphi_{\bar{q}}^{(\mathcal{J})}\left(\bar{0}+\sum_{j\in\mathcal{J}(r)}t_j\bar{e}_j\right)$$
$$\prod_{j\in\mathcal{J}(r)}\frac{1}{n_j}\sum_{k_j=0}^{n_j-1}\sum_{\nu_j=k_j+1}^{\infty}q_j^{\nu_j}\cos\nu_jt_j\,dt_j.$$

Since the functions $\varphi_{\bar{q}}^{(\mathcal{J})}$ satisfy the condition $|\varphi_{\bar{q}}^{(\mathcal{J})}(\bar{x})|\le 1$ almost everywhere, and

$$\int_{\mathbb{T}^1}\left|\sum_{k_j=0}^{n_j-1}\sum_{\nu_j=k_j+1}^{\infty}q_j^{\nu_j}\cos\nu_jt_j\right|dt_j$$
$$=\int_{\mathbb{T}^1}\left|\frac{\partial\mathscr{P}(q_j;t_j)}{\partial q_j}\right|dt_j = O(1)\frac{1}{(1-q_j)^3},\quad i\in\overline{\{1,d\}},$$

then

$$f(\bar{0})-\sigma_{\bar{n}}[f](\bar{0}) = \frac{1}{\pi}\sum_{i=1}^{d}\frac{1}{n_i}\int_{\mathbb{T}^1}\varphi_{q_i}^{(i)}\left(\bar{0}+t_i\bar{e}_i\right)\sum_{k_i=0}^{n_i-1}$$
$$\sum_{\nu_i=k_i+1}^{\infty}q_i^{\nu_i}\cos\nu_it_i\,dt_i$$
$$+O(1)\left(\sum_{r=2}^{d}\sum_{\mathcal{J}(r)\subset\overline{\{1,d\}}}\prod_{j\in\mathcal{J}(r)}\frac{1}{n_j(1-q_j)^3}\right).$$

Denote by $\varphi_{q_i}^{*(i)}(\bar{x})$, $\bar{x}\in\mathbb{T}^d$ an arbitrary continuation on the set $\mathbb{T}^d$ of the function $\varphi_{q_i}^{*(i)}(x_i)$, $x_i\in\mathbb{T}^1$, and denote by $f_i^*(\bar{x})$, $\bar{x}\in\mathbb{T}^d$ the function, such that

$$S\left[\varphi_{q_i}^{*(i)}\right](\bar{x}) = \sum_{\substack{\bar{k}\in\mathbb{Z}_+^d,\\ k_i\ne 0}}2^{-\gamma(\bar{k})}q_i^{-k_i}\sum_{\bar{s}\in\{0;1\}^d}a_{\bar{k}}^{\bar{s}}\left[f_i^*\right]\prod_{i=1}^{d}\cos\left(k_ix_i-\frac{s_i\pi}{2}\right).$$

Let $f^*(\bar{x}) := \sum_{i=1}^{d}f_i^*(\bar{x})$. It's clear that $f^*(\bar{x})\in C^{\bar{q}}\left(\mathbb{T}^d\right)$. Therefore, we have

$$f^*(\bar{0})-\sigma_{\bar{n}}[f^*](\bar{0}) = \frac{1}{\pi}\sum_{i=1}^{d}\frac{1}{n_i}\int_{\mathbb{T}^1}\varphi_{q_i}^{*(i)}(t_i)\sum_{k_i=0}^{n_i-1}\sum_{\nu_i=k_i+1}^{\infty}q_i^{\nu_i}\cos\nu_it_i\,dt_i$$
$$+O(1)\left(\sum_{r=2}^{d}\sum_{\mathcal{J}(r)\subset\overline{\{1,d\}}}\prod_{j\in\mathcal{J}(r)}\frac{1}{n_j(1-q_j)^3}\right). \tag{8}$$

Combining Equations 5, 7, and 8, we obtain equality (Equation 2). The proof is complete.

**Remark 1.** *Formula Equation 2 is asymptotically exact for any* $\bar{q} \in (0;1)^d$.

**Remark 2.** In the case $d = 2$, formula Equation 2 is simplified as follows:

$$\sup\left\{\|f - \sigma_{\bar{n}}[f]\| : f \in C^{\bar{q}}\left(\mathbb{T}^2\right)\right\} = \frac{4}{\pi}\sum_{i=1,2}\frac{A(q_i)}{n_i}$$

$$+O(1)\left(\sum_{i=1,2}\frac{q_i^{n_i}}{n_i(1-q_i)^3} + \prod_{j=1,2}\frac{1}{n_j(1-q_j)^3}\right).$$

# 3 Conclusion

In this study, we propose an approach to define the multidimensional analogs of classes of Poisson integrals, which allows us to take into account the rate of decrease of each sequence that determine the class. The problem connected with the search for upper bounds of approximation errors with respect to a fixed class of functions and with the choice of an approximation tool is considered. In the certain case, our approach turned out to be effective for obtaining exact asymptotic. The key point in this approach is to construct the function $f^*(\bar{x}) \in C^{\bar{q}}\left(\mathbb{T}^d\right)$ that implements the upper bound.

Our study may be useful for solving the upper bound problem in other particular cases. In particular, our ideas can be used to obtain the corresponding asymptotic equalities on classes, which in one-dimensional cases are determined by the Poisson kernels $\tilde{\mathscr{P}}_q(t) = \sum_{k=1}^{\infty} \sin kt$, $\mathscr{P}_q^{\beta}(t) = \sum_{k=0}^{\infty} \cos\left(kt + \frac{\beta\pi}{2}\right)$, $\beta \in \mathbb{R}$, etc.

# Data availability statement

The original contributions presented in the study are included in the article/supplementary material; further inquiries can be directed to the corresponding author.

# Author contributions

OR: Writing – review & editing, Writing – original draft.

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# Conflict of interest

The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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