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Mellin convolutions of products and ratios

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Usually, convolution refers to Laplace convolution in the literature, but Mellin convolutions can yield very ueful results. This aspect is illustrated in the coming sections. This study deals with Mellin convolutions of products and ratios. Functions belonging to the pathway family of functions are considered. Several types of integral representations, their equivalent representations in terms of G and H-functions, and their equivalent computable series representations are examined in this study.

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1 Introduction

Laplace convolutions involving products of two functions are widely known in the literature (see [1, 2]), but Mellin convolutions of products and ratios are not widely used in the literature. Here, we examine Mellin convolutions of products and ratios involving some functions belonging to the pathway family of functions. The pathway family of functions was introduced by Mathai [3] for real-valued scalar functions with the arguments being rectangular matrices in the real domain. These results were extended to the complex domain in Mathai and Provost [4]. Here, we consider the real scalar variable case first, and then, some generalizations are also considered. Let $x_1 > 0$ and $x_2 > 0$ be real positive scalar variables associated with the functions $f_1(x_1)$ and $f_2(x_2)$, respectively, and with the joint function $f_1(x_1)f_2(x_2)$ the product. Then, the Mellin transforms of f_1 and f_2, with Mellin parameter s, and denoted by $M_{f_1}(s)$ and $M_{f_2}(s)$, are the following:

$$M_{f_1}(s) = \int_0^\infty x_1^{s-1} f_1(x_1)\mathrm{d}x_1 \text{ and } M_{f_2}(s) = \int_0^\infty x_2^{s-1} f_2(x_2)\mathrm{d}x_2 \tag{1.1}$$

whenever the integrals are convergent. Let $u = x_1x_2$. Let $g_1(u)$ be the function associated with u. Then, the Mellin transform of $g_1(u)$ is the following:

$$M_{g_1}(s) = \int_0^\infty u^{s-1} g_1(u)\mathrm{d}u = \int_0^\infty \int_0^\infty (x_1x_2)^{s-1} f_1(x_1)f_2(x_2)\mathrm{d}x_1 \wedge \mathrm{d}x_2 = M_{f_1}(s)M_{f_2}(s) \tag{1.2}$$

since the joint function of x_1 and x_2 is assumed to be the product $f_1(x_1)f_2(x_2)$. Consider the transformation

$$u = x_1x_2, v = x_2 \text{ or } v = x_1 \Rightarrow \mathrm{d}x_1 \wedge \mathrm{d}x_2 = \frac{1}{v}\mathrm{d}u \wedge \mathrm{d}v$$

and the joint function of u and v is $\frac{1}{v}f_1(\frac{u}{v})f_2(v)$ or $\frac{1}{v}f_1(v)f_2(\frac{u}{v})$. Then, $g_1(u)$ is

the following:

$$g_1(u) = \int_v \frac{1}{v} f_1(\frac{u}{v}) f_2(v) \mathrm{d}v = \int_v \frac{1}{v} f_1(v) f_2(\frac{u}{v}) \mathrm{d}v. \quad (1.3)$$

But, from Equation 1.2, $g_1(u)$ is available as the inverse Mellin transform of $M_{f_1}(s)M_{f_2}(s)$. That is,

$$g_1(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M_{f_1}(s) M_{f_2}(s) u^{-s} \mathrm{d}s, i = \sqrt{(-1)} \quad (1.4)$$

whenever the inverse Mellin transform exists. From Equations 1.3, 1.4, we have

$$\begin{aligned} g_1(u) &= \int_v \frac{1}{v} f_1(\frac{u}{v}) f_2(v) \mathrm{d}v = \int_v \frac{1}{v} f_1(v) f_2(\frac{u}{v}) \mathrm{d}v \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M_{f_1}(s) M_{f_2}(s) u^{-s} \mathrm{d}s \end{aligned} \quad (1.5)$$

This Equation 1.5 is the Mellin convolution of a product property.

When $x_1 > 0$ and $x_2 > 0$ are real scalar random variables with the densities $f_1(x_1)$ and $f_2(x_2)$, respectively, and when x_1 and x_2 are statistically independently distributed, then

$$\begin{aligned} M_{f_1}(s) = E[x_1^{s-1}], M_{f_2}(s) = E[x_2^{s-1}], \\ E[(x_1x_2)^{s-1}] = E[x_1^{s-1}]E[x_2^{s-1}] = M_{f_1}(s)M_{f_2}(s) \end{aligned} \quad (1.6)$$

due to statistical independence of x_1 and x_2, where $E[\cdot]$ denotes the expected value of $[\cdot]$. The density of $u = x_1x_2$, denoted by $g_1(u)$, is available as the first part of Equation 1.5 if we use transformation of variables and as the second part of Equation 1.5 if we use the result of arbitrary moments uniquely determining the corresponding density. Whenever $f_1(x_1)$ and $f_2(x_2)$ are statistical densities, we can also give a statistical interpretation of the Mellin convolution of a product as the unique density of the product of independently distributed real scalar positive random variables x_1 and x_2. Thus, the results in Equations 1.1–1.5 can be given statistical interpretations also whenever the functions f_1 and f_2 are statistical densities.

For the real scalar positive variables case, the pathway family consists of the generalized type-1 beta family given by

$$\begin{aligned} p_1(x) = \frac{\delta a^{\frac{\alpha}{\delta}} \Gamma(\frac{\alpha}{\delta} + \beta)}{\Gamma(\frac{\alpha}{\delta})\Gamma(\beta)} x^{\alpha-1}(1 - ax^\delta)^{\beta-1}, 0 \le x \le 1, a > 0, \\ \alpha > 0, \beta > 0, 1 - ax^\delta > 0 \end{aligned} \quad (1.7)$$

and zero elsewhere, with the standard form given by

$$p_{11}(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1 - x)^{\beta-1}, 0 \le x \le 1, \alpha > 0, \beta > 0$$

and zero elsewhere; the generalized type-2 beta model is given by

$$p_2(x) = \frac{\delta a^{\frac{\alpha}{\delta}} \Gamma(\frac{\alpha}{\delta} + \beta)}{\Gamma(\frac{\alpha}{\delta})\Gamma(\beta)} x^{\alpha-1}(1 + ax^\delta)^{-(\frac{\alpha}{\delta}+\beta)}, 0 \le x < \infty \quad (1.8)$$

for $a > 0, \alpha > 0, \beta > 0, \delta > 0$ and zero elsewhere, with the standard form given by

$$p_{21}(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1 + x)^{-(\alpha+\beta)}, 0 \le x < \infty, \alpha > 0, \beta > 0$$

and zero elsewhere; and the generalized gamma model is given by

$$p_3(x) = \frac{\delta a^{\frac{\alpha}{\delta}}}{\Gamma(\frac{\alpha}{\delta})} x^{\alpha-1} \mathrm{e}^{-ax^\delta}, 0 \le x < \infty, a > 0, \delta > 0, \alpha > 0 \quad (1.9)$$

and zero elsewhere with the standard form given by

$$p_{31}(x) = \frac{a^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-ax}, 0 \le x < \infty, a > 0, \alpha > 0$$

and zero elsewhere. In statistical densities, the parameters are usually real, and hence, the conditions are stated for the real parameters but the functions are available when the parameters are in the complex domain also. In that case, the same conditions hold on the real parts of the parameters. We will examine Mellin convolutions of products and ratios involving $f_1(x_1)$ and $f_2(x_2)$ belonging to the models (Equations 1.7–1.9) so that the results obtained will be readily applicable.

The concept of Mellin convolution of a ratio is coming from the following considerations. As before, let $x_1 > 0$ and $x_2 > 0$ be real scalar positive variables with their associated functions $f_1(x_1)$ and $f_2(x_2)$, respectively, and with the joint function $f_1(x_1)f_2(x_2)$ the product. Let $u = \frac{x_2}{x_1}$ the ratio and let $g_2(u)$ be the function corresponding to u. Then, proceeding as before we have

$$E[u^{s-1}] = E[(\frac{x_2}{x_1})^{s-1}] = E[x_2^{s-1}]E[x_1^{-s+1}] = M_{f_2}(s)M_{f_1}(2 - s). \quad (1.10)$$

Let $u = \frac{x_2}{x_1}, v = x_2 \Rightarrow \mathrm{d}x_1 \wedge \mathrm{d}x_2 = -\frac{v}{u^2}\mathrm{d}u \wedge \mathrm{d}v, x_2 = v, x_1 = \frac{v}{u}$. In addition, $u = \frac{x_2}{x_1}, v = x_1 \Rightarrow \mathrm{d}x_1 \wedge \mathrm{d}x_2 = v\mathrm{d}u \wedge \mathrm{d}v, x_1 = v, x_2 = uv$. Then, the Mellin conovlution of a ratio property is the following:

$$\begin{aligned} g_2(u) &= \int_v \frac{v}{u^2} f_1(\frac{v}{u}) f_2(v) \mathrm{d}v = \int_v v f_1(v) f_2(uv) \mathrm{d}v \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M_{f_1}(2 - s) M_{f_2}(s) u^{-s} \mathrm{d}s. \end{aligned} \quad (1.11)$$

Again, we will examine Mellin convolution of a ratio involving functions belonging to the pathway family (Equations 1.7–1.9) for the real scalar variable case first.

This study is organized as follows: Section 2 deals with Mellin convolution of a product involving functions in the standard form in Equations 1.7–1.9. One general form is illustrated at the end. Section 3 gives Mellin convolutions of a ratio involving the standard forms in Equations 1.7–1.9. Then, one illustration is given at the end for a general case. Section 4 examines some generalizations to the real matrix-variate case.

2 Mellin convolution of a product

Here, we consider only Mellin convolutions of products of two real-valued scalar functions of real scalar positive variables. The results can be extended to Mellin convolution of a product involving three or more real-valued scalar functions.

Problem 2.1. Type-2 beta vs. gamma

To start with, we consider

$$f_1(x_1) = \frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} x_1^{\alpha}(1+x_1)^{-(\alpha+1+\beta)}$$
$$x_1 \geq 0,\ f_2(x_2) = \frac{a^{\rho+1}}{\Gamma(\rho+1)} x_2^{\rho} \mathrm{e}^{-ax_2}, x_2 \geq 0,$$

for $\alpha > -1, \rho > -1, \beta > 0, a > 0$ and let f_1 and f_2 be zero elsewhere. Let the joint function be $f_1(x_1)f_2(x_2)$ and $u = x_1x_2$ with the corresponding function $g_1(u)$. Then, for $c = \frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} \frac{a^{\rho+1}}{\Gamma(\rho+1)}$,

$$\begin{aligned} g_1(u) &= \int_0^{\infty} \frac{1}{v} f_1(\frac{u}{v}) f_2(v) \mathrm{d}v \\ &= c \int_0^{\infty} \frac{1}{v} (\frac{u}{v})^{\alpha} (1+\frac{u}{v})^{-(\alpha+1+\beta)} v^{\rho} \mathrm{e}^{-av} \mathrm{d}v \\ &= c\, u^{\alpha} \int_0^{\infty} v^{\rho+\beta}(v+u)^{-(\alpha+1+\beta)} \mathrm{e}^{-av} \mathrm{d}v, 0 \leq u < \infty \quad (2.1) \\ g_1(u) &= \int_0^{\infty} \frac{1}{v} f_1(v) f_2(\frac{u}{v}) \mathrm{d}v \\ &= c\, u^{\rho} \int_0^{\infty} v^{\alpha-\rho-1}(1+v)^{-(\alpha+1+\beta)} \mathrm{e}^{-\frac{au}{v}} \mathrm{d}v, 0 \leq u < \infty. \end{aligned} \quad (2.2)$$

Also, the Mellin transforms are the following:

$$\begin{aligned} M_{f_1}(s) &= \frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} \int_0^{\infty} x_1^{\alpha+s-1}(1+x_1)^{-(\alpha+1+\beta)} \mathrm{d}x_1 \\ &= \frac{\Gamma(\alpha+s)}{\Gamma(\alpha+1)} \frac{\Gamma(\beta-s+1)}{\Gamma(\beta)}, \Re(\alpha+s) > 0, \Re(\beta-s+1) > 0 \end{aligned} \quad (2.3)$$

$$\begin{aligned} M_{f_2}(s) &= \frac{a^{\rho+1}}{\Gamma(\rho+1)} \int_0^{\infty} x_2^{\rho+s-1} \mathrm{e}^{-ax_2} \mathrm{d}x_2 \\ &= \frac{\Gamma(\rho+s)}{\Gamma(\rho+1)} \frac{a}{a^s}, \Re(\rho+s) > 0 \end{aligned} \quad (2.4)$$

where $\Re(\cdot)$ denotes the real part of $(\cdot)$. Then,

$$\begin{aligned} g_1(u) &= \frac{1}{\Gamma(\alpha+1)\Gamma(\beta)} \frac{a}{\Gamma(\rho+1)} \\ &\quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\alpha+s)\Gamma(\rho+s)\Gamma(\beta+1-s)(au)^{-s} \mathrm{d}s \quad (2.5) \\ &= \frac{a}{\Gamma(\alpha+1)\Gamma(\beta)\Gamma(\rho+1)} G_{1,2}^{2,1}\left[au|_{\alpha,\rho}^{-\beta}\right], 0 \leq u < \infty \quad (2.6) \end{aligned}$$

where the $G(\cdot)$ is the G-function. For the theory and applications of the G-function, see for example, Mathai [5]. The inverse Mellin transform in Equation 2.5 can be evaluated by using residue calculus. For $\alpha - \rho \neq \pm\nu, \nu = 0, 1, \ldots$ the poles of the integrand are simple. Then, the function is available as the sum of the residues at the poles of $\Gamma(\alpha+s)$ and $\Gamma(\rho+s)$. These produce two confluent hypergeometric series, and the sum of these residues is the explicit value of the G-function for $0 \leq u < \infty$. $g_1(u)$ above gives various representations of the density of u. Since the density is unique, we can compare the different representations of the unique density and obtain several results on the unique density or we can obtain several results on the G-function. Different types of results on the G-function are given as Theorem 2.1 below. Since the procedure is the same, we will not give the derivations in detail for the remaining theorems in Section 2; only the results will be listed as theorems. The proofs are parallel to the steps given above.

Theorem 2.1. *For $a > 0, u \geq 0, \alpha > -1, \rho > 0, \beta > 0$*

$$\begin{aligned} G_{1,2}^{2,1}\left[au|_{\alpha,\rho}^{-\beta}\right] &= \Gamma(\alpha+1+\beta)a^{\rho}u^{\alpha} \\ &\quad \int_0^{\infty} v^{\rho+\beta}(v+u)^{-(\alpha+1+\beta)} \mathrm{e}^{-av} \mathrm{d}v \\ &= \Gamma(\alpha+1+\beta)a^{\rho}u^{\rho} \\ &\quad \int_0^{\infty} v^{\alpha-\rho-1}(1+v)^{-(\alpha+1+\beta)} \mathrm{e}^{-a\frac{u}{v}} \mathrm{d}v \\ &= (au)^{\alpha}\Gamma(\rho-\alpha)\Gamma(1+\alpha+\beta) \\ &\quad {}_1F_1(\beta+1+\alpha; 1+\alpha-\rho : au) \\ &\quad + (au)^{\rho}\Gamma(1+\beta+\rho)\Gamma(\alpha-\rho) \\ &\quad {}_1F_1(\beta+1+\rho; 1+\rho-\alpha; au), \end{aligned}$$

for $0 \leq au < \infty, \alpha - \rho \neq \pm\nu, n = 0, 1, \ldots$.

Note that the integral can be evaluated at specific values of a and u such as $u = 1; a = 1; a = 1, u = 1$, giving rise to simpler integrals. The condition $\alpha - \rho \neq \pm\nu$ is needed only to write the G-function in terms of a confluent hypergeometric series. Each integral above multiplied by $\frac{1}{\Gamma(\alpha+1)\Gamma(\beta)} \frac{a}{\Gamma(\rho+1)}$ produces a statistical density $g_1(u)$ for u also since our starting f_1 and f_2 are statistical densities. We have taken non-negative integrable functions with total integral unity for convenience. The results hold for other types of functions provided the Mellin transforms exist and the product of the Mellin transforms is invertible. A direct generalization of Problem 2.1 can be achieved by replacing $1+x_1$ in f_1 by $1+a_1x_1^{\delta_1}, a_1 > 0, \delta_1 > 0$ and replacing the exponent ax_2 in f_2 by $ax_2^{\delta_2}, \delta_2 > 0$. Then, instead of the G-function, one will end up with a H-function and then one will obtain results on this H-function. For the theory and applications of the H-function, see for example, Mathai et al. [6]. Since $x_j \geq 0$ in Problem 2.1, we can obtain parallel results for $\delta_j < 0$ also.

Problem 2.2. Type-2 beta vs. type-2 beta

Let

$$f_j(x_j) = \frac{\Gamma(\alpha_j+1+\beta_j)}{\Gamma(\alpha_j+1)\Gamma(\beta_j)} x_j^{\alpha_j}(1+x_j)^{-(\alpha_j+1+\beta_j)},$$

for $0 \leq x_j < \infty, \alpha_j > -1, \beta_j > 0, j = 1, 2$ and let f_j be zero elsewhere for $j = 1, 2$. Let the joint function of x_1 and x_2 be $f_1(x_1)f_2(x_2)$ and $u = x_1x_2$ with the corresponding function $g_1(u)$. Then, proceeding as in Problem 2.1, we have the following results:

Theorem 2.2. *For $\alpha_j > -1, \beta_j > 0, j = 1, 2$ and for $c = \left\{\prod_{j=1}^{2} \frac{\Gamma(\alpha_j+1+\beta_j)}{\Gamma(\alpha_j+1)\Gamma(\beta_j)}\right\}$,*

$$\begin{aligned} g_1(u) &= c\, u^{\alpha_1} \int_0^{\infty} v^{\beta_1+\alpha_2}(u+v)^{-(\alpha_1+1+\beta_1)}(1+v)^{-(\alpha_2+1+\beta_2)} \mathrm{d}v, \\ &\quad 0 \leq u < \infty \\ &= c\, u^{\alpha_2} \int_0^{\infty} v^{\alpha_1+\beta_2}(1+v)^{-(\alpha_1+1+\beta_1)}(v+u)^{-(\alpha_2+1+\beta_2)} \mathrm{d}v, \\ &\quad 0 \leq u < \infty \\ &= \left\{\prod_{j=1}^{2} \frac{1}{\Gamma(\alpha_j+1)\Gamma(\beta_j)}\right\} G_{2,2}^{2,2}\left[u|_{\alpha_1,\alpha_2}^{-\beta_1,-\beta_2}\right], 0 \leq u < \infty \end{aligned}$$

$$
\begin{aligned}
&G_{2,2}^{2,2}\left[u|_{\alpha_1,\alpha_2}^{-\beta_1,-\beta_2}\right]\\
&=\Gamma(\alpha_2-\alpha_1)\Gamma(\beta_1+1+\alpha_1)\Gamma(\beta_2+1+\alpha_1)u^{\alpha_1}\\
&\times {}_2F_1(\beta_1+1+\alpha_1,\beta_2+1+\alpha_1;1+\alpha_1-\alpha_2;u)\\
&+\Gamma(\alpha_1-\alpha_2)\Gamma(\beta_1+1+\alpha_2)\Gamma(\beta_2+1+\alpha_2)u^{\alpha_2}\\
&\times {}_2F_1(\beta_1+1+\alpha_2,\beta_2+1+\alpha_2;1+\alpha_2-\alpha_1;u)\\
&\textit{for } 0\le u<1,\alpha_1-\alpha_2\ne\pm\nu,\nu=0,1,\ldots\\
&\Gamma(\alpha_1+1+\beta_1)\Gamma(\alpha_2+\beta_1+1)\Gamma(\beta_2-\beta_1)u^{-\beta_1-1}\\
&\times {}_2F_1(\alpha_1+\beta_1+1,\alpha_2+\beta_1+1;1+\beta_1-\beta_2;\frac{1}{u})\\
&+\Gamma(\alpha_1+\beta_2+1)\Gamma(\alpha_2+\beta_2+1)\Gamma(\beta_1-\beta_2)u^{-\beta_2-1}\\
&\times {}_2F_1(\alpha_1+\beta_2+1,\alpha_2+\beta_2+1;1+\beta_2-\beta_1;\frac{1}{u})\\
&\textit{for } \beta_1-\beta_2\ne\pm\nu,\nu=0,1,\ldots,1\le u<\infty
\end{aligned}
$$

A direct generalization of Problem 2.2 is to replace $1+x_j$ in f_j by $1+a_jx_j^{\delta_j}, a_j>0,\delta_j>0, j=1,2$. Then, we will end up with a H-function, instead of the G-function above, and then parallel results will be available on this H-function. Since $x_j\ge 0$ in f_j here, one can allow δ_j to be negative also. Parallel results will be available for $j=1,2$.

Problem 2.3. Type-1 beta vs. type-1 beta

Let

$$
f_j(x_j)=\frac{\Gamma(\alpha_j+1+\beta_j)}{\Gamma(\alpha_j+1)\Gamma(\beta_j)}x_j^{\alpha_j}(1-x_j)^{\beta_j-1},
$$

$$
0\le x_j\le 1,\alpha_j>-1,\beta_j>0
$$

and zero elsewhere for $j=1,2$. Let the joint function be $f_1(x_1)f_2(x_2)$ and let $u=x_1x_2$ with the corresponding function denoted as $g_1(u)$. Then, proceeding as in Problem 2.1, one has the following results, for $\alpha_j>-1,\beta_j>0,j=1,2,c=\prod_{j=1}^2\frac{\Gamma(\alpha_j+1+\beta_j)}{\Gamma(\alpha_j+1)\Gamma(\beta_j)}$:

Theorem 2.3. *For $\alpha_j>-1,\beta_j>0,j=1,2$ and for $0\le u\le 1$*

$$
\begin{aligned}
&G_{2,2}^{2,0}[u|_{\alpha_1,\alpha_2}^{\alpha_1+\beta_1,\alpha_2+\beta_2}]\\
&=\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}u^{\alpha_1}\int_{v=u}^1 v^{\alpha_2-\alpha_1-\beta_1}(v-u)^{\beta_1-1}(1-v)^{\beta_2-1}\mathrm{d}v,\\
&0\le u\le 1\\
&=\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}u^{\alpha_2}\int_{v=u}^1 v^{\alpha_1-\beta_2-\alpha_2}(v-u)^{\beta_2-1}(1-v)^{\beta_1-1}\mathrm{d}v,\\
&0\le u\le 1\\
&=\frac{\Gamma(\alpha_2-\alpha_1)}{\Gamma(\beta_1)\Gamma(\alpha_2+\beta_2-\alpha_1)}u^{\alpha_1}\\
&\times {}_2F_1(1-\beta_1,1+\alpha_1-\alpha_2-\beta_2;1+\alpha_1-\alpha_2;u)\\
&+\frac{\Gamma(\alpha_1-\alpha_2)}{\Gamma(\beta_2)\Gamma(\alpha_1+\beta_1-\alpha_2)}u^{\alpha_2}\\
&\times {}_2F_1(1-\beta_2,1+\alpha_2-\alpha_1-\beta_1;1+\alpha_2-\alpha_1;u),0\le u\le 1\\
&\textit{for } \alpha_1-\alpha_2\ne\pm\nu,\nu=0,1,\ldots,0\le u<1
\end{aligned}
$$

This G-function multiplied by $\left\{\prod_{j=1}^2\frac{\Gamma(\alpha_j+1+\beta_j)}{\Gamma(\alpha_j+1)}\right\}$ gives a statistical density $g_1(u)$ for u also. A direct generalization is available by replacing $1-x_j$ in f_j by $1-a_jx_j^{\delta_j}$ for $a_j>0,\delta_j>0$, with the condition $1-a_jx_j^{\delta_j}>0, j=1,2$. Then, one will end up with a H-function, instead of the G-function above, and parallel results will be available for this H-function. It may be observed that a constant multiple of this H-function is a statistical density for u also.

Problem 2.4. Type-1 beta vs. gamma

Let

$$
f_1(x_1)=\frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)}x_1^{\alpha}(1-x_1)^{\beta-1},0\le x_1\le 1,
$$

$$
f_2(x_2)=\frac{a^{\gamma+1}}{\Gamma(\gamma+1)}x_2^{\gamma}\mathrm{e}^{-ax_2},x_2\ge 0
$$

for $\alpha>-1,\beta>0,\gamma>-1,a>0$ and let f_1 and f_2 be zero elsewhere and let the joint function be $f_1(x_1)f_2(x_2)$ and let $u=x_1x_2$ with the corresponding function denoted as $g_1(u)$. Then, proceeding as in Problem 2.1, we have the following results:

Theorem 2.4. *For $\alpha>-1,\gamma>-1,a>0,\beta>0$*

$$
\begin{aligned}
&G_{1,2}^{2,0}\left[au|_{\alpha,\gamma}^{\alpha+\beta}\right]=\frac{a^{\gamma}}{\Gamma(\beta)}u^{\alpha}\\
&\int_{v=u}^{\infty}v^{\gamma-\alpha-\beta}(v-u)^{\beta-1}\mathrm{e}^{-av}\mathrm{d}v,0\le u<\infty\\
&=\frac{a^{\gamma}}{\Gamma(\beta)}u^{\gamma}\int_0^1 v^{\alpha-\gamma-1}(1-v)^{\beta-1}\mathrm{e}^{-a\frac{u}{v}}\mathrm{d}v,0\le u<\infty\\
&=\frac{\Gamma(\gamma-\alpha)}{\Gamma(\beta)}(au)^{\alpha}{}_1F_1(1-\beta;1+\alpha-\gamma:-au)\\
&+\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha+\beta-\gamma)}(au)^{\gamma}{}_1F_1(1+\gamma-\alpha-\beta;\\
&1+\gamma-\alpha;-au),0\le u<\infty\\
&\textit{for } \alpha-\gamma\ne\pm\nu,\nu=0,1,\ldots,0\le u<\infty
\end{aligned}
$$

Note that this G-function multiplied by $\frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)}\frac{a}{\Gamma(\gamma+1)}$ produces a statistical density $g_1(u)$ for u also. A direct generalization is available by replacing $1-x_1$ in f_1 by $1-a_1x_1^{\delta_1}$ with $1-a_1x_1^{\delta_1}>0,a_1>0,\delta_1>0$ and replacing the exponent ax_2 in f_2 by $ax_2^{\delta_2}$ with $\delta_2>0$. Then, one can obtain parallel results on a H-function. Here, δ_2 can be negative also so that the integrals and the corresponding H-function will exist. If δ_1 is allowed to be negative, then it will create problems with the support there and the problem will be complicated.

Problem 2.5. Type-1 beta vs. type-2 beta

Let

$$
f_1(x_1)=\frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)}x_1^{\alpha}(1-x_1)^{\beta-1},
$$

$$
f_2(x_2)=\frac{\Gamma(\gamma+1+\delta)}{\Gamma(\gamma+1)\Gamma(\delta)}x_2^{\gamma}(1+x_2)^{-(\gamma+1+\delta)},
$$

for $0\le x_1\le 1,x_2\ge 0,\alpha>-1,\gamma>-1,\delta>0,\beta>0$ and let f_1 and f_2 be zero elsewhere. Let the joint function be $f_1(x_1)f_2(x_2)$ and let $u=x_1x_2$ with the corresponding function denoted as $g_1(u)$. Then, proceeding as in Problem 2.1, we will have the following results:

Theorem 2.5. *For* $\alpha > -1, \gamma > -1, \beta > 0, \delta > 0$

$$G_{2,2}^{2,1}\left[u|_{\alpha,\gamma}^{-\delta,\alpha+\beta}\right] = \frac{\Gamma(\gamma+1+\delta)}{\Gamma(\beta)}u^{\alpha}$$
$$\int_{v=u}^{\infty} v^{\gamma-\alpha-\beta}(v-u)^{\beta-1}(1+v)^{-(\gamma+1+\delta)}\mathrm{d}v,$$
$$0 \le u < \infty$$
$$= \frac{\Gamma(\gamma+1+\delta)}{\Gamma(\beta)}u^{\gamma}$$
$$\int_{v=0}^{1} v^{\alpha+\delta}(1-v)^{\beta-1}(v+u)^{-(\gamma+1+\delta)}\mathrm{d}v,$$
$$0 \le u < \infty$$
$$= \begin{cases} \frac{\Gamma(\gamma-\alpha)\Gamma(1+\delta+\alpha)}{\Gamma(\beta)}u^{\alpha}{}_2F_1(1+\delta+\alpha, 1-\beta; \\ 1+\alpha-\gamma; -u) \\ +\frac{\Gamma(\alpha-\gamma)\Gamma(1+\delta+\gamma)}{\Gamma(\alpha+\beta-\gamma)}u^{\gamma}{}_2F_1(1+\delta \\ +\gamma, 1+\gamma-\alpha-\beta; 1+\gamma-\alpha; -u) \\ \textit{for } 0 \le u < 1, \alpha-\gamma \ne \pm\nu, \nu = 0,1,\ldots \\ \frac{\Gamma(\alpha+1+\delta)\Gamma(\gamma+1+\delta)}{\Gamma(\alpha+\beta+1+\delta)}u^{-1-\delta}{}_2F_1(\alpha+1+\delta, \\ \gamma+1+\delta; \alpha+\beta+1+\delta; -\frac{1}{u}) \\ \textit{for } 1 \le u < \infty. \end{cases}$$

This G-function multiplied by $\frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)}\frac{1}{\Gamma(\gamma+1)\Gamma(\delta)}$ produces a statistical density $g_1(u)$ for u also. A direct generalization is available by replacing $1-x_1$ in f_1 by $1-a_1x_1^{\delta_1}$ with $a_1 > 0, \delta_1 > 0, 1-a_1x_1^{\delta_1} > 0$ and replacing $1+x_2$ in f_2 by $1+a_2x_2^{\delta_2}$ with $a_2 > 0, \delta_2 > 0$. Then, one will obtain parallel results on a H-function. Also, note that in this Problem 2.5, if the type-2 beta density is replaced by a general density, then the first integral representation gives Erdélyi-Kober fractional integral of the second kind of order β and parameter α, see also Mathai and Haubold [7] and Fortin et al. [8, 9].

Problem 2.6. Gamma vs. gamma

Let

$$f_j(x_j) = \frac{a_j^{\alpha_j+1}}{\Gamma(\alpha_j+1)}x_j^{\alpha_j}\mathrm{e}^{-a_jx_j}, x_j \ge 0, \alpha_j > -1, a_j > 0, j = 1,2$$

and let the joint function be $f_1(x_1)f_2(x_2)$ and $u = x_1x_2$ with the corresponding function denoted as $g_1(u)$. Then, proceeding as in Problem 2.1, we will have the following results:

Theorem 2.6. *For* $a_j > 0, \alpha_j > -1, j = 1,2$

$$G_{0,2}^{2,0}\left[a_1a_2u|_{\alpha_1,\alpha_2}\right] = a_1^{\alpha_1}a_2^{\alpha_2}u^{\alpha_1}$$
$$\int_{v=0}^{\infty} v^{\alpha_2-\alpha_1-1}\mathrm{e}^{-a_1\frac{u}{v}-a_2v}\mathrm{d}v, 0 \le u < \infty$$
$$= a_1^{\alpha_1}a_2^{\alpha_2}u^{\alpha_2}$$
$$\int_0^{\infty} v^{\alpha_1-\alpha_2-1}\mathrm{e}^{-a_1v-a_2\frac{u}{v}}\mathrm{d}v, 0 \le u < \infty$$
$$= (a_1a_2u)^{\alpha_1}\Gamma(\alpha_2-\alpha_1)$$
$${}_0F_1(\ ;1+\alpha_1-\alpha_2; a_1a_2u)$$
$$+ (a_1a_1u)^{\alpha_2}\Gamma(\alpha_1-\alpha_2)$$
$${}_0F_1(\ ;1+\alpha_2-\alpha_1; a_1a_2u), 0 \le u < \infty$$
$$\textit{for } \alpha_1-\alpha_2 \ne \pm\nu, \nu = 0,1,\ldots$$

A direct generalization is available by replacing the exponent a_jx_j by $a_jx_j^{\delta_j}, \delta_j > 0, j = 1,2$. Then, we will obtain parallel results on a H-function. Here, one can allow $\delta_j < 0, j = 1,2$ also, which will then produce parallel results on a H-function. Since we have started with statistical densities for f_1 and f_2, we may observe that a constant multiple of this G-function in Theorem 2.6 is a statistical density $g_1(u)$ for u also. When $\delta_1 = 1$ we have Krätzel integral and associated with it is the Krätzel transform, see Mathai and Haubold [10]. When $\delta_1 = \delta_2 = 1$, that is the case considered in Theorem 2.6, the integrand in the integral representations, normalized, gives inverse Gaussian density. The unconditional density in the Bayesian analysis when the prior density is gamma and when the scale parameter has a prior gamma distribution then the unconditional density has the structure in the general as well as in the particular case where $\delta_1 = \delta_2 = 1$. In this particular case, one has the Bessel integral given by the G-function representation in the simple poles cases also.

For the sake of illustration, one direct generalization of a Mellin convolution of a product will be derived here in detail. This will be listed as Problem 2.7.

Problem 2.7. Generalized type-2 beta vs. generalized type-2 beta

Let

$$f_j(x_j) = \frac{\delta_j a_j^{\frac{\alpha_j+1}{\delta_j}}\Gamma(\frac{\alpha_j+1}{\delta_j}+\beta_j)}{\Gamma(\frac{\alpha_j+1}{\delta_j})\Gamma(\beta_j)}x_j^{\alpha_j}(1+a_jx_j^{\delta_j})^{-(\frac{\alpha_j+1}{\delta_j}+\beta_j)},$$

for $x_j \ge 0, a_j > 0, \delta_j > 0, \alpha_j > -1, j = 1,2$ and f_j is zero otherwise for $j = 1,2$. Let the joint function be $f_1(x_1)f_2(x_2)$ and let $u = x_1x_2$ and let the function corresponding to u be denoted as $g_1(u)$. Then,

$$E[u^{s-1}] = E[x_1^{s-1}]E[x_2^{s-1}]$$
$$E[x_j^{s-1}] = \frac{\delta_j a_j^{\frac{\alpha_j+1}{\delta_j}}\Gamma(\frac{\alpha_j+1}{\delta_j}+\beta_j)}{\Gamma(\frac{\alpha_j+1}{\delta_j})\Gamma(\beta_j)}$$
$$\int_0^{\infty} x_j^{\alpha_j+s-1}(1+a_jx_j^{\delta_j})^{-(\frac{\alpha_j+1}{\delta_j}+\beta_j)}\mathrm{d}x_j$$
$$= \frac{a_j^{\frac{1}{\delta_j}}}{a_j^{\frac{s}{\delta_j}}}\frac{\Gamma(\frac{\alpha_j+s}{\delta_j})}{\Gamma(\frac{\alpha_j+1}{\delta_j})}\frac{\Gamma(\beta_j-\frac{s-1}{\delta_j})}{\Gamma(\beta_j)}. \tag{2.7}$$

For convenience, let us consider the case $\delta_1 = \delta_2 = \delta$. Then, for $c_1 = \prod_{j=1}^{2}\frac{a_j^{\frac{1}{\delta}}}{\Gamma(\frac{\alpha_j+1}{\delta})\Gamma(\beta_j)}$ and from Equation 2.7 by taking the inverse Mellin transform, we have $g_1(u)$ as the following:

$$g_1(u) = c_1\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}[\prod_{j=1}^{2}\Gamma(\frac{\alpha_j+s}{\delta})\Gamma(\beta_j-\frac{s-1}{\delta})][(a_1a_2)^{\frac{1}{\delta}}u]^{-s}\mathrm{d}s$$
$$= c_1H_{2,2}^{2,2}\left[(a_1a_2)^{\frac{1}{\delta}}u|_{(\frac{\alpha_j}{\delta},\frac{1}{\delta}),j=1,2}^{(1-\beta_j-\frac{1}{\delta},\frac{1}{\delta}),j=1,2}\right], 0 \le u < \infty. \tag{2.8}$$

The inverse Mellin transform in Equation 2.8 will be evaluated by using residue calculus. Note that for $\alpha_1-\alpha_2 \ne \pm\nu, \nu = 0,1,\ldots$, the poles of $\Gamma(\frac{\alpha_1+s}{\delta})\Gamma(\frac{\alpha_2+s}{\delta})$ are simple. The poles of $\Gamma(\frac{\alpha_j+s}{\delta})$ are at $\frac{\alpha_j+s}{\delta} = -\nu, \nu = 0,1,\ldots \Rightarrow s = -\alpha_j-\delta\nu, \nu = 0,1,\ldots$. The residue

at $s = -\alpha_1 - \delta\nu$ coming from $\Gamma(\frac{\alpha_1+s}{\delta})$ is $\lim_{s\to-\alpha_1-\delta\nu}(s+\alpha_1+\delta\nu)\Gamma(\frac{\alpha_1+s}{\delta}) = \delta\frac{(-1)^\nu}{\nu!}$. The rest of the steps are parallel to those in the derivations in Problem 2.1. For $0 \le a_1a_2u^\delta < 1$, we will end up with two Gauss' hypergeometric series, and the continuation part for $1 \le a_1a_1u^\delta < \infty$ will give another two Gauss' hypergeometric series. If we consider the transformation of variables, then $g_1(u)$ has the following forms where $c = \prod_{j=1}^2 \frac{\delta a_j^{\frac{\alpha_j+1}{\delta}}\Gamma(\frac{\alpha_j+1}{\delta}+\beta_j)}{\Gamma(\frac{\alpha_j+1}{\delta})\Gamma(\beta_j)}$:

$$\begin{aligned} g_1(u) &= c\int_v \frac{1}{v}(\frac{u}{v})^{\alpha_1}[1+a_1(\frac{u}{v})^\delta]^{-(\frac{\alpha_1+1}{\delta}+\beta_1)}v^{\alpha_2} \\ &\quad (1+a_2v)^{-(\frac{\alpha_2+1}{\delta}+\beta_2)}\mathrm{d}v \\ &= c\,u^{\alpha_1}\int_0^\infty v^{\alpha_2-\alpha_1-1}[1+\frac{a_1u^\delta}{v^\delta}]^{-(\frac{\alpha_1+1}{\delta}+\beta_1)} \\ &\quad (1+a_2v)^{-(\frac{\alpha_2+1}{\delta}+\beta_2)}\mathrm{d}v. \end{aligned} \tag{2.9}$$

The second form is available by interchanging (α_1,β_1) and (α_2,β_2) in Equation 2.9. Note that

$$g_1(u) = \delta^2\left\{\prod_{j=1}^2 a_j^{\frac{\alpha_j}{\delta}}\Gamma(\frac{\alpha_j+1}{\delta}+\beta_j)\right\}\times\ (\text{the H-function})$$

where the H-function has the following explicit forms:

$$H = H_{2,2}^{2,2}\left[(a_1a_2)^{\frac{1}{\delta}}u\Big|_{(\frac{\alpha_j}{\delta},\frac{1}{\delta}),j=1,2}^{(1-\beta_j-\frac{1}{\delta},\frac{1}{\delta}),j=1,2}\right]$$

$$= \begin{cases} \delta\Gamma(\frac{\alpha_2-\alpha_1}{\delta})\Gamma(\beta_1+\frac{\alpha_1+1}{\delta})\Gamma(\beta_2+\frac{\alpha_1+1}{\delta})[(a_1a_2)^{\frac{1}{\delta}}u]^{\alpha_1} \\ \times{}_2F_1(\beta_1+\frac{\alpha_1+1}{\delta},\beta_2+\frac{\alpha_1+1}{\delta};1+\frac{\alpha_1-\alpha_2}{\delta};a_1a_1u^\delta) \\ +\delta\Gamma(\frac{\alpha_1-\alpha_2}{\delta})\Gamma(\beta_1+\frac{\alpha_2+1}{\delta})\Gamma(\beta_2+\frac{\alpha_2+1}{\delta})[(a_1a_2)^{\frac{1}{\delta}}u]^{\alpha_2} \\ \times{}_2F_1(\beta_1+\frac{\alpha_2+1}{\delta},\beta_2+\frac{\alpha_2+1}{\delta};1+\frac{\alpha_2-\alpha_1}{\delta};a_1a_2u^\delta), \\ \text{for } 0\le a_1a_2u^\delta<1 \text{ and for } \alpha_1-\alpha_2\ne\pm\nu,\nu=0,1,\ldots \\ \delta\Gamma(\beta_1+\frac{\alpha_1+1}{\delta})\Gamma(\beta_1+\frac{\alpha_2+1}{\delta})\Gamma(\beta_2-\beta_1)[(a_1a_2)^{\frac{1}{\delta}}u]^{-(\beta_1\delta+1)} \\ \times{}_2F_1(\beta_1+\frac{\alpha_1+1}{\delta},\beta_1+\frac{\alpha_2+1}{\delta};1+\beta_1-\beta_2;\frac{1}{a_1a_2u^\delta}) \\ +\delta\Gamma(\beta_2+\frac{\alpha_1+1}{\delta})\Gamma(\beta_2+\frac{\alpha_2+1}{\delta})\Gamma(\beta_1-\beta_2) \\ [(a_1a_2)^{\frac{1}{\delta}}u]^{-(\beta_2\delta+1)} \\ \times{}_2F_1(\beta_2+\frac{\alpha_1+1}{\delta},\beta_2+\frac{\alpha_2+1}{\delta};1+\beta_2-\beta_1;\frac{1}{a_1a_2u^\delta}) \\ \text{for } a_1a_2u^\delta\ge1 \text{ and for } \beta_1-\beta_2\ne\pm\nu,\nu=0,1,\ldots \end{cases}$$

This completes the calculations and the representations of $g_1(u)$ in two different integral representations and in terms of a H-function with explicit computable series form representations.

3 Mellin convolution of a ratio

Here, we examine a ratio. Let $x_1 > 0$ and $x_2 > 0$ be real scalar positive variables associated with the functions $f_1(x_1)$ and $f_2(x_2)$, respectively. Let the joint function be $f_1(x_1)f_2(x_2)$. Let $u = \frac{x_2}{x_1}$ the ratio. Let the Mellin transforms of f_1 and f_2 be $M_{f_1}(s)$ and $M_{f_2}(s)$, respectively with the Mellin parameter s. Let the function associated with the ratio u be denoted by $g_2(u)$. Then, the Mellin transform of $g_2(u)$ is available from the joint function as $M_{f_2}(s)M_{f_1}(2-s)$ which can easily be seen if x_1 and x_2 are statistically independently distributed random variables with the densities $f_1(x_1)$ and $f_2(x_2)$, respectively. Then,

$$E[u^{s-1}] = E[(\frac{x_2}{x_1})^{s-1}] = E[x_2^{s-1}]E[x_1^{-s+1}] = M_{f_2}(s)M_{f_1}(2-s) \tag{3.1}$$

whenever the Mellin transforms exist. Then, from the inverse Mellin transform, $g_2(u)$ is available as the following:

$$g_2(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_{f_2}(s)M_{f_1}(2-s)u^{-s}\mathrm{d}s, i=\sqrt{(-1)} \tag{3.2}$$

whenever the integral is convergent. We can also reach $g_2(u)$ through transformation of variables. Let $x_2 = v$ and $u = \frac{x_2}{x_1}$, then $x_1 = \frac{v}{u}$ and $\mathrm{d}x_1\wedge\mathrm{d}x_2 = -\frac{v}{u^2}\mathrm{d}u\wedge\mathrm{d}v$. If one takes $x_1 = v$, then $x_2 = uv$ and $\mathrm{d}x_1\wedge\mathrm{d}x_2 = v\mathrm{d}u\wedge\mathrm{d}v$. Then, we have two different integral representations for $g_2(u)$, namely,

$$g_2(u) = \int_v \frac{v}{u^2}f_1(\frac{v}{u})f_2(v)\mathrm{d}v \tag{3.3}$$

$$= \int_v vf_1(v)f_2(uv)\mathrm{d}v. \tag{3.4}$$

Then, Equations 3.2–3.4 give three different representations for the same function $g_2(u)$. We will consider various functions belonging to the pathway family of functions for convenience.

Problem 3.1. Gamma vs. gamma

Let

$$f_j(x_j) = \frac{a_j^{\alpha_j+1}}{\Gamma(\alpha_j+1)}x_j^{\alpha_j}\mathrm{e}^{-a_jx_j}, x_j\ge0, \alpha_j>-1, a_j>0, j=1,2$$

and let the joint function be $f_1(x_1)f_2(x_2)$ and let $c = \prod_{j=1}^2\frac{a_j^{\alpha_j+1}}{\Gamma(\alpha_j+1)}$. Then, from Equation 3.3

$$\begin{aligned} g_2(u) &= c\int\frac{v}{u^2}(\frac{v}{u})^{\alpha_1}\mathrm{e}^{-a_1\frac{v}{u}}v^{\alpha_2}\mathrm{e}^{-a_2v}\mathrm{d}v \\ &= c\Gamma(\alpha_1+\alpha_2+2)u^{\alpha_2}(a_1+a_2u)^{-(\alpha_1+\alpha_2+2)} \end{aligned} \tag{3.5}$$

and from Equation 3.4

$$\begin{aligned} g_2(u) &= c\int_v vv^{\alpha_1}\mathrm{e}^{-a_1v}(uv)^{\alpha_2}\mathrm{e}^{-a_2uv}\mathrm{d}v \\ &= c\,u^{\alpha_2}\Gamma(\alpha_1+\alpha_2+2)(a_1+a_2u)^{-(\alpha_1+\alpha_2+2)}, \end{aligned} \tag{3.6}$$

for $0\le u<\infty$, which is the same as the one in Equation 3.5. Also,

$$\begin{aligned} M_{f_2}(s) &= \frac{a_2^{\alpha_2+1}}{\Gamma(\alpha_2+1)}\int_0^\infty x_2^{\alpha_2+s-1}\mathrm{e}^{-a_2x_2}\mathrm{d}x_2 \\ &= \frac{a_2}{a_2^s}\frac{\Gamma(\alpha_2+s)}{\Gamma(\alpha_2+1)}, \Re(\alpha_2+s)>0 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} M_{f_1}(2-s) &= \frac{a_1^{\alpha_1+1}}{\Gamma(\alpha_1+1)}\int_0^\infty x_1^{\alpha_1-s+1}\mathrm{e}^{-a_1x_1}\mathrm{d}x_1 \\ &= \frac{a_1^s}{\Gamma(\alpha_1+1)a_1}\Gamma(2+\alpha_1-s), \Re(2+\alpha_1-s)>0. \end{aligned} \tag{3.8}$$

Then,

$$g_2(u) = \frac{a_2}{a_1}\frac{1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\alpha_2+s)\Gamma(2+\alpha_1-s)(\frac{a_2u}{a_1})^{-s}\mathrm{d}s = \frac{a_2}{a_1\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}G_{1,1}^{1,1}\left[\frac{a_2u}{a_1}|_{\alpha_2}^{-1-\alpha_1}\right], 0 \le u < \infty. \tag{3.9}$$

This G-function can be evaluated as the sum of the residues at the poles of $\Gamma(\alpha_2+s)$ for $0 \le \frac{a_2u}{a_1} < 1$ and as the sum of the residues at the poles of $\Gamma(2+\alpha_1-s)$ for $1 \le \frac{a_2u}{a_1} < \infty$. Thus, we have

$$g_2(u) = \frac{\Gamma(\alpha_1+\alpha_2+2)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} \times \frac{1}{u}\begin{cases}(\frac{a_2u}{a_1})^{\alpha_2+1}{}_1F_0(2+\alpha_1+\alpha_2;\ ;-\frac{a_2u}{a_1}), 0 \le \frac{a_2u}{a_1} < 1\\ (\frac{a_1}{a_2u})^{\alpha_1+1}{}_1F_0(2+\alpha_1+\alpha_2;\ ;-\frac{a_1}{a_2u}), 1 \le \frac{a_2u}{a_1} < \infty.\end{cases} \tag{3.10}$$

Then, from Equations 3.5, 3.6, 3.10, we have the following theorem:

Theorem 3.1. *For* $a_j > 0, \alpha_j > -1, c = \left\{\prod_{j=1}^2 \frac{a_j^{\alpha_j+1}}{\Gamma(\alpha_1+1)}\right\}$,

$$g_2(u) = c\Gamma(\alpha_1+\alpha_2+2)u^{\alpha_2}(a_1+a_2u)^{-(\alpha_1+\alpha_2+2)}, 0 \le u < \infty = \frac{\Gamma(\alpha_1+\alpha_2+2)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} \times \frac{1}{u}\begin{cases}(\frac{a_2u}{a_1})^{\alpha_2+1}{}_1F_0(2+\alpha_1+\alpha_2;\ ;-\frac{a_2u}{a_1}), 0 \le \frac{a_2u}{a_1} < 1\\ (\frac{a_1}{a_2u})^{\alpha_1+1}{}_1F_0(2+\alpha_1+\alpha_2;\ ;-\frac{a_1}{a_2u}), 1 \le \frac{a_2u}{a_1} < \infty,\end{cases}$$

where $g_2(u)$ is a statistical density for u also.

Generalization is available by replacing a_jx_j in $f_j(x_j)$ by $a_jx_j^{\delta_j}, \delta_j > 0, j = 1,2$. In this case, one will end up with a H-function instead of the G-function above. Then, various integral and series representations of that H-function will be available. Here, we will be able to consider $\delta_j < 0, j = 1,2$ also giving rise to parallel results. Since the derivations of the results in Problem 3.1 are given in detail, the remaining results will be stated without the detailed derivations. The derivations will be parallel to those provided in Problem 3.1.

Problem 3.2. Gamma vs. type-2 beta

Let

$$f_1(x_1) = \frac{a_1^{\alpha_1+1}}{\Gamma(\alpha_1+1)}x_1^{\alpha_1}\mathrm{e}^{-a_1x_1},$$
$$f_2(x_2) = \frac{\Gamma(\alpha_2+1+\beta_2)}{\Gamma(\alpha_2+1)\Gamma(\beta_2)}x_2^{\alpha_2}(1+x_2)^{-(\alpha_2+1+\beta_2)}$$

for $\alpha_j > -1, 0 \le x_j < \infty, j = 1,2, a_1 > 0, \beta_2 > 0$ and let the joint function be $f_1(x_1)f_2(x_2)$. Let $u = \frac{x_2}{x_1}$ with the corresponding function denoted as $g_2(u)$. Then, following through the derivations parallel to those in Problem 3.1, we have the following results:

Theorem 3.2. *For* $\alpha_j > -1, j = 1,2, \beta_2 > 0, a_1 > 0$

$$G_{2,1}^{1,2}\left[\frac{u}{a_1}|_{\alpha_2}^{-\beta_2,-1-\alpha_1}\right] = a_1^{\alpha_1+2}\Gamma(\alpha_2+\beta_2+1)u^{-\alpha_1-2}\int_0^\infty v^{\alpha_1+\alpha_2+1}(1+v)^{-(1+\alpha_2+\beta_2)}\mathrm{e}^{-a_1\frac{v}{u}}\mathrm{d}v, 0 \le u < \infty = a_1^{\alpha_1+2}\Gamma(\alpha_2+\beta_2+1)u^{\alpha_2}\int_0^\infty v^{\alpha_1+\alpha_2+1}(1+uv)^{-(1+\alpha_2+\beta_2)}\mathrm{e}^{-a_1v}\mathrm{d}v, 0 \le u < \infty = \Gamma(\alpha_2+\beta_2+1)\Gamma(1+\alpha_1-\beta_2)(\frac{a_1}{u})^{1+\beta_2}{}_1F_1(1+\alpha_2+\beta_2;\beta_2-\alpha_1;\frac{a_1}{u}) + \Gamma(2+\alpha_1+\alpha_2)\Gamma(\beta_2-\alpha_1-1)(\frac{a_1}{u})^{2+\alpha_1}{}_1F_1(2+\alpha_1+\alpha_2;2+\alpha_1-\beta_2;\frac{a_1}{u}), 0 \le u < \infty$$
for $\beta_2-\alpha_1-1 \ne \pm\nu, \nu = 0,1,\ldots$

Generalizations are possible by replacing the exponent a_1x_1 in f_1 by $a_1x_1^{\delta_1}, \delta_1 > 0$ and replacing $1+x_2$ in f_2 by $1+a_2x_2^{\delta_2}, a_2 > 0, \delta_2 > 0$. Then, one will end up with a H-function, and results will be available for this H-function. In this case, $\delta_j < 0, j = 1,2$ will also work and parallel results will be available.

Problem 3.3. Type-2 beta vs. gamma

Let

$$f_1(x_1) = \frac{\Gamma(\gamma+1+\delta)}{\Gamma(\gamma+1)\Gamma(\delta)}x_1^{\gamma}(1+x_1)^{-(\gamma+1+\delta)},$$
$$f_2(x_2) = \frac{a^{\alpha+1}}{\Gamma(\alpha+1)}x_2^{\alpha}\mathrm{e}^{-ax_2}$$

for $\gamma > -1, \alpha > -1, a > 0, \delta > 0, 0 \le x_j < \infty, j = 1,2$ and let the joint function be $f_1(x_1)f_2(x_2)$. Let $u = \frac{x_2}{x_1}$. Then, proceeding as in the derivations in Problem 3.1, we will have the following results:

Theorem 3.3. *For* $\alpha > -1, \gamma > -1, \delta > 0, a > 0$

$$G_{1,2}^{2,1}\left[au|_{\alpha,\delta-1}^{-1-\gamma}\right] = \Gamma(\gamma+1+\delta)a^{\alpha}u^{\delta-1}\int_0^\infty v^{\alpha+\gamma+1}(u+v)^{-(\gamma+1+\delta)}\mathrm{e}^{-av}\mathrm{d}v, 0 \le u < \infty = \Gamma(\gamma+1+\delta)a^{\alpha}u^{\alpha}\int_0^\infty v^{\alpha+\gamma+1}(1+v)^{-(\gamma+1+\delta)}\mathrm{e}^{-auv}\mathrm{d}v, 0 \le u < \infty = \Gamma(\delta-1-\alpha)\Gamma(2+\gamma+\alpha)(au)^{\alpha}{}_1F_1(2+\gamma+\alpha;2+\alpha-\delta;au) + \Gamma(1+\gamma+\delta)\Gamma(1+\alpha-\delta)(au)^{\delta-1}{}_1F_1(1+\gamma+\delta;\delta-\alpha;au), 0 \le u < \infty$$
for $\delta-1-\alpha \ne \pm\nu, \nu = 0,1,\ldots$

Generalizations are possible by replacing $1+x_1$ in f_1 by $1+b_1x_1^{\delta_1}, b_1 > 0, \delta_1 > 0$ and by replacing the exponent in f_2 by $ax_2^{\delta_2}, \delta_2 > 0$. This will produce a H-function and several results on this H-function. In this case, $\delta_j < 0, j = 1,2$ will also work, resulting in parallel results.

Problem 3.4. Gamma vs. type-1 beta

Let

$$f_1(x_1)=\frac{a^{\gamma+1}}{\Gamma(\gamma+1)}x_1^{\gamma}\mathrm{e}^{-ax_1}, f_2(x_2)=\frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)}x_2^{\alpha}(1-x)^{\beta-1}$$

for $a>0,\beta>0,\gamma>-1,\alpha>-1,x_1\geq 0, 0\leq x_2\leq 1$, and f_1 and f_2 are zero otherwise. Let the joint function be $f_1(x_1)f_2(x_2)$ and let $u=\frac{x_2}{x_1}$. Then, proceeding as in the derivations for Problem 3.1, we have the following results:

Theorem 3.4. *For $a>0,\beta>0,\alpha>-1,\gamma>-1$*

$$\begin{aligned}G_{2,1}^{1,1}\left[\frac{u}{a}|_{\alpha}^{-1-\gamma,\alpha+\beta}\right]&=\frac{a^{\gamma+2}}{\Gamma(\beta)}u^{-\gamma-2}\\&\int_0^1 v^{\alpha+\gamma+1}(1-v)^{\beta-1}\mathrm{e}^{-a\frac{v}{u}}\mathrm{d}v, 0<u<\infty\\&=\frac{a^{\gamma+2}}{\Gamma(\beta)}u^{\alpha}\\&\int_0^{\frac{1}{u}} v^{\alpha+\gamma+1}(1-uv)^{\beta-1}\mathrm{e}^{-av}\mathrm{d}v, 0\leq u<\infty\\&=\frac{\Gamma(2+\alpha+\gamma)}{\Gamma(2+\alpha+\gamma+\beta)}(\frac{u}{a})^{-\gamma-2}\\&{}_1F_1(2+\alpha+\gamma;2+\alpha+\gamma+\beta;-\frac{a}{u}),\end{aligned}$$

for $0\leq u<\infty$. Generalization can be achieved by replacing the exponent ax_1 in f_1 by $ax_1^{\delta_1},\delta_1>0$ and replacing $1-x_2$ in f_2 by $1-a_2x_2^{\delta_2}, a_2>0,\delta_2>0, 1-a_2x_2^{\delta_2}>0$. Then, one will end up with a H-function and several results on this H-function.

Problem 3.5. Type-1 beta vs. gamma

Let

$$f_1(x_1)=\frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)}x_1^{\alpha}(1-x_1)^{\beta-1}, f_2(x_2)=\frac{a^{\gamma+1}}{\Gamma(\gamma+1)}x_2^{\gamma}\mathrm{e}^{-ax_2}$$

for $\alpha>-1,\gamma>-1,a>0,\beta>0,x_2\geq 0, 0\leq x_1\leq 1$, and f_1 and f_2 are zero otherwise. Let the joint function be $f_1(x_1)f_2(x_2)$ and let $u=\frac{x_2}{x_1}$. Then, proceeding as in the derivations in Problem 3.1, we have the following results:

Theorem 3.5. *For $\alpha>-1,\gamma>-1,a>0,\beta>0$*

$$\begin{aligned}G_{1,2}^{1,1}\left[au|_{\gamma,-1-\alpha-\beta}^{-1-\alpha}\right]&=\frac{a^{\gamma}}{\Gamma(\beta)}u^{-\beta-\alpha-1}\\&\int_{v=0}^{u} v^{\alpha+\gamma+1}(u-v)^{\beta-1}\mathrm{e}^{-av}\mathrm{d}v, 0<u<\infty\\&=\frac{a^{\gamma}}{\Gamma(\beta)}u^{\gamma}\int_0^1 v^{\alpha+\gamma+1}(1-v)^{\beta-1}\mathrm{e}^{-auv}\mathrm{d}v,\\&0<u<\infty\\&=a^{\gamma}\frac{\Gamma(2+\alpha+\gamma)}{\Gamma(2+\alpha+\gamma+\beta)}u^{\gamma}\\&{}_1F_1(2+\alpha+\gamma;2+\alpha+\gamma+\beta;-au), 0\leq u<\infty.\end{aligned}$$

A direct generalization is achieved by replacing $1-x_1$ in f_1 by $1-a_1x_1^{\delta_1}, a_1>0,\delta_1>0, 1-a_1x_1^{\delta_1}>0$ and by replacing the exponent ax_2 in f_2 by $ax_2^{\delta_2},\delta_2>0$. Then, one will end up with a H-function, and several results will be available on this H-function.

It may be observed that if in f_1, α is replaced by $\alpha-1$, and if f_2 is a general function, then the first integral representation in Theorem 3.5 is also Erdélyi-Kober fractional integral of the first kind of order β and parameter α, see also Mathai and Haubold [7].

Problem 3.6. Type-1 beta vs. type-1 beta

Let

$$f_j(x_j)=\frac{\Gamma(\alpha_j+1+\beta_j)}{\Gamma(\alpha_j+1)\Gamma(\beta_j)}x_j^{\alpha_j}(1-x_j)^{\beta_j-1},$$
$$0\leq x_j\leq 1,\alpha_j>-1,\beta_j>0,j=1,2.$$

Let the joint function be $f_1(x_1)f_2(x_2)$ and let $u=\frac{x_2}{x_1}$. Then, proceeding as in the derivation of Problem 3.1, we have the following results:

Theorem 3.6. *For $\alpha_j>-1,\beta_j>0,j=1,2$*

$$G=G_{2,2}^{1,1}\left[u|_{\alpha_2,-1-\alpha_1-\beta_1}^{-1-\alpha_1,\alpha_2+\beta_2}\right]$$
$$=\begin{cases}\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}u^{\alpha_2}\int_0^{\frac{1}{u}} v^{\alpha_1+\alpha_2+1}(1-v)^{\beta_1-1}(1-uv)^{\beta_2-1}\mathrm{d}v,\\0\leq u<1\\\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}u^{-\alpha_1-\beta_1-1}\int_0^1 v^{\alpha_1+\alpha_2+1}(u-v)^{\beta_1-1}(1-v)^{\beta_2-1}\mathrm{d}v,\\1\leq u<\infty\end{cases}$$

$$G=\begin{cases}\frac{\Gamma(2+\alpha_1+\alpha_2)}{\Gamma(2+\alpha_1+\alpha_2+\beta_1)\Gamma(\beta_2)}u^{\alpha_2}{}_2F_1(1-\beta_2,2+\alpha_1+\alpha_2;\\2+\alpha_1+\alpha_2+\beta_1;u), 0\leq u<1\\\frac{\Gamma(2+\alpha_1+\alpha_2)}{\Gamma(2+\alpha_1+\alpha_2+\beta_2)\Gamma(\beta_1)}u^{-2-\alpha_1}{}_2F_1(1-\beta_1,2+\alpha_1+\alpha_2;\\2+\alpha_1+\alpha_2+\beta_2;\frac{1}{u}),\\\text{for } 1\leq u<\infty.\end{cases}$$

A direct generalization will be available by replacing $1-x_j$ by $1-a_jx_j^{\delta_j}, a_j>0,\delta_j>0, 1-a_jx_j^{\delta_j}>0, j=1,2$. Then, we will end up with a H-function, and several results will be available on this H-function. One general case of the ratio $u=\frac{x_2}{x_1}$ will be discussed here for the sake of illustration. This will be listed here as Problem 3.7.

Problem 3.7. Generalized type-2 beta vs. generalized gamma

Let

$$f_1(x_1)=\delta\frac{a^{\frac{\alpha+1}{\delta}}\Gamma(\frac{\alpha+1}{\delta}+\beta)}{\Gamma(\frac{\alpha+1}{\delta})\Gamma(\beta)}x_1^{\alpha}(1+ax_1^{\delta})^{-(\frac{\alpha+1}{\delta}+\beta)},$$
$$f_2(x_2)=\frac{\rho b^{\frac{\gamma}{\rho}}}{\Gamma(\frac{\gamma+1}{\rho})}x_2^{\rho}\mathrm{e}^{-bx_2^{\rho}}$$

for $\alpha>-1,\gamma>-1,\delta>0,\rho>0,a>0,b>0,\beta>0$, and f_1 and f_2 are zero elsewhere. Let $u=\frac{x_2}{x_1}$ with the associated function $g_2(u)$. The Mellin transforms of f_1 and f_2 are the following:

$$M_{f_2}(s)=\frac{b^{\frac{1}{\rho}}}{\Gamma(\frac{\gamma+1}{\rho})}\Gamma(\frac{\gamma+s}{\rho})(b^{\frac{1}{\rho}})^{-s}, \Re(\frac{\gamma+s}{\rho})>0$$

and

$$M_{f_1}(2-s)=\frac{1}{a^{\frac{1}{\delta}}\Gamma(\frac{\alpha+1}{\delta})\Gamma(\beta)}\Gamma(\frac{2+\alpha}{\delta}-\frac{s}{\delta})\Gamma(\beta-\frac{1}{\delta}+\frac{s}{\delta})(a^{\frac{1}{\delta}})^{s}$$

for $\Re(2+\alpha-s)>0, \Re(\beta\delta-1+s)>0$. Then

$$M_{f_1}(2-s)M_{f_2}(s)=\frac{b^{\frac{1}{\rho}}}{a^{\frac{1}{\delta}}\Gamma(\frac{\alpha+1}{\delta})\Gamma(\frac{\gamma+1}{\rho})\Gamma(\beta)}$$
$$\times\Gamma(\frac{\gamma}{\rho}+\frac{s}{\rho})\Gamma(\beta-\frac{1}{\delta}+\frac{s}{\delta})\Gamma(\frac{2+\alpha}{\delta}-\frac{s}{\delta})\left(\frac{b^{\frac{1}{\rho}}}{a^{\frac{1}{\delta}}}\right)^{-s}.$$

Let

$$c=\frac{\delta\rho a^{\frac{\alpha+1}{\delta}}b^{\frac{\gamma+1}{\rho}}\Gamma(\frac{\alpha+1}{\delta}+\beta)}{\Gamma(\frac{\alpha+1}{\delta})\Gamma(\beta)\Gamma(\frac{\gamma+1}{\rho})}, c_1=\frac{b^{\frac{1}{\rho}}}{a^{\frac{1}{\delta}}\Gamma(\frac{\alpha+1}{\delta})\Gamma(\beta)\Gamma(\frac{\gamma+1}{\rho})}.$$

Then, from the inverse Mellin transform we have

$$\begin{aligned}g_2(u)&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}M_{f_1}(2-s)M_{f_2}(s)u^{-s}\mathrm{d}s\\&=c_1\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(\frac{\gamma}{\rho}+\frac{s}{\rho})\Gamma(\beta-\frac{1}{\delta}+\frac{s}{\delta})\\&\Gamma(\frac{2+\alpha}{\delta}-\frac{s}{\delta})\left[\frac{b^{\frac{1}{\rho}}u}{a^{\frac{1}{\delta}}}\right]^{-s}\mathrm{d}s\\&=c_1H_{1,2}^{2,1}\left[\frac{b^{\frac{1}{\rho}}u}{a^{\frac{1}{\delta}}}\Big|_{(\frac{\gamma}{\rho},\frac{1}{\rho}),(\beta-\frac{1}{\delta},\frac{1}{\delta})}^{(1-\frac{2+\alpha}{\delta},\frac{1}{\delta})}\right],0\le u<\infty.\end{aligned}$$

From the integral representations of $g_2(u)$, we have the following forms and for $\delta=\rho$ and for $\gamma+\rho\nu\neq\beta\delta+\delta\lambda, \nu=0,1,...,\lambda=0,1,...$ the poles of the integrand are simple, and then, we have the following series representations for the H-function for $\delta=\rho$. We will state this as a theorem.

Theorem 3.7. *For the conditions, c, c_1 stated above, we have*

$$\begin{aligned}g_2(u)&=cu^{-\alpha-2}\int_0^\infty v^{\alpha+\gamma+1}[1+a(\frac{v}{u})^\delta]^{-(\frac{\alpha+1}{\delta}+\beta)}\mathrm{e}^{-bv^\rho}\mathrm{d}v\\&=cu^{-\gamma}\int_0^\infty v^{\alpha+\gamma+1}[1+av^\delta]^{-(\frac{\alpha+1}{\delta}+\beta)}\mathrm{e}^{-b(uv)^\rho}\mathrm{d}v\\&=c_1\ \textit{H-function above}\end{aligned}$$

where for $\rho=\delta$,

$$H=H_{1,2}^{2,1}\left[(\frac{b}{a})^{\frac{1}{\delta}}u\Big|_{(\frac{\gamma}{\delta},\frac{1}{\delta}),(\beta-\frac{1}{\delta},\frac{1}{\delta})}^{(1-\frac{2+\alpha}{\delta},\frac{1}{\delta})}\right]$$
$$=\delta\begin{cases}\Gamma(\beta-\frac{\gamma+1}{\delta})\Gamma(\frac{2+\alpha+\gamma}{\delta})[(\frac{b}{a})^{\frac{1}{\delta}}u]^\gamma{}_1F_1(\frac{2+\alpha+\gamma}{\delta};\\1+\frac{\gamma+1}{\delta}-\beta;\frac{bu^\delta}{a})\\+\Gamma(\frac{\gamma+1}{\delta}-\beta)\Gamma(\frac{\alpha+1}{\delta}+\beta)\left[(\frac{b}{a})^{\frac{1}{\delta}}u\right]^{\beta\delta-1}{}_1F_1(\frac{\alpha+1}{\delta}+\beta;\\1+\beta-\frac{\alpha+1}{\delta};\frac{b}{a}u^\delta),\\\text{for } 0\le u<\infty,\gamma-\delta\beta+1\neq\pm\nu,\nu=0,1,...\end{cases}$$

4 Generalization to matrix-variate cases

Here, all the matrices appearing are $p\times p$ real symmetric positive definite when in the real domain or $p\times p$ Hermitian positive definite when in the complex domain. Let $X_1>O$ and $X_2>O$ be $p\times p$ real symmetric positive definite matrices with the associated functions $f_1(X_1)$ and $f_2(X_2)$, respectively, and with the joint function $f_1(X_1)f_2(X_2)$, capital letters representing matrices and lower case letters denoting scalar variables where $f_j(X_j)$ is a real-valued scalar function of $X_j, j=1,2$. When $f_j(X_j)\ge 0$ for all X_j, in the domain of X_j, with $\int_{X_j}f_j(X_j)\mathrm{d}X_j=1$, then $f_j(X_j)$ is a statistical density also. Since we are dealing with symmetric matrices, we will be defining symmetric product corresponding to the scalar case x_1x_2 as $X_2^{\frac{1}{2}}X_1X_2^{\frac{1}{2}}$ where $X_2^{\frac{1}{2}}$ is the symmetric positive definite square root of X_2, and symmetric ratio corresponding to $\frac{x_2}{x_1}$ as $X_2^{\frac{1}{2}}X_1^{-1}X_2^{\frac{1}{2}}$ where X_1^{-1} is the regular inverse of X_1. We can also define symmetric product corresponding to x_1x_2 as $X_1^{\frac{1}{2}}X_2X_1^{\frac{1}{2}}$ and symmetric ratio corresponding to $\frac{x_2}{x_1}$ as $X_1^{-\frac{1}{2}}X_2X_1^{-\frac{1}{2}}$ also.

4.1 Symmetric product in the real matrix-variate case

Let $U=X_2^{\frac{1}{2}}X_1X_2^{\frac{1}{2}}$ with $V=X_2$ so that $X_1=V^{-\frac{1}{2}}UV^{-\frac{1}{2}}$. We can show that

$$\mathrm{d}X_1\wedge\mathrm{d}X_2=|V|^{-\frac{p+1}{2}}\mathrm{d}U\wedge\mathrm{d}V \tag{4.1}$$

see, for example, Mathai [11], where $|(\cdot)|$ denotes the determinant of $(\cdot)$. Let $X_j=(x_{jik})$ with the (i,k)-th element x_{jik} for $i=1,...,p, k=1,...,p$. Since $X_j=X_j'$, a prime denoting the transpose, there are only $p(p+1)/2$ distinct elements in X_j and only $p(p+1)/2$ differentials $\mathrm{d}x_{jik}, i\le k$ or $i\ge k$. Then, $\mathrm{d}X_j=\wedge_{i\le k}\mathrm{d}x_{jik}$. Note that the wedge product in Equation 4.1 remains the same if we take $V=X_1$ instead of X_2. For illustrative purposes, we will discuss one problem involving M-convolution of a product. For M-convolutions and M-transforms, see Mathai [11].

Problem 4.1. Real $p\times p$ matrix-variate gamma vs. real $p\times p$ matrix-variate gamma

Let

$$f_j(X_j)=\frac{|B_j|^{\alpha_j}}{\Gamma_p(\alpha_j)}|X_j|^{\alpha_j-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(B_jX_j)}, X_j>O, B_j>O, j=1,2$$

where $X_j=X_j'>O$ is a real matrix of a $p(p+1)/2$ distinct (functionally independent) real scalar variables x_{jik} as elements, $B_j=B_j'>O$ is a $p\times p$ constant positive definite matrix, $\mathrm{tr}(\cdot)$ means the trace of $(\cdot)$, and $\Gamma_p(\alpha_j)$ is a real matrix-variate gamma defined as

$$\Gamma_p(\alpha)=\pi^{\frac{p(p-1)}{4}}\Gamma(\alpha)\Gamma(\alpha-\frac{1}{2})...\Gamma(\alpha-\frac{p-1}{2}),\Re(\alpha)>\frac{p-1}{2} \tag{4.2}$$

$$=\int_{Y>O}|Y|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(Y)}\mathrm{d}Y, Y>O,\Re(\alpha)>\frac{p-1}{2} \tag{4.3}$$

and thus, the real matrix-variate gamma $\Gamma_p(\alpha)$ is associated with the real matrix-variate gamma integral in Equation 4.3. Hence, $\Gamma_p(\alpha)$ is called as the real matrix-variate gamma. Let $g_2(U)$ be the real-valued scalar function associated with the matrix $U=$

$X_2^{\frac{1}{2}}X_1X_2^{\frac{1}{2}}$. Then, from the joint function $f_1(X_1)f_2(X_2)$, we have for $c=\prod_{j=1}^{2}\frac{|B_j|^{\alpha_j}}{\Gamma_p(\alpha_j)}$,

$$g_2(U)=c\int_{V>O}|V|^{-\frac{p+1}{2}}|V^{-\frac{1}{2}}UV^{-\frac{1}{2}}|^{\alpha_1-\frac{p+1}{2}}|V|^{\alpha_2-\frac{p+1}{2}}$$
$$\mathrm{e}^{-\mathrm{tr}(B_1V^{-\frac{1}{2}}UV^{-\frac{1}{2}})-\mathrm{tr}(B_2V)}\mathrm{d}V$$
$$=c|U|^{\alpha_1-\frac{p+1}{2}}\int_{V>O}|V|^{\alpha_2-\alpha_1-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(B_1V^{-\frac{1}{2}}UV^{-\frac{1}{2}})-\mathrm{tr}(B_2V)}\mathrm{d}V \tag{4.4}$$
$$=c|U|^{\alpha_2-\frac{p+1}{2}}\int_{V>O}|V|^{\alpha_1-\alpha_2-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(B_1V)-\mathrm{tr}(B_2V^{-\frac{1}{2}}UV^{-\frac{1}{2}})}\mathrm{d}V. \tag{4.5}$$

Note that Equations 4.4, 4.5 can be taken as real matrix-variate generalization of Bessel integrals, Krätzel integral, Bayesian structures, etc. The integrand, normalized can also be taken as a real matrix-variate inverse Gaussian density when $f_j(X_j), j=1,2$ are densities. In this case, $g_2(U)$ is the unique density of the symmetric product U also. All the pairs of functions that we considered in Section 2 can be generalized to the corresponding matrix-variate cases except Problem 2.7. In Problem 2.7, generalization to the matrix-variate case is possible for some specific values of δ_1 and δ_2.

5 Concluding remarks

In Sections 2 and 3, elementary functions belonging to the pathway family are considered so that the results in Sections 2 and 3 will be readily applicable to various practical problems. Immediate generalizations of all pairs in Sections 2 and 3 are also useful in situations in real scalar variable cases. All problems in Sections 2 and 3, except Problems 2.7 and 3.7, can be generalized to the corresponding matrix-variate cases in the real and complex domains. In all those cases, when $f_j(X_j), j=1,2$ are statistical densities, then $g_1(U)$ and $g_2(U)$ will be unique densities corresponding to symmetric product and symmetric, ratios respectively. There are not many results in the literature on Mellin convolutions of products and ratios involving more than two real scalar variables. Let $u=x_1x_2...x_k$ with the joint function $f_1(x_1)...f_k(x_k)$. Then, the integral representations and series representations of $g_1(u)$ can be obtained in many different ways which will produce various results. When it comes to the ratio, then ratios can be defined in many different ways when more than two real scalar variables are involved. Then, the corresponding $g_2(u)$ can take different forms and shapes.

"Convolution" in mathematical literature often means Laplace convolution. Mellin convolutions of products and ratios used to appear very rarely in mathematical literature. In the area of special functions, when the functions are defined through Mellin-Barnes representations, Mellin transforms and Mellin convolutions appear automatically. The class of hypergeometric functions appears in very many practical problems as models and as series solutions of differential equations. Hypergeometric function ${}_pF_q(\cdot)$ has a Mellin-Barnes representation. Mittag-Leffler function is considered as the queen function in the area of fractional calculus. These functions also have Mellin-Barnes representations. Wright's function is another prominent function in the literature having Mellin-Barnes representation. Generalized scalar variable special functions such as G and H-functions have Mellin-Barnes representations. In all these cases, Mellin convolutions of products and ratios appear. Products and ratios of random variables appear in many real-life situations. In industrial production processes, the money value of the output has the structure of the product of two real scalar positive random variables, namely, quantity of output and price per unit. In a series of papers, starting from 2009, Mathai has shown that all types fractional integrals introduced by various authors from time to time are nothing but Mellin convolutions of products and ratios of real scalar variables. In the overview paper Mathai [12], it is shown that the whole area of fractional calculus, including fractional integrals and fractional derivatives etc., can be studied by using Mellin convolutions of products and ratios. It is shown that the approach through Mellin convolutions enable us to go to the Mellin transforms thereby to the corresponding inverse Mellin transforms which provide explicit evaluations of fractional integrals. It is also shown that when the arbitrary functions in fractional integrals belong to some general classes, then the inverse Mellin transform will end up in G and H-functions. Then, one could make use of the G-function differential equation to construct differential equations for fractional integrals. It is also shown in Mathai [12] that the approach through statistical distribution theory or Mellin convolutions enables us to extend fractional integrals to functions of matrix arguments and to the complex domains.

Explicit computable series representations for various categories of G-functions, including the general G-function, are given in Mathai [5] book. The same techniques work in obtaining explicit computable series forms for the H-function also. With the help of these computable representations, computer packages are also produced. Programs are available in MAPLE and MATHEMATICA for analytical and numerical computations of G and H-functions Hence, numerical computations and graphs are not attempted in the present study.

In the case of Mellin convolutions of products involving two functions, it is shown that when these two functions belong to generalized gamma densities, then one can obtain Krätzel integrals and Krätzel transforms, reaction-rate probability integrals in nuclear reaction-rate theory, and inverse Gaussian density in stochastic processes. The large contributions of Mathai and Haubold in astrophysics area started with the evaluation of a reaction-rate probability integral by using the density of product of real scalar positive random variables, which is nothing but the Mellin convolution of a product involving generalized gamma densities. A summary of the study in this area of astrophysics is available in the monograph Mathai and Haubold [13], and an updated version of this book is scheduled to appear in early 2025 from Springer.

In the matrix-variate case, Mathai [11] defined M-convolutions and M-transforms, corresponding to Mellin convolutions and Mellin transforms in the real scalar case. Jacobians of matrix transformations and functions of matrix arguments in the real and complex domains are given there, along with the necessary conditions for the existence of various items. Detailed existence

conditions for G-functions are given in Mathai [5]. These M-convolutions are shown to be densities of symmetric product and symmetric ratios of matrices when the basic functions involved are matrix-variate statistical densities in the real or complex domain. Distributions of symmetric products and symmetric ratios of real positive definite or Hermitian positive definite matrix-variate random variables appear in a large number of practical situations such as in neutrino problems [8], matrix-variate fractional calculus [12], the analysis of multiple and multi-look data in radar, and sonar [14]. Matrix-variate statistical distributions including singular distributions in the real and complex domains and their applications in various areas are given in Mathai et al. [15]. In this book, detailed existence conditions of the various results in the real and complex domains are also given.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

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References

1. Luchko Y. Integral transforms of the Mellin convolution type and their generating operators. *Integral Transf Spec Funct*. (2008) 19:809–51. doi: 10.1080/10652460802091617

2. Mainardi F, Pagnini G. Mellin-Barnes integrals for stable distributions and their convolutions. *Fract Calcu Appl Analy*. (2008) 11:443–56.

3. Mathai AM. A pathway to matrix-variate gamma and normal densities. *Linear Algeb Appl*. (2005) 396:317–28. doi: 10.1016/j.laa.2004.09.022

4. Mathai AM, Provost SB. Some complex matrix-variate statistical distributions on rectangular matrices. *Linear Algeb. Appl.* (2006) 40:198–216. doi: 10.1016/j.laa.2005.07.016

5. Mathai AM. *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*. Oxford: Oxford University Press, Oxford (1993).

6. Mathai AM, Saxena RK, Haubold HJ. *The H-function: Theory and Applications*. New York: Springer (2010).

7. Mathai AM, Haubold HJ. Erdélyi-Kober fractional calculus from a statistical perspective, inspired by solar neutrino physics. In: *Springer Briefs in Mathematical Physics 31*. Singapore: Springer Nature (2018).

8. Fortin J-F, Giasson N, Marleau L. Probability density function for neutrino masses and mixings. *Phys Rev*. (2016) 94:115004. doi: 10.1103/PhysRevD.94.115004

9. Fortin J-F, Giasson N, Marleau L, Pelletier-Dumont J. Mellin transform approach to rephasing invariants. *Phys Rev*. (2020) 102:036001. doi: 10.1103/PhysRevD.102.036001

10. Mathai AM, Haubold HJ. Mathematical aspects of Krätzel integral and Krätzel transform. *Mathematics* (2020) 8:526. doi: 10.3390/math8040526

11. Mathai AM. *Jacobians of Matrix Transformations and Functions of Matrix Arguments*. New York: World Scientific Publishing (1997).

12. Mathai AM. Statistical distribution theory and fractional calculus. *Stats*. (2024) 7:1259–95. doi: 10.3390/stats7040074

13. Mathai AM, Haubold HJ. *Modern Problems in Nuclear and Neutrino Astrophysics*. Berlin: Akademie-Verlag (1988).

14. Deng X. *Texture Analysis and Physical Interpretation of Polarimetric SAR Data* (Ph.D thesis). Barcelona: Universitat Politecnica de Catelunya (2016).

15. Mathai AM, Provost SB, Haubold HJ. *Multivariate Statistical Analysis in the Real and Complex Domains*. Berlin: Springer Nature (2022).