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New analytical wave solutions of fractional order DMBBM and Bateman-Burgers equations

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The purpose of this article is to explore a new method for solving one of the nonlinear partial differential equations (NPDE) which is difficult to solve. The dispersive modified Benjamin-Bona-Mahony (DMBBM) equation and the solutions of space-time fractional Bateman-Burgers equation is solved by a travelling wave analysis method as the Riccati sub-equation. The solutions of space-time fractional DMBBM equation and the solutions of space-time fractional Bateman-Burgers equation can be expressed in the forms of exponential functions, trigonometric functions, rational functions, and hyperbolic functions. The singular wave, singular kink wave, and periodic wave are the representations of the solution graphs.

KEYWORDS

dispersive modified Benjamin-Bona-Mahony (DMBBM) equation, Bateman-Burgers equation, nonlinear partial differential equation, travelling wave analysis method, Riccati sub-equation

1 Introduction

The nonlinear partial differential equations (NPDE) are the partial differential equation (PDE) which have nonlinear terms. The NPDEs are well known for their application of mathematics and physics. Investigating the analytical solutions of NPDEs plays important role in nonlinear science. In fact, NPDEs of physical problems are difficult to solve analytically.

At present, there are many alternative methods could obtain exact travelling wave solutions of NPDEs such as Riccati sub-equation (1, 15–17, 19), Kudryashov method (2, 18, 20), F -expansion method (3), $\frac{G}{G}$ -expansion method (4–6) and hyperbolic tangent method.

One of interested NPDEs is the nonlinear dispersive modified Benjamin-Bona-Mahony (DMBBM) equation (7, 8) which was proposed by Benjamin, Bona and Mahony in 1972. It describes the role of nonlinear dispersion. The DMBBM equation is presented as

$$u_t + u_x - \alpha u^2 u_x + u_{xxx} = 0, \quad (1)$$

where α is a nonzero positive constant and $u = u(x, t)$. However, this method has complicated solutions and still difficult to solve even though the analytical solutions of DMBBM equation are established by using the direct reduction methods and some transformations. The article aims to investigate the travelling wave analysis methods to solve the analytical solutions of the DMBBM (Equation 1) and the Bateman-Burgers (Equation 2) in other ways.

In 1915, Harry Bateman presented the Bateman-Burgers equation which is illustrious in the field of applied mathematics such as fluid mechanics, traffic flow and gas dynamics. Bateman-Burgers equation is presented as

$$u_x + uu_x = \delta u_{xx}. \quad (2)$$

where u is a function of variables x and t , δ is the viscosity of a fluid (1).

1.1 Jumarie's Riemann-Liouville derivative

The Jumatie's Riemann-Liouville derivative of order β respect to t is defined as follows (9),

$$D_t^\beta f(t) = \begin{cases} f(t), & \beta = 0, \\ \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-\xi)^{-\beta} (f(\xi) - f(0)) d\xi, & 0 < \beta < 1 \\ \frac{d^n}{dt^n} D_t^{\beta-n} f(t), & n \leq \beta \leq n+1, n \geq 1. \end{cases} \quad (3)$$

The important properties of fractional Riemann-Liouville derivatives are following (10),

$$D_t^\beta t^k = \frac{\Gamma(k+1)}{\Gamma(k-\beta+1)} t^{k-\beta}, k \geq 0, \quad (4)$$

$$D_t^\beta (f(t)g(t)) = f(t)D_t^\beta g(t) + g(t)D_t^\beta f(t), \quad (5)$$

$$D_t^\beta f[g(t)] = D_t^\beta f[g(t)](g'(t))^\beta = f'_g[g(t)]D_t^\beta g(t), \quad (6)$$

By combining the fractional Jumatie's Riemann-Liouville derivative (Equations 3–6) with the algorithm of the Riccati sub-equation method, we have found numerous new exact travelling wave solutions of the fractional nonlinear space-time DMBBM equation and the fractional nonlinear space-time Bateman-Burgers equation. Wave behavior graphs in contour, two-dimensional, and three-dimensional plots are demonstrated. Additionally, compared to results from previous approaches, the acquired results showed greater variability.

The researcher intends to use the Riccati sub-equation method to determine the exact solutions and analyse the behavior of the fractional nonlinear space-time DMBBM equation and the fractional nonlinear space-time Bateman-Burger equation. This method will provide the exact solution in three different forms: rational functions, generalised hyperbolic functions, and generalised triangular functions. There are a total of 15 distinct solutions. The researcher analysed the behavior of the solution by examining the graph of the solutions in three different formats: contour, two-dimensional graph, and three-dimensional graph. Using Jumarie's Riemann-Liouville Derivative and its properties, the above procedure must convert nonlinear fractional PDE to nonlinear PDE. Then, it must use the wave transform to convert nonlinear PDE to nonlinear ODE, and finally, the Riccati sub-equation technique to solve the problem further.

2 Methodology

The simple equation method with the Riccati sub-equation for solving the DMBBM equation and Bateman-Burgers equation are

illustrated as the following procedures. To begin with, the general form of (1 + 1) dimensional NPDEs.

$$\Psi(u, u_x, u_t, u_{xx}, u_{xt}, \dots) = 0 \quad (7)$$

where u is the functional of x and t . Considering two independent variables x and t , the nonlinear conformable fractional partial differential equation is as follows:

$$H(u, D_t^\beta u, D_x^\beta u, D_t^{2\beta} u, D_x^{2\beta} u, D_t^\beta D_x^\beta u, \dots) = 0, t > 0, 0 < \beta \leq 1, \quad (8)$$

where $u = u(x, t)$ and H is a polynomial expression that contains both greatest order and nonlinear factors in u , along with fractional derivatives.

2.1 Step 1. Wave transformation

Setting the wave variable $\xi = \frac{kx^\psi}{\Gamma(1+\psi)} - \frac{\omega t^\psi}{\Gamma(1+\psi)}$ where k , and

ω are nonzero with positive direction of travelling wave when $\omega > 0$ and a negative direction of the travelling wave when $\omega < 0$, and $U(\xi) = u(x, t)$ (11). Nonlinear Equation 8 can be transformed into an ODE

$$\Phi\left(U, \frac{dU}{d\xi}, \frac{d^2U}{d\xi^2}, \frac{d^3U}{d\xi^3}, \dots\right) = 0 \quad (9)$$

where Φ is a polynomial that contains its derivatives for both largest order and nonlinear factors in U .

2.2 Step 2. Solution supposition

2.2.1 Riccati sub-equation method

Recall that the solution of the simple equation (SE) method may be expressed in the form (11).

$$U(\xi) = \sum_{i=0}^N a_i V^i(\xi) \quad (10)$$

where a_i are the real constants with a_N is nonzero. The Riccati sub-equation method is used to find $V(\xi)$ defined as (12, 13):

$$V'(\xi) = \rho + V^2(\xi), \quad (11)$$

Where ρ is an arbitrary constant. There are general three cases solutions of Equation 11 shown as follows:

Case I: when $\rho < 0$,

$$V_1(\xi) = -\sqrt{-\rho} \tanh_{pq}(\sqrt{-\rho}\xi), \quad (12)$$

$$V_2(\xi) = -\sqrt{-\rho} \coth_{pq}(\sqrt{-\rho}\xi), \quad (13)$$

$$V_3(\xi) = -\sqrt{-\rho} \tanh_{pq}(2\sqrt{-\rho}\xi) \pm i\sqrt{-\rho} \operatorname{sech}_{pq}(2\sqrt{-\rho}\xi), \quad (14)$$

$$V_4(\xi) = -\sqrt{-\rho} \coth_{pq}(2\sqrt{-\rho}\xi) \pm \sqrt{-\rho} \operatorname{csch}_{pq}(2\sqrt{-\rho}\xi), \quad (15)$$

$$V_5(\xi) = -\frac{1}{2} \left[\sqrt{-\rho} \tanh_{pq} \left(\frac{\sqrt{-\rho}}{2} \xi \right) + \sqrt{-\rho} \coth_{pq} \left(\frac{\sqrt{-\rho}}{2} \xi \right) \right], \quad (16)$$

$$V_6(\xi) = \frac{\sqrt{-(Q^2 + R^2)}\rho - Q\sqrt{-\rho} \cosh_{pq}(2\sqrt{-\rho}\xi)}{Q \sinh_{pq}(2\sqrt{-\rho}\xi) + R}, \quad (17)$$

$$V_7(\xi) = -\frac{\sqrt{-(R^2 - Q^2)}\rho - Q\sqrt{-\rho} \sinh_{pq}(2\sqrt{-\rho}\xi)}{Q \cosh_{pq}(2\sqrt{-\rho}\xi) + R}, \quad (18)$$

where Q, R are nonzero constants with $R^2 - Q^2 > 0$.

Case II: when $\rho > 0$,

$$V_8(\xi) = \sqrt{\rho} \tan_{pq}(\sqrt{\rho}\xi), \quad (19)$$

$$V_9(\xi) = -\sqrt{\rho} \cot_{pq}(\sqrt{\rho}\xi), \quad (20)$$

$$V_{10}(\xi) = -\sqrt{\rho} \tan_{pq}(2\sqrt{\rho}\xi) \pm \sqrt{\rho} \sec_{pq}(2\sqrt{\rho}\xi), \quad (21)$$

$$V_{11}(\xi) = -\sqrt{\rho} \cot_{pq}(2\sqrt{\rho}\xi) \pm \sqrt{\rho} \csc_{pq}(2\sqrt{\rho}\xi), \quad (22)$$

$$V_{12}(\xi) = \frac{1}{2} \left[\sqrt{\rho} \tan_{pq} \left(\frac{\sqrt{\rho}}{2} \xi \right) - \sqrt{\rho} \cot_{pq} \left(\frac{\sqrt{\rho}}{2} \xi \right) \right], \quad (23)$$

$$V_{13}(\xi) = \frac{\pm \sqrt{(Q^2 - R^2)}\rho - Q\sqrt{\rho} \cos_{pq}(2\sqrt{\rho}\xi)}{Q \sin_{pq}(2\sqrt{\rho}\xi) + R}, \quad (24)$$

$$V_{14}(\xi) = -\frac{\pm \sqrt{(Q^2 - R^2)}\rho - Q\sqrt{\rho} \sin_{pq}(2\sqrt{\rho}\xi)}{Q \cos_{pq}(2\sqrt{\rho}\xi) + R}, \quad (25)$$

where Q, R are nonzero constants with $Q^2 - R^2 > 0$.

Case III: when $\rho > 0$,

$$V_{15}(\xi) = -\frac{1}{\xi + b}, \quad (26)$$

where b is a constant.

The generalised hyperbolic functions in Equations 12–18, with p and q are arbitrary constants and $p, q > 0$, are defined as follow,

$$\sinh_{pq}(\theta) = \frac{pe^\theta - qe^{-\theta}}{2}, \quad (27)$$

$$\cosh_{pq}(\theta) = \frac{pe^\theta + qe^{-\theta}}{2}, \quad (28)$$

$$\tanh_{pq}(\theta) = \frac{pe^\theta - qe^{-\theta}}{pe^\theta + qe^{-\theta}}, \quad (29)$$

$$\coth_{pq}(\theta) = \frac{pe^\theta + qe^{-\theta}}{pe^\theta - qe^{-\theta}}, \quad (30)$$

$$\operatorname{sech}_{pq}(\theta) = \frac{2}{pe^\theta + qe^{-\theta}}, \quad (31)$$

$$\operatorname{csch}_{pq}(\theta) = \frac{2}{pe^\theta - qe^{-\theta}}, \quad (32)$$

where θ is an independent variable.

The generalised triangular functions in Equations 19–25, with p and q are arbitrary constants and $p, q > 0$, are defined as follow,

$$\sin_{pq}(\theta) = \frac{pe^{i\theta} - qe^{-i\theta}}{2i}, \quad (33)$$

$$\cos_{pq}(\theta) = \frac{pe^{i\theta} + qe^{-i\theta}}{2}, \quad (34)$$

$$\tan_{pq}(\theta) = -i \frac{pe^{i\theta} - qe^{-i\theta}}{pe^{i\theta} + qe^{-i\theta}}, \quad (35)$$

$$\cot_{pq}(\theta) = i \frac{pe^{i\theta} + qe^{-i\theta}}{pe^{i\theta} - qe^{-i\theta}}, \quad (36)$$

$$\sec_{pq}(\theta) = \frac{2}{pe^{i\theta} + qe^{-i\theta}}, \quad (37)$$

$$\csc_{pq}(\theta) = \frac{2i}{pe^{i\theta} - qe^{-i\theta}}, \quad (38)$$

where θ is an independent variable. Substituting the appropriate solutions for case I, II, or III in [Equations 12–38](#) leads to the general three-case solutions of [Equation 11](#). Step 3 Finding the integer.

2.3 Step 3. Finding the integer N

The integer N ([11](#)) in [Equation 10](#) of this method can be obtained by balancing the highest order derivative and nonlinear terms, which are appeared in [Equation 9](#).

2.4 Step 4. Solution obtaining

Collect all the coefficient terms with the same order of V^i , ($i = 0, 1, 2, \dots$). Set them to zero ([11](#)). The parameters a_i , ($i = 0, 1, 2, \dots, N$) and ω are obtained. Thus, the analytical solutions of [Equation 7](#) are composed.

3 Application

The travelling wave behaviors were investigated by the fractional nonlinear space–time DMBBM equation and the fractional nonlinear space–time Bateman-Burgers equation shown here.

3.1 The fractional nonlinear space–time DMBBM equation

In this chapter, we apply some travelling wave analysis methods in the previous chapter to solve the analytical solutions of the DMBBM equation with the following methods using simple equation. The nonlinear space–time fractional DMBBM equation,

$$D_t^\psi u + D_x^\psi u - \alpha u^2 D_x^\psi u + D_x^{3\psi} u = 0, \quad (39)$$

where α is a nonzero positive constant. [Equation 39](#) is transformed into [Equation 40](#) as the following ([14](#)):

$$-\omega \frac{dU}{d\xi} + k \frac{dU}{d\xi} - \alpha k U^2 \frac{dU}{d\xi} + k^3 \frac{d^3 U}{d\xi^3} = 0. \quad (40)$$

Integrating concerning ξ choosing of integration as zero, we obtain the following ODE,

$$3(k - \omega)U - \alpha k U^3 + 3k^3 U'' = 0. \quad (41)$$

Balance the highest order derivative and the nonlinear term. So, we have $N = 1$. The solution is expressed as

$$U(\xi) = a_0 + a_1 V(\xi). \quad (42)$$

Differentiating [Equation 42](#) yields,

$$U'(\xi) = a_1 V'(\xi), \quad (43)$$

$$U''(\xi) = 2a_1 \rho V(\xi) + 2a_1 V^3(\xi), \quad (44)$$

and

$$U^2 = a_0^2 + 2a_0 a_1 V(\xi) + a_1^2 V^2(\xi), \quad (45)$$

$$U^3 = a_0^3 + 3a_0^2 a_1 V(\xi) + 3a_0 a_1^2 V^2(\xi) + a_1^3 V^3(\xi). \quad (46)$$

Substitute [Equations 42, 44–46](#) into [Equation 41](#),

$$(3ka_0 - 3\omega a_0 - \alpha ka_0^3) + (3ka_1 - 3\omega a_1 - 3\alpha ka_0^2 a_1 + 6k^3 a_1 \rho) V(\xi)$$

$$-3\alpha ka_0 a_1^2 V^2(\xi) + (6k^3 a_1 - \alpha ka_1^3) V^3(\xi) = 0$$

Setting the coefficients of V^i to zero, we obtain

$$V^0(\xi): 3ka_0 - 3\omega a_0 - \alpha ka_0^3 = 0, \quad (47)$$

$$V^1(\xi): 3ka_1 - 3\omega a_1 - 3\alpha ka_0^2 a_1 + 6k^3 a_1 \rho = 0, \quad (48)$$

$$V^2(\xi): -3\alpha ka_0 a_1^2 = 0, \quad (49)$$

$$V^3(\xi): 6k^3 a_1 - \alpha ka_1^3 = 0. \quad (50)$$

Solving the system of [Equations 47–50](#), we get

$$a_0 = 0, \quad a_1 = \eta \sqrt{\frac{6k}{\alpha}} \quad \text{and} \quad \omega = k + 2k^3 \rho \quad (51)$$

where $\eta = \pm 1$. The exact solutions of the fractional DMBBM equation are:

Type I: when $\rho < 0$,

$$\text{case1: } u_1(x, t) = -\eta \sqrt{\frac{-6k\rho}{\alpha}} \tanh_{pq}(\sqrt{-\rho}\xi), \quad (52)$$

$$\text{case2: } u_2(x, t) = -\eta \sqrt{\frac{-6k\rho}{\alpha}} \coth_{pq}(\sqrt{-\rho}\xi), \quad (53)$$

$$\text{case3: } u_3(x, t) = -\eta \sqrt{\frac{-6k\rho}{\alpha}} \tanh_{pq}(2\sqrt{-\rho}\xi) \pm i\eta \sqrt{\frac{-6k\rho}{\alpha}} \operatorname{sech}_{pq}(2\sqrt{-\rho}\xi), \quad (54)$$

$$\text{case4: } u_4(x, t) = -\eta \sqrt{\frac{-6k\rho}{\alpha}} \coth_{pq}(2\sqrt{-\rho}\xi) \pm i\eta \sqrt{\frac{-6k\rho}{\alpha}} \operatorname{csch}_{pq}(2\sqrt{-\rho}\xi), \quad (55)$$

$$\text{case5: } u_5(x, t) = -\frac{\eta}{2} \sqrt{\frac{6k}{\alpha}} \left[\sqrt{-\rho} \tanh_{pq}\left(\frac{\sqrt{-\rho}}{2}\xi\right) + \sqrt{-\rho} \coth_{pq}\left(\frac{\sqrt{-\rho}}{2}\xi\right) \right], \quad (56)$$

$$\text{case6: } u_6(x, t) = \eta \sqrt{\frac{6k}{\alpha}} \left[\frac{\sqrt{-(Q^2 + R^2)}\rho - Q\sqrt{-\rho} \cosh_{pq}(2\sqrt{-\rho}\xi)}{Q \sinh_{pq}(2\sqrt{-\rho}\xi) + R} \right], \quad (57)$$

$$\text{case7: } u_7(x, t) = -\eta \sqrt{\frac{6k}{\alpha}} \left[\frac{\sqrt{-(R^2 - Q^2)}\rho - Q\sqrt{-\rho} \sinh_{pq}(2\sqrt{-\rho}\xi)}{Q \cosh_{pq}(2\sqrt{-\rho}\xi) + R} \right], \quad (58)$$

where Q, R are nonzero constants with $R^2 - Q^2 > 0$.
Type II: when $\rho > 0$,

$$\text{case8: } u_8(x, t) = \eta \sqrt{\frac{6k\rho}{\alpha}} \tan_{pq}(\sqrt{\rho}\xi), \quad (59)$$

$$\text{case9: } u_9(x, t) = -\eta \sqrt{\frac{6k\rho}{\alpha}} \cot_{pq}(\sqrt{\rho}\xi), \quad (60)$$

$$\text{case10: } u_{10}(x, t) = -\eta \sqrt{\frac{6k\rho}{\alpha}} \tan_{pq}(2\sqrt{\rho}\xi) \pm \eta \sqrt{\frac{6k\rho}{\alpha}} \sec_{pq}(2\sqrt{\rho}\xi), \quad (61)$$

$$\text{case11: } u_{11}(x, t) = -\eta \sqrt{\frac{6k\rho}{\alpha}} \cot_{pq}(2\sqrt{\rho}\xi) \pm \eta \sqrt{\frac{6k\rho}{\alpha}} \csc_{pq}(2\sqrt{\rho}\xi), \quad (62)$$

$$\text{case12: } u_{12}(x, t) = \frac{\eta}{2} \sqrt{\frac{6k}{\alpha}} \left[\sqrt{\rho} \tan_{pq}\left(\frac{\sqrt{\rho}}{2}\xi\right) - \sqrt{\rho} \cot_{pq}\left(\frac{\sqrt{\rho}}{2}\xi\right) \right], \quad (63)$$

$$\text{case13: } u_{13}(x, t) = \eta \sqrt{\frac{6k}{\alpha}} \left[\frac{\pm \sqrt{(Q^2 - R^2)}\rho - Q\sqrt{\rho} \cos_{pq}(2\sqrt{\rho}\xi)}{Q \sin_{pq}(2\sqrt{\rho}\xi) + R} \right], \quad (64)$$

$$\text{case14: } u_{14}(x, t) = -\eta \sqrt{\frac{6k}{\alpha}} \left[\frac{\pm \sqrt{(Q^2 - R^2)}\rho - Q\sqrt{\rho} \sin_{pq}(2\sqrt{\rho}\xi)}{Q \cos_{pq}(2\sqrt{\rho}\xi) + R} \right], \quad (65)$$

where Q, R are nonzero constants with $Q^2 - R^2 > 0$.
Type III: when $\rho > 0$,

$$\text{case15: } u_{15}(x, t) = -\sqrt{\frac{6k}{\alpha}} \left(\frac{\eta}{\xi + b} \right), \quad (66)$$

where b is a constant, $\xi = \frac{kx^\psi}{\Gamma(1+\psi)} - \frac{(k + 2k^3\rho)t^\psi}{\Gamma(1+\psi)}$. The

analytical solutions of the fractional DMBBM equation are shown in [Equations 51–66](#). Replacing [Equations 59, 60, and 66](#) by setting the parameters $\alpha = 0.5, b = 1, k = 1$, $0 \leq x \leq 60$ and $0 \leq y \leq 60$, the wave of the solutions of the space–time fractional DMBBM equation by Riccati sub-equation can be demonstrated as the following [Figures 1–3](#).

3.2 The fractional nonlinear space–time Bateman-Burgers equation

Given,

$$D_t^\psi + u D_x^\psi u - \delta D_x^{2\psi} u = 0, \quad (67)$$

where δ is the viscosity of a fluid. Transform [Equation 67](#) into the differential equation, setting the wave variable $\xi = \frac{kx^\psi}{\Gamma(1+\psi)} - \frac{\beta t^\psi}{\Gamma(1+\psi)}$ and $U(\xi) = u(x, t)$ yields,

$$-\beta \frac{dU}{d\xi} + kU \frac{dU}{d\xi} - \delta k^2 \frac{d^2U}{d\xi^2} = 0. \quad (68)$$

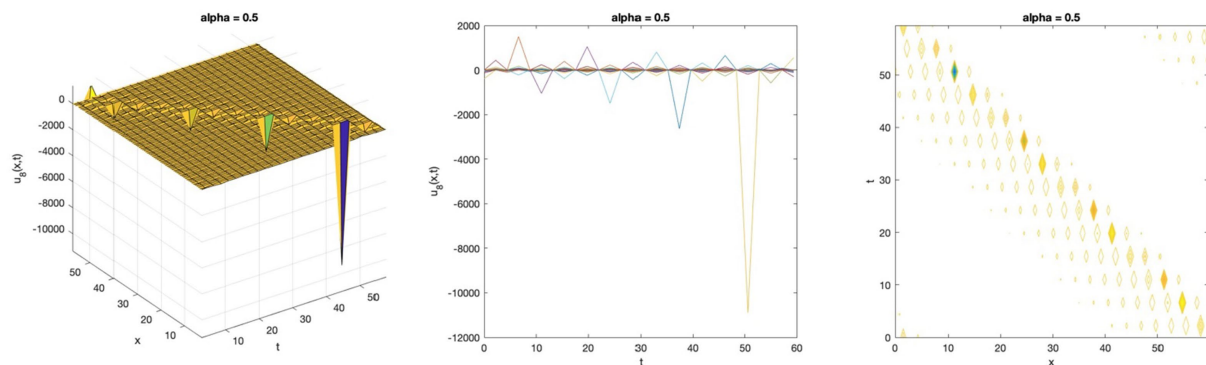


FIGURE 1

Periodic wave solution profile of the space-time fractional DMBBM equation by Riccati sub-equation for Equation 59.

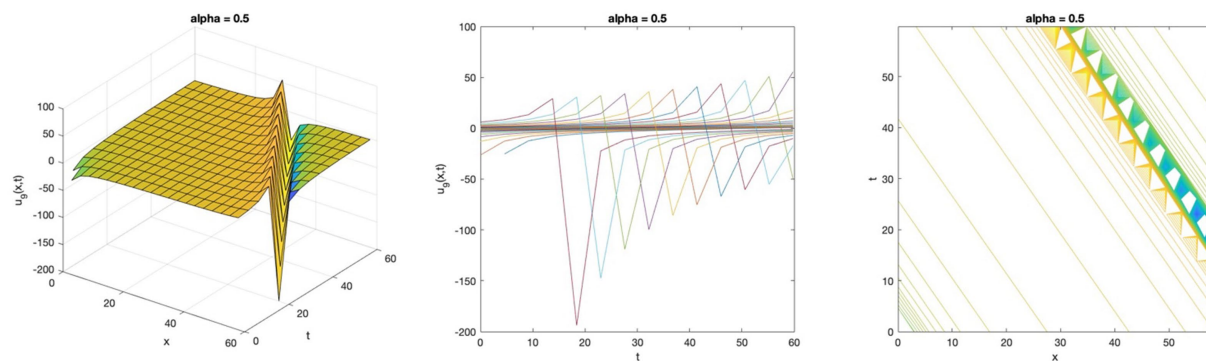


FIGURE 2

Singular solution profile of the space-time fractional DMBBM equation by Riccati sub-equation for Equation 60.

Balance the highest order derivative and the nonlinear term. So, we have $N = 1$. The solution is expressed as

$$U(\xi) = a_0 + a_1 V(\xi). \quad (69)$$

$$V^3(\xi): ka_1^2 - 2\delta k^2 a_1 = 0. \quad (73)$$

Solving the system of Equations 70–73, we get

$$a_0 = \frac{\beta}{k} \text{ and } a_1 = 2\delta k \quad (74)$$

Substitute Equations 69, 43, and 44 into Equation 68, we get

$$(k\rho a_0 a_1 - \beta \rho a_1) + (k\rho a_1^2 - 2\delta \rho k^2 a_1)V + (ka_0 a_1 - \beta a_1)V^2 + (ka_1^2 - 2\delta k^2 a_1)V^3 = 0.$$

Setting coefficients of V^i to zero, we obtain

$$V^0(\xi): k\rho a_0 a_1 - \beta \rho a_1 = 0, \quad (70)$$

$$V^1(\xi): k\rho a_1^2 - 2\delta \rho k^2 a_1 = 0 \quad (71)$$

$$V^2(\xi): ka_0 a_1 - \beta a_1 = 0, \quad (72)$$

The exact solutions of the fractional Bateman-Burgers equation are:

Type I: when $\rho < 0$,

$$\text{case1: } u_1(x, t) = \frac{\beta}{k} - 2\delta k \sqrt{-\rho} \tanh_{pq}(\sqrt{-\rho} \xi), \quad (75)$$

$$\text{case2: } u_2(x, t) = \frac{\beta}{k} - 2\delta k \sqrt{-\rho} \coth_{pq}(\sqrt{-\rho} \xi), \quad (76)$$

$$\text{case3: } u_3(x, t) = \frac{\beta}{k} - 2\delta k \sqrt{-\rho} \tanh_{pq}(2\sqrt{-\rho} \xi) \pm 2\delta k i \sqrt{-\rho} \operatorname{sech}_{pq}(2\sqrt{-\rho} \xi), \quad (77)$$

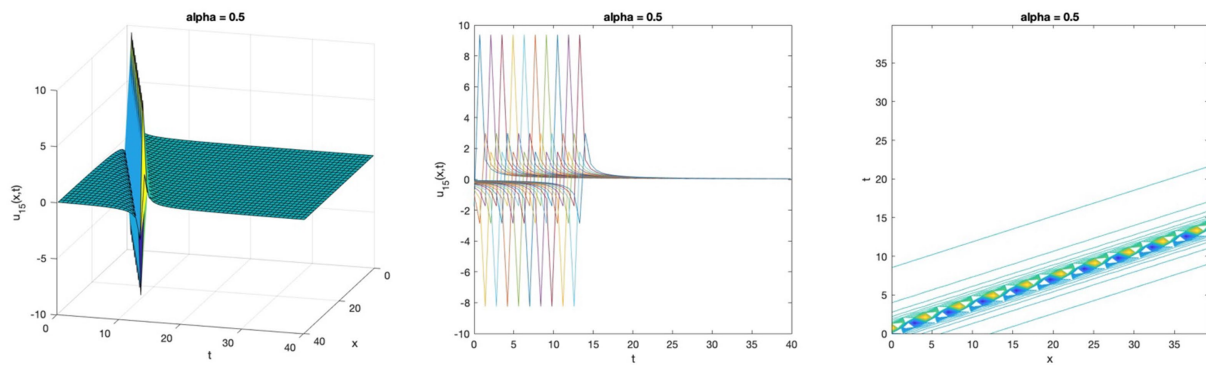


FIGURE 3

Singular Kink wave solution profile of the space-time fractional DMBBM equation by Riccati sub-equation for Equation 66.

$$\text{case4: } u_4(x, t) = \frac{\beta}{k} - 2\delta k \sqrt{-\rho} \coth_{pq} \left(2\sqrt{-\rho} \xi \right) \pm 2\delta k \sqrt{-\rho} \operatorname{csch}_{pq} \left(2\sqrt{-\rho} \xi \right), \quad (78)$$

$$\text{case11: } u_{11}(x, t) = \frac{\beta}{k} - 2\delta k \sqrt{\rho} \cot_{pq} \left(2\sqrt{\rho} \xi \right) \pm 2\delta k \sqrt{\rho} \operatorname{csc}_{pq} \left(2\sqrt{\rho} \xi \right), \quad (85)$$

$$\text{case5: } u_5(x, t) = \frac{\beta}{k} - \delta k \left[\sqrt{-\rho} \tanh_{pq} \left(\frac{\sqrt{-\rho}}{2} \xi \right) + \sqrt{-\rho} \coth_{pq} \left(\frac{\sqrt{-\rho}}{2} \xi \right) \right], \quad (79)$$

$$\text{case12: } u_{12}(x, t) = \frac{\beta}{k} + \delta k \left[\sqrt{\rho} \tan_{pq} \left(\frac{\sqrt{\rho}}{2} \xi \right) - \sqrt{\rho} \cot_{pq} \left(\frac{\sqrt{\rho}}{2} \xi \right) \right], \quad (86)$$

$$\text{case6: } u_6(x, t) = \frac{\beta}{k} + 2\delta k \left[\frac{\sqrt{-(Q^2 + R^2)} \rho - Q \sqrt{-\rho} \cosh_{pq} \left(2\sqrt{-\rho} \xi \right)}{Q \sinh_{pq} \left(2\sqrt{-\rho} \xi \right) + R} \right], \quad (80)$$

$$\text{case13: } u_{13}(x, t) = \frac{\beta}{k} + 2\delta k \left[\frac{\pm \sqrt{(Q^2 - R^2)} \rho - Q \sqrt{\rho} \cos_{pq} \left(2\sqrt{\rho} \xi \right)}{Q \sin_{pq} \left(2\sqrt{\rho} \xi \right) + R} \right], \quad (87)$$

$$\text{case7: } u_7(x, t) = \frac{\beta}{k} - 2\delta k \left[\frac{\sqrt{(R^2 - Q^2)} \rho - Q \sqrt{-\rho} \sinh_{pq} \left(2\sqrt{-\rho} \xi \right)}{Q \cosh_{pq} \left(2\sqrt{-\rho} \xi \right) + R} \right], \quad (81)$$

$$\text{case14: } u_{14}(x, t) = \frac{\beta}{k} - 2\delta k \left[\frac{\pm \sqrt{(Q^2 - R^2)} \rho - Q \sqrt{\rho} \sin_{pq} \left(2\sqrt{\rho} \xi \right)}{Q \cos_{pq} \left(2\sqrt{\rho} \xi \right) + R} \right], \quad (88)$$

where Q, R are nonzero constants with $R^2 - Q^2 > 0$.
Type II: when $\rho > 0$,

$$\text{case8: } u_8(x, t) = \frac{\beta}{k} + 2\delta k \tan_{pq} \left(\sqrt{\rho} \xi \right), \quad (82)$$

$$\text{case9: } u_9(x, t) = \frac{\beta}{k} - 2\delta k \sqrt{\rho} \cot_{pq} \left(\sqrt{\rho} \xi \right), \quad (83)$$

$$\text{case10: } u_{10}(x, t) = \frac{\beta}{k} - 2\delta k \sqrt{\rho} \tan_{pq} \left(2\sqrt{\rho} \xi \right) \pm 2\delta k \sqrt{\rho} \sec_{pq} \left(2\sqrt{\rho} \xi \right), \quad (84)$$

where Q, R are nonzero constants with $Q^2 - R^2 > 0$.
Type III: when $\rho > 0$,

$$\text{case15: } u_{15}(x, t) = \frac{\beta}{k} - \left(\frac{2\delta k}{\xi + b} \right), \quad (89)$$

where b is a constant, $\xi = \frac{kx^\psi}{\Gamma(1+\psi)} - \frac{\beta t^\psi}{\Gamma(1+\psi)}$. The analytical

solutions of the fractional Bateman-Burgers equation are shown in Equations 74–89.

Replacing Equations 82, 83, and 89 by setting the parameters $\delta = 0.5, b = 1, k = 1, 0 \leq x \leq 80$ and $0 \leq y \leq 80$, the wave of the solutions of the space-time fractional Bateman-Burgers equation by Riccati sub-equation can be demonstrated as the following Figures 4–6.

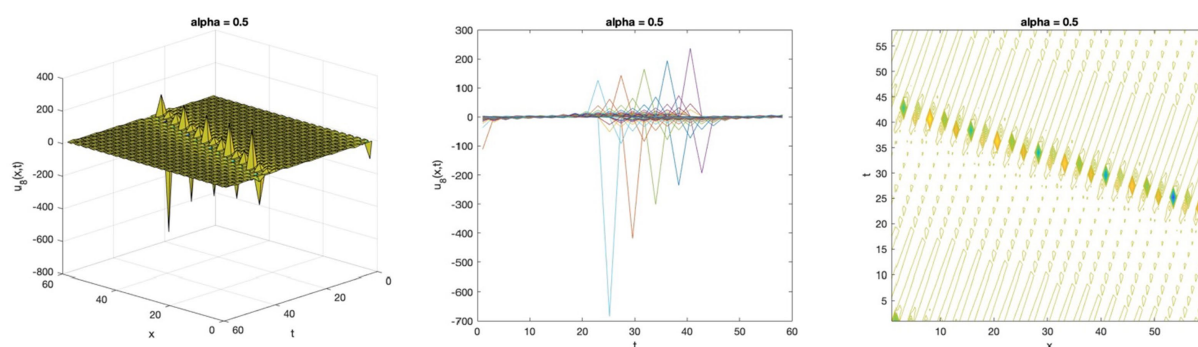


FIGURE 4

Periodic wave solution profile of the space-time fractional Bateman-Burgers equation by Riccati sub-equation for Equation 82.

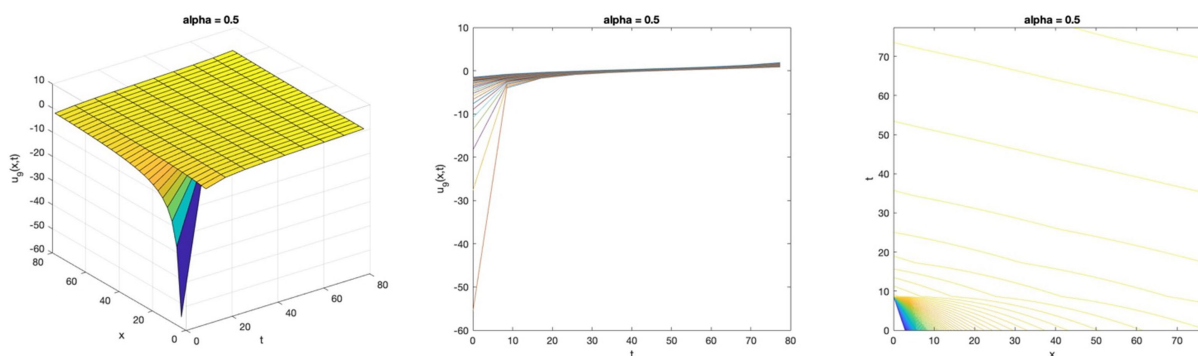


FIGURE 5

Singular wave solution profile of the space-time fractional Bateman-Burgers equation by Riccati sub-equation for Equation 83.

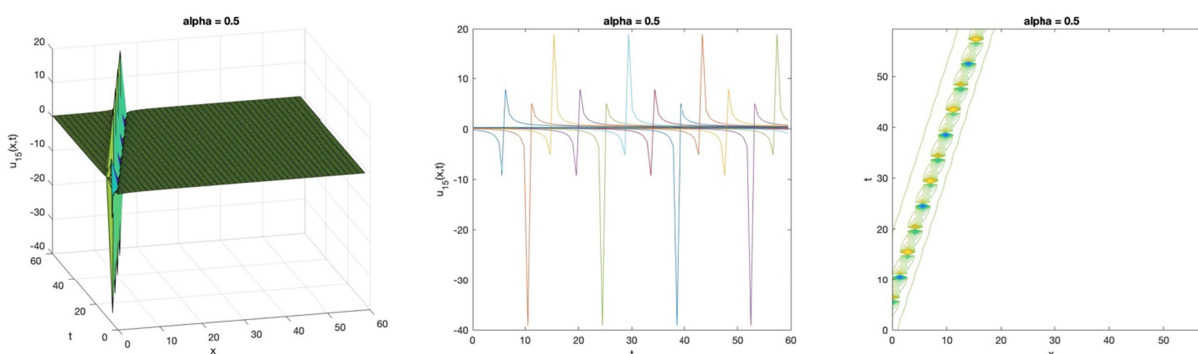


FIGURE 6

Singular Kink wave solution profile of the space-time fractional Bateman-Burgers equation by Riccati sub-equation for Equation 89.

4 Discussion and results

The fifteen novel solutions found using the Riccati sub-equation method for the two equations are as follows: seven generalised

hyperbolic functions, seven generalised triangular functions, and one rational function. The new solutions are more varied than the ones that were previously available: the space-time fractional modified Benjamin-Bona-Mahony (mBBM) equation, which was solved using

the modified Kudryashov method, has four solutions (15), while the $(1 + 1)$ -dimensional nonlinear dispersive modified Benjamin-Bona-Mahony (DMBBM) equation, which was solved using the Bernoulli equations, has four solutions (8). In addition, the solution graphs of the equation display periodic waves which are solutions to travelling waves that are periodic, in Figures 1, 4. Singular waves, as seen in Figures 2, 5, and kink waves, as depicted in Figures 3, 6.

5 Conclusion

In this article, the simple equation method has been successfully implemented. The wave solutions for the space-time fractional DMBBM equation and the space-time fractional Bateman-Burgers equation are found using the Riccati sub-equation. This method releases fifteen solutions with independent parameters in the form of generalised triangular functions and generalised hyperbolic functions which are considered to describe some intricate physical phenomena. This research shows some physical wave behavior graphs in three-dimensional plots. Both Figures 1, 4 are exhibit periodic wave solutions which increase and decrease, Figures 2, 5 show singular solutions and singular kink wave solutions were presented in Figures 3, 6. Based on the results, the Riccati sub-equation methodology is an effective, dependable, and efficient method that provides new solutions to these equations and shows the wave impacts of those solutions in different ways.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

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Author contributions

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