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Hyers-Ulam, Rassias, and Mittag-Leffler stability for quantum difference equations in β-calculus

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This paper investigates first-order nonlinear quantum difference equations governed by a general β-difference operator, encompassing the Jackson q-difference and Hahn difference operators as special cases. We establish sufficient conditions for the existence and uniqueness of solutions using fixed-point theory and examine their solvability under specific assumptions to ensure well-posedness. Particular attention is given to various notions of stability, including Hyers-Ulam, Hyers-Ulam-Rassias, and Mittag-Leffler type stability. Under suitable Lipschitz conditions, we derive explicit error bounds characterizing each type of stability, with Mittag-Leffler stability demonstrated to be of exponential order α. Several illustrative examples are included to validate the theoretical findings within the framework of quantum calculus and discrete dynamical systems.

KEYWORDS

nonlinear quantum difference equations, quantum calculus, Hyers-Ulam stability, generalized Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Mittag-Leffler stability, Banach Fixed Point Theorem

1 Introduction

Quantum calculus, often referred to as q-calculus, generalizes classical calculus by replacing the concepts of limits and infinitesimals with difference quotients involving a parameter q. While its origins can be traced back to the works of Euler and Gauss, quantum calculus gained significant attention in the 20th century due to its connections with basic hypergeometric functions, special functions, and quantum groups [1, 2]. A central tool in quantum calculus is the first-order quantum differential operator, which plays a crucial role in various areas of modern mathematical and physical research. In quantum mechanics, it serves as an alternative to classical derivatives, enabling discrete models of physical systems where traditional continuous models may not be effective [3]. Additionally, quantum calculus has contributed to the development of q-analogs of orthogonal polynomials, such as the Askey-Wilson and q-Hermite polynomials, which are fundamental in the theory of special functions and have extensive analytical applications [3]. Furthermore, fractional calculus has incorporated the q-difference operator to define new forms of fractional derivatives and integrals, enabling the modeling of systems with memory and hereditary characteristics in discrete settings [4]. In this context, quantum estimates are explored within the framework of fractional calculus, employing the quantum

Hahn integral operator and Jackson's q-integral to solve boundary value problems [4]. Recent studies, including [5], have demonstrated the use of the generalized Mittag-Leffler function in the kernel of fractional operators, offering a comprehensive framework for analyzing fractional differential equations, and enhancing the accuracy and efficiency of the results in applications across various scientific and engineering domains. As highlighted in the work by Zhou et al. [6], a new strategy has been proposed to derive inequalities through the use of the Hilfer generalized proportional fractional integral operators, which establishes important connections between fractional differential equations and statistical theory, offering a powerful tool for modeling real-world problems. The study of quantum difference equations has deep roots in the theory of difference equations and functional analysis. Historically, quantum difference equations emerged as generalizations of q-difference equations, which were first introduced by Jackson [7] and Hahn [8]. These early works established the foundation for the development of q-calculus, a generalization of classical calculus. Quantum difference equations, particularly those involving operators such as the Jackson q-difference and Hahn difference operators, have proven crucial in describing discrete systems in quantum mechanics, where time evolution and state transitions are inherently discrete. Furthermore, the introduction of the β-difference operator has expanded the scope of these equations, allowing for the modeling of more general discrete systems. These operators form the basis for various stability results in quantum calculus, including the Hyers-Ulam, generalized Hyers-Ulam, and Mittag-Leffler-type stability, which are of central importance in the study of perturbation theory and solution uniqueness. The historical significance of these operators is underscored by their application in a wide range of fields, from quantum field theory and signal processing to dynamical systems and discrete time models. Thus, this paper builds upon a rich legacy of work, contributing new results on the stability of quantum difference equations in the context of modern mathematics. In Wongcharoen et al. [9], the authors explore the existence and uniqueness of solutions for a boundary value problem involving a nonlinear fractional q-difference equation, subject to novel boundary conditions that combine fractional Hadamard and quantum integrals. Similarly, in Daribayev et al. [10], the authors present a quantum algorithm for solving the one-dimensional heat equation with Dirichlet boundary conditions, utilizing quantum gates and the Trotter-Suzuki decomposition to simulate heat propagation and assess the impact of quantum simulations on heat conduction modeling. One important operator in quantum difference calculus is the β-difference operator, defined by

$$\mathscr{D}_\beta g(t) = \frac{g(\beta(t)) - g(t)}{\beta(t) - t},$$

for each t such that $\beta(t) \neq t$. When $\beta(t) = t$, the operator reduces to the classical derivative $g'(t)$, provided that $g'(t)$ exists. This operator generalizes both the Jackson q-difference operator and the Hahn difference operator, two fundamental tools in discrete analysis. In this study, β is a continuous function on an interval $\mathcal{I}$, and g is a function mapping $\mathcal{I}$ into a Banach space $\mathbb{U}$. A function g is β-differentiable on $\mathcal{I}$ if it is classically differentiable at each point where $\beta(t) = t$. For example, if $\beta(t) = qt$ with $q \in (0,1)$, the operator $\mathscr{D}_q$ becomes the Jackson q-difference operator, denoted D_q, and if $\beta(t) = qt + \omega$ with $q \in (0,1)$ and $\omega > 0$, it becomes the Hahn difference operator, denoted $D_{q,\omega}$. The study of quantum difference equations has garnered significant interest due to their ability to handle non-differentiable functions, thus eliminating the need for redundant proofs for both q-difference and Hahn difference equations. For a comprehensive treatment of the theory of quantum difference equations, we refer to [11]. While quantum difference equations have emerged as a powerful tool for handling non-differentiable functions, another significant aspect of mathematical analysis, namely the stability of functional equations, has also seen substantial development in recent years. The concept of stability in functional equations was first introduced by Ulam in 1940, who posed the problem of determining conditions for the existence of linear mappings near approximately linear mappings [12, 13]. Hyers extended this by proving the stability of the Cauchy functional equation in Banach spaces [14], leading to the concept of Hyers-Ulam stability, which has since been widely studied due to its applications in various fields, including control theory and numerical analysis. In 1978, Rassias further extended this concept by introducing Hyers-Ulam-Rassias stability [15], broadening its scope to more general conditions and applications in differential equations, recurrence relations, and dynamic systems. The Hyers-Ulam stability of first-order linear differential equations has been extensively studied in the literature. Notably, it was investigated in [16, 17], where foundational results were established. These findings were later extended and generalized by Miura and collaborators in a series of works [18–20], broadening the scope of applicability to more general settings. In addition to differential equations, Hyers-Ulam stability has also been explored in the context of discrete systems. In Popa [21], the author investigates the Hyers-Ulam stability of linear recurrence equations with constant coefficients, establishing conditions under which small perturbations in initial values or coefficients lead to bounded deviations in the solution. Further studies on difference equations, which share structural similarities with recurrence relations, can be found in [22, 23], where various stability properties are examined. In [24], the authors present a comprehensive study on the stability of multi-term delay fractional differential equations, with a particular focus on the analysis of integro-multipoint boundary conditions, offering valuable insights into the dynamic behavior of these equations under various conditions. More recently, there has been a growing interest in the investigation of Hyers–Ulam stability within the framework of dynamic equations on time scales. In [25], the authors investigate the existence and stability results for multi-term fractional delay differential equations, specifically focusing on nonlocal multi-point and multi-strip boundary conditions. Several contributions in this area have been made, including works such as [26–37], which provide a unified approach to differential and difference equations under a common mathematical framework. For additional references and further developments, see also [38, 39]. This paper aims to investigate the existence and uniqueness of solutions for quantum differential equations and explore the Hyers-Ulam and Hyers-Ulam-Rassias stability of nonlinear quantum difference equations.

The outline of the paper is the following: In Section 2, we introduce the mathematical framework and provide the necessary preliminary definitions. In Section 3, we present the main problem along with the existence and uniqueness results. Section 4 is dedicated to stability results, including Hyers-Ulam stability, Hyers-Ulam-Rassias stability, and Mittag-Leffler-type stability for quantum difference equations. In Section 5, we present illustrative examples. Finally, we conclude in Section 6.

2 Mathematical framework and preliminary definitions

We assume that β is a continuous, increasing function on $\mathcal{I}$, with a unique fixed point $t_0 \in \mathcal{I}$, satisfying the following conditions:

$$\beta(t) \leq t \quad \text{for all } t \in \mathcal{I}, \quad t \geq t_0,$$

and

$$\beta(t) \geq t \quad \text{for all } t \in \mathcal{I}, \quad t \leq t_0.$$

Additionally, $\mathbb{U}$ represents a Banach space equipped with the norm $\|\cdot\|$.

A key concept in quantum difference calculus is the β-interval, defined as:

$$[a,b]_\beta = \{\beta^k(a) \mid k \in \mathbb{N}_0\} \cup \{\beta^k(b) \mid k \in \mathbb{N}_0\} \cup \{t_0\}, \quad \text{where } a,b \in \mathcal{I}.$$

For a point $d \in [a,b]_\beta$, the following behaviors are observed:

- If $d > t_0$, then $\beta^k(d)$ decreases monotonically to t_0 as $k \to \infty$.
- If $d < t_0$, then $\beta^k(d)$ increases monotonically to t_0 as $k \to \infty$.

This behavior leads to the definition $\beta^\infty(t) = t_0$ for $t \geq t_0$, which provides a natural extension for the behavior of β beyond its fixed point. For a more comprehensive understanding of quantum difference calculus, we direct the reader to [40–42]. In this section, we outline the essential definitions and pivotal theorems that form the foundation of our analysis, ensuring a clear and structured framework for the subsequent discussions and results.

Example 1. 1. $D_\beta t^n = \sum_{k=0}^{n-1} (\beta(t))^{n-k-1} t^k, \quad t \in (-1,1),\ n \geq 1.$
2. $D_\beta \frac{1}{t} = \frac{-1}{t\beta(t)},\ t \neq 0, \beta(t) \neq 0.$
3. Let $f(t) = at + b$, where $a, b \in \mathbb{R}$, and let $\beta(t) = t + h$ for some constant $h \neq 0$.

The β-difference operator applied to $f(t)$ is:

$$D_\beta f(t) = \frac{f(\beta(t)) - f(t)}{\beta(t) - t} = \frac{a(t+h) + b - (at+b)}{h} = \frac{a(t+h-t)}{h} = a.$$

Thus, $D_\beta f(t) = a$.

Definition 1. Let $g : \mathcal{I} \longrightarrow \mathbb{U}$ and $c, d \in \mathcal{I}$. The β-integral of the function g over the interval from $[c,d]$ is defined by

$$\int_c^d g(t) d_\beta t = \int_{t_0}^d g(t) d_\beta t - \int_{t_0}^c g(t) d_\beta t, \tag{1}$$

where

$$\int_{t_0}^h g(t) d_\beta t = \sum_{k=0}^{\infty} (\beta^k(h) - \beta^{k+1}(h)) g(\beta^k(h)), \quad h \in \mathcal{I}, \tag{2}$$

and this series is assumed to converge at both $h = c$ and $h = d$. The function g is said to be β-integrable on $\mathcal{I}$ if the series converges for all $c, d \in \mathcal{I}$. Clearly, if g is continuous at $t_0 \in \mathcal{I}$, then g is β-integrable on $\mathcal{I}$, as shown in [41].

Definition 2. Let $b \in \mathbb{I}, b > t_0$, and let $(\mathbb{U}, \|\cdot\|)$ be a Banach space. The space of continuous functions from $[t_0, b]$ to $\mathbb{U}$ is denoted by

$$C([t_0,b], \mathbb{U}) = \{\varphi : [t_0,b] \to \mathbb{U} \mid \varphi \text{ is continuous}\},$$

equipped with the supremum norm

$$\|\varphi\|_\infty = \sup_{t \in [t_0,b]} \|\varphi(t)\|.$$

Definition 3. The β-exponential function $\mathcal{E}_\zeta(t,t_0)$ is defined as the solution to the following differential equation:

$$\mathscr{D}_\beta \mathcal{E}_\zeta(t,t_0) = \zeta(t)\mathcal{E}_\zeta(t,t_0), \quad \mathcal{E}_\zeta(t_0,t_0) = 1.$$

Theorem 1. [41] Let $g : \mathcal{I} \longrightarrow \mathbb{U}$ be a function that is continuous at t_0. Define the function

$$G(t) = \int_{t_0}^t g(s) d_\beta s, \quad t \in \mathcal{I}.$$

Then, G is continuous at t_0, the β-derivative of $G(t)$, denoted $\mathscr{D}_\beta G(t)$, exists for every $t \in \mathcal{I}$, and we have

$$\mathscr{D}_\beta G(t) = g(t).$$

Theorem 2. [41] Suppose $g : \mathcal{I} \to \mathbb{U}$ is β-differentiable on $\mathcal{I}$. Then, the following identity holds:

$$\int_c^d \mathscr{D}_\beta g(\eta)\, d_\beta \eta = g(d) - g(c), \quad c, d \in \mathcal{I}.$$

Theorem 3. [41] If $\mathcal{Z} : \mathcal{I} \to \mathbb{U}$ is continuous at t_0, then the series

$$\sum_{k=0}^{\infty} \|(\beta^k(t) - \beta^{k+1}(t)) \mathcal{Z}(\beta^k(t))\|$$

is uniformly convergent on every compact interval $\mathbb{I} \subseteq \mathcal{I}$ that contains t_0.

Finally, the following β-Hölder inequality in quantum calculus was established in [43], and further details can be found in [42].

Theorem 4. (β-Hölder inequality) If $f \in L^p([a,b]_\beta, \mathbb{R})$ and $g \in L^q([a,b]_\beta, \mathbb{U})$, where $p > 1$ and $q = \frac{p}{p-1}$, then $fg \in L^1([a,b]_\beta, \mathbb{U})$, and the following inequality holds:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q,$$

or equivalently,

$$\int_a^b \|f(t)g(t)\| d_\beta t \leq \left(\int_a^b |f(t)|^p d_\beta t \right)^{\frac{1}{p}} \left(\int_a^b \|g(t)\|^q d_\beta t \right)^{\frac{1}{q}}.$$

In particular, when $p = q = 2$, this reduces to the β-Cauchy-Schwarz inequality.

Theorem 5. [42] Let $y, f, \mathcal{Z}$ be continuous real-valued functions on $\mathcal{I}$, with $\mathcal{Z} \geq 0$. If the following inequality holds:

$$y(t) \leq f(t) + \int_{t_0}^{t} y(\eta)\mathcal{Z}(\eta)d_\beta\eta, \quad t \in \mathcal{I},$$

then it implies that

$$y(t) \leq f(t) + \int_{t_0}^{t} \mathcal{E}_{\beta,\mathcal{Z}}(t,\beta(\eta))f(\eta)\mathcal{Z}(\eta)d_\beta\eta, \quad t \in \mathcal{I}.$$

Corollary 1. [42] Let $\mathcal{Z}(t) \geq 0$ and $\mu \in \mathbb{R}$. If the inequality

$$y(t) \leq \mu + \int_{t_0}^{t} y(\eta)\mathcal{Z}(\eta)d_\beta\eta, \quad t \in \mathcal{I},$$

holds, then it follows that

$$y(t) \leq \mu\mathcal{E}_{\beta,\mathcal{Z}}(t,t_0).$$

Definition 4. [44] [Discrete fractional Grönwall inequality] Let $\alpha \in (0,1]$, $T \in \mathbb{N}$, and let $\{y(n)\}_{n=0}^{T}$, $\{a(n)\}_{n=0}^{T}$ be nonnegative real sequences. Suppose $b \geq 0$ is a constant and the inequality

$$y(n) \leq a(n) + b\sum_{k=0}^{n-1}\frac{(n-k)^{\alpha-1}}{\Gamma(\alpha)}y(k), \quad \text{for all } n = 1,2,\ldots,T$$

holds. Then the function $y(n)$ satisfies the bound:

$$y(n) \leq a(n) + \sum_{k=0}^{n-1}\frac{(n-k)^{\alpha-1}}{\Gamma(\alpha)}a(k)E_\alpha\left(b(n-k)^\alpha\right), \quad \forall n = 1,\ldots,T,$$

where $E_\alpha(z)$ is the one-parameter Mittag–Leffler function defined by

$$E_\alpha(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}.$$

In particular, if $a(k) \leq A(k-t_0)^\alpha$ for some constant $A > 0$, then

$$y(n) \leq A(n-t_0)^\alpha E_\alpha\left(b(n-t_0)^\alpha\right).$$

Theorem 6. [40] If $\mathcal{Z}: \mathcal{I} \to \mathbb{U}$ is continuous at t_0, then the following hold:

1. The β-derivative with respect to η, denoted $\mathscr{D}_{\beta,\eta}$, satisfies

$$\mathscr{D}_{\beta,\eta}\mathcal{E}_{\beta,\mathcal{Z}}(t,\eta) = -\mathcal{Z}(\eta)\mathcal{E}_{\beta,\mathcal{Z}}(t,\beta(\eta)).$$

2. The following integral identity holds:

$$\int_{t_0}^{t} \mathcal{E}_{\beta,\mathcal{Z}}(t,\beta(\eta))\mathcal{Z}(\eta)\,d\eta = \mathcal{E}_{\beta,\mathcal{Z}}(t,t_0) - 1.$$

Lemma 1. [41] Let $f: \mathcal{I} \to \mathbb{U}$ and $g: \mathcal{I} \to \mathbb{R}$ be β-integrable functions on $\mathcal{I}$. If

$$\|f(t)\| \leq g(t) \quad \text{for all } t \in [a,b]_\beta,\ a,b \in \mathcal{I} \text{ and } a \leq b,$$

then for $x, y \in [a,b]_\beta$ with $x < t_0 < y$, the following inequalities hold:

$$\left\|\int_{t_0}^{y} f(t)\,d_\beta t\right\| \leq \int_{t_0}^{y} g(t)\,d_\beta t,$$

$$\left\|\int_{t_0}^{x} f(t)\,d_\beta t\right\| \leq -\int_{t_0}^{x} g(t)\,d_\beta t,$$

and

$$\left\|\int_{x}^{y} f(t)\,d_\beta t\right\| \leq \int_{x}^{y} g(t)\,d_\beta t.$$

Moreover, if $g(t) \geq 0$ for all $t \in [a,b]_\beta$, then the following inequalities hold:

$$\int_{t_0}^{y} g(t)\,d_\beta t \geq 0 \quad \text{and} \quad \int_{x}^{y} g(t)\,d_\beta t \geq 0.$$

We now make the following assumptions that will be used later.

- $\mathcal{M}_1: \zeta \in C(\mathbb{I},\mathbb{R})$ and $f \in C(\mathbb{I},\mathbb{U})$, where $C(\mathbb{I},\mathbb{U})$ denotes the space of continuous functions from $\mathbb{I}$ to $\mathbb{U}$.
- $\mathcal{M}_2$: The function $\mathcal{F}: \mathbb{I}\times\mathbb{U}\times\mathbb{U} \to \mathbb{U}$ is Lipschitz continuous with respect to its second and third arguments, with a Lipschitz constant $L_\mathcal{F} \geq 0$. Additionally, the function $h: \mathbb{U} \to \mathbb{U}$ is Lipschitz continuous with a Lipschitz constant $L_h \geq 0$.
  Specifically:
  - $\|\mathcal{F}(t,u,v)\| \leq L_\mathcal{F}(\|u\| + \|v\|)$,
  - $\|f(t)\| \leq C_f$ for all $t \in \mathbb{I}$,
  - $\|h(u)\| \leq L_h\|u\|$.

  Specifically, for all $x_1, x_2, y_1, y_2 \in \mathbb{U}$ and for all $t \in \mathbb{I}$, we have:

$$\|\mathcal{F}(t,x_1,h(x_1)) - \mathcal{F}(t,x_2,h(x_2))\| \leq \kappa\|x_1 - x_2\|,$$

  where $\kappa := L_\mathcal{F}(1+L_h)$.
- $\mathcal{M}_3$: For every $\mathscr{U}_0 \in \mathbb{U}$, Equation 4 has a solution.
- $\mathcal{M}_4$: The constant $\theta = \sup_{t\in\mathbb{I}}\int_{t_0}^{t}|\zeta(t)|d_\beta s$ satisfies:

$$\theta + \kappa(b-t_0) < 1. \tag{3}$$

- $\mathcal{M}_5: \kappa(b-a)\mathcal{E}_\zeta(b,a) < 1$.
- $\mathcal{M}_6: 1 - \left(\theta + \kappa + C_f\right)(b-t_0) \geqslant \frac{\|\mathscr{U}_0\|}{R}$, with $R > 0$.

3 Existence and uniqueness of solutions

In this section, we aim to establish sufficient conditions for the existence and uniqueness of solutions to first-order nonlinear quantum difference equations of the form:

$$\begin{aligned}\mathscr{D}_\beta\mathscr{U}(t) &= \zeta(t)\mathscr{U}(t) + \mathcal{F}(t,\mathscr{U}(t),h(\mathscr{U}(t))) + f(t), \quad t \in \mathbb{I},\\ \mathscr{U}(t_0) &= \mathscr{U}_0,\end{aligned} \tag{4}$$

where $\mathcal{D}_\beta$ denotes the β-derivative, ζ is a given function, and $\mathcal{F}:\mathbb{I}\times\mathbb{U}\times\mathbb{U}\to\mathbb{U}$ represents a nonlinear interaction term. The function f corresponds to an external forcing term, defined on the interval $\mathbb{I}=[t_0,b]$ with $b>t_0$. Equation 4 represents a crucial advancement in modeling quantum systems with nonlinear interactions and discrete time steps. Its general form encompasses a wide range of quantum dynamics, making it a powerful tool for understanding complex behaviors in discrete quantum systems.

The main objective of this section is to derive the conditions under which a solution exists and is unique. We also aim to express the solution in integral form, which can provide additional insights into its behavior.

Theorem 7. The function $\mathscr{U}$ is a solution to Equation 4 if and only if it satisfies the following integral equation:

$$\mathscr{U}(t)=\mathscr{U}_0+\int_{t_0}^{t}\left[\zeta(s)\mathscr{U}(s)+\mathcal{F}(s,\mathscr{U}(s),h(\mathscr{U}(s)))+f(s)\right]d_\beta s, \quad (5)$$

where $\mathscr{U}_0\in\mathbb{U}$ is the initial value at $t=t_0$.

Proof. (Forward direction): Assume that $\mathscr{U}$ satisfies Equation 4, i.e.,

$$\mathcal{D}_\beta\mathscr{U}(t)=\zeta(t)\mathscr{U}(t)+\mathcal{F}(t,\mathscr{U}(t),h(\mathscr{U}(t)))+f(t),\quad t\in\mathbb{I},\mathscr{U}(t_0)=\mathscr{U}_0.$$

By integrating both sides from t_0 to t, we obtain:

$$\int_{t_0}^{t}\mathcal{D}_\beta\mathscr{U}(s)d_\beta s=\int_{t_0}^{t}\left[\zeta(s)\mathscr{U}(s)+\mathcal{F}(s,\mathscr{U}(s),h(\mathscr{U}(s)))+f(s)\right]d_\beta s,$$

which leads to the following equation:

$$\mathscr{U}(t)=\mathscr{U}_0+\int_{t_0}^{t}\left[\zeta(s)\mathscr{U}(s)+\mathcal{F}(s,\mathscr{U}(s),h(\mathscr{U}(s)))+f(s)\right]d_\beta s.$$

Thus, $\mathscr{U}$ satisfies the integral form (Equation 5).

(Reverse Direction): Now, assume that $\mathscr{U}$ is given by Equation 5, i.e.,

$$\mathscr{U}(t)=\mathscr{U}_0+\int_{t_0}^{t}\left[\zeta(s)\mathscr{U}(s)+\mathcal{F}(s,\mathscr{U}(s),h(\mathscr{U}(s)))+f(s)\right]d_\beta s.$$

Taking the $\mathcal{D}_\beta$-derivative of both sides, and applying the fundamental theorem of quantum calculus, we obtain:

$$\mathcal{D}_\beta\mathscr{U}(t)=\zeta(t)\mathscr{U}(t)+\mathcal{F}(t,\mathscr{U}(t),h(\mathscr{U}(t)))+f(t),$$

and since $\mathscr{U}(t_0)=\mathscr{U}_0$, it is obvious that $\mathscr{U}$ satisfies the original equation (Equation 4).

In Theorem 8, we establish the existence and uniqueness results for the problem (Equation 4) by applying the standard tools of fixed-point theory. Let us first transform the problem (Equation 4) into a fixed-point problem: $\mathcal{H}\mathscr{U}=\mathscr{U}$ where $\mathcal{H}:C(\mathbb{I},\mathbb{U})\to C(\mathbb{I},\mathbb{U})$ is the fixed operator defined by

$$(\mathcal{H}\mathscr{U})(t)=\mathscr{U}_0+\int_{t_0}^{t}\zeta(s)\mathscr{U}(s)+\mathcal{F}(s,\mathscr{U}(s),h(\mathscr{U}(s)))+f(s)d_\beta s.$$

Theorem 8. Under the assumptions $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_4$, if $\mathscr{U}_0\in\mathbb{U}$, the quantum difference equation (Equation 4) has a unique solution on the interval $\mathbb{I}$.

Proof. Let $\mathscr{U}_0\in\mathbb{U}$. Using Theorem 7, define the operator $\mathcal{H}:C(\mathbb{I},\mathbb{U})\to C(\mathbb{I},\mathbb{U})$ by:

$$(\mathcal{H}\mathscr{U})(t)=\mathscr{U}_0+\int_{t_0}^{t}\left[\zeta(s)\mathscr{U}(s)+\mathcal{F}(s,\mathscr{U}(s),h(\mathscr{U}(s)))+f(s)\right]d_\beta s.$$

To prove that $\mathcal{H}$ is a contraction, let $\mathscr{U}_1,\mathscr{U}_2\in C(\mathbb{I},\mathbb{U})$. Then, we have:

$$\begin{aligned}\|(\mathcal{H}\mathscr{U}_1)(t)-(\mathcal{H}\mathscr{U}_2)(t)\| &= \Big\|\int_{t_0}^{t}\big[\zeta(s)(\mathscr{U}_1(s)-\mathscr{U}_2(s))\\ &\quad+\mathcal{F}(s,\mathscr{U}_1(s),h(\mathscr{U}_1(s)))\\ &\quad-\mathcal{F}(s,\mathscr{U}_2(s),h(\mathscr{U}_2(s)))\big]d_\beta s\Big\|\\ &\le\int_{t_0}^{t}|\zeta(s)|\,\|\mathscr{U}_1(s)-\mathscr{U}_2(s)\|d_\beta s\\ &\quad+\int_{t_0}^{t}\|\mathcal{F}(s,\mathscr{U}_1(s),h(\mathscr{U}_1(s)))\\ &\quad-\mathcal{F}(s,\mathscr{U}_2(s),h(\mathscr{U}_2(s)))\|\,d_\beta s.\end{aligned}$$

By the Lipschitz condition $\mathcal{M}_2$, we get

$$\begin{aligned}\|(\mathcal{H}\mathscr{U}_1)(t)-(\mathcal{H}\mathscr{U}_2)(t)\|&\le\int_{t_0}^{t}|\zeta(s)|\|\mathscr{U}_1(s)-\mathscr{U}_2(s)\|\,d_\beta s\\ &\quad+\int_{t_0}^{t}\kappa\|\mathscr{U}_1(s)-\mathscr{U}_2(s)\|d_\beta s,\quad t\in\mathbb{I}.\end{aligned}$$

After simple calculation, we obtain

$$\|(\mathcal{H}\mathscr{U}_1)(t)-(\mathcal{H}\mathscr{U}_2)(t)\|\le(\theta+\kappa(b-t_0))\|\mathscr{U}_1-\mathscr{U}_2\|_\infty,\quad t\in\mathbb{I}. \quad (6)$$

Since $\theta+\kappa(b-a)<1$ by condition $\mathcal{M}_4$, $\mathcal{H}$ is a contraction mapping. By the Banach Fixed Point Theorem, $\mathcal{H}$ has a unique fixed point $\mathscr{U}$ in $C(\mathbb{I},\mathbb{U})$, which satisfies the integral equation (Equation 5). This implies that $\mathscr{U}$ is the unique solution of the quantum difference equation (Equation 4).

Theorem 9. Under the assumptions $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_4$, if $\mathscr{U}_0\in\mathbb{U}$, then Equation 4 has at least one solution $\mathscr{U}\in C(\mathbb{I},\mathbb{U})$.

Proof. We prove the existence of solutions using the Leray-Schauder nonlinear alternative theorem.

Step 1: Uniform Boundedness of $\mathcal{H}$.
Firstly, we aim to demonstrate that the operator $\mathcal{H}:C(\mathbb{I},\mathbb{U})\to C(\mathbb{I},\mathbb{U})$, maps bounded sets into bounded sets within $\mathbb{X}=C(\mathbb{I},\mathbb{U})$. For a positive number R, we consider a closed ball $B_R=\{\mathscr{U}\in\mathbb{X}:\|\mathscr{U}\|\leqslant R\}$ be bounded set in $\mathbb{X}$. For $\mathscr{U}\in C(\mathbb{I},\mathbb{U})$ with $\|\mathscr{U}\|_\infty\le R$, we estimate:

$$\begin{aligned}\|(\mathcal{H}\mathscr{U})(t)\|&\le\|\mathscr{U}_0\|+\int_{t_0}^{t}\big(|\zeta(s)|\,\|\mathscr{U}(s)\|+L_{\mathcal{F}}(\|\mathscr{U}(s)\|\\ &\quad+L_h\|\mathscr{U}(s)\|)+C_f\big)\,d_\beta s\\ &\le\|\mathscr{U}_0\|+R\left(\theta+\kappa+C_f\right)(b-t_0).\end{aligned}$$

Then using $\mathcal{M}_6$, we get

$$\|(\mathcal{H}\mathscr{U})(t)\|\le R\quad\text{for all }t\in\mathbb{I}.$$

This indicates that the set $\mathcal{H}(B_R)$ is uniformly bounded. Thus, $\mathcal{H}$ maps bounded sets to bounded sets.

Step 2: Continuity of $\mathcal{H}$.

We now claim that $\mathcal{H}$ is continuous. To prove this, we consider a sequence $\{\mathcal{U}_n\}$ in B_R that converges to $\mathcal{U}$ and show that $\mathcal{H}\mathcal{U}_n \to \mathcal{H}\mathcal{U}$ as $n \mapsto \infty$.

Using Equation 6, we have

$$\|(\mathcal{H}\mathcal{U}_n)(t) - (\mathcal{H}\mathcal{U})(t)\| \leq \left(\theta + \kappa(b - t_0)\right) \|\mathcal{U}_n - \mathcal{U}\|_\infty.$$

As $n \to \infty$, $\|\mathcal{U}_n - \mathcal{U}\|_\infty \to 0$, then $\|\mathcal{H}\mathcal{U}_n - \mathcal{H}\mathcal{U}\| \to 0$ as $n \to \infty$.

Hence, $\mathcal{H}$ is continuous.

Step 3: Equicontinuity of $\mathcal{H}$.

We claim that $\mathcal{H}$ maps bounded set into a set of equicontinuous functions.

Let $t_1, t_2 \in \mathbb{I}$ with $t_1 < t_2$. For $\|\mathcal{U}\|_\infty \leq R$:

$$\begin{aligned}\|(\mathcal{H}\mathcal{U})(t_2) - (\mathcal{H}\mathcal{U})(t_1)\| &\leq \int_{t_1}^{t_2} \big\|\zeta(s)\mathcal{U}(s) + \mathcal{F}(s, \mathcal{U}(s), h(\mathcal{U}(s))) \\ &\quad + f(s)\big\|\, d_\beta s \\ &\leq \int_{t_1}^{t_2} \left(|\zeta(s)|\, R + L_{\mathcal{F}}(1 + L_h)R \right.\\ &\quad \left. + \|f\|_\infty\right) d_\beta s \\ &\leq \left(R\,(2\theta + \kappa) + C_f\right)(t_2 - t_1).\end{aligned}$$

The right-hand side tends to zero as $|t_2 - t_1| \to 0$, independent of $\mathcal{U}$. Hence, $\mathcal{H}$ is equicontinuous.

Step 4: Application of Leray-Schauder's Theorem.

Now, we need to prove that there exists an open set $\mathcal{A} \subseteq \mathbb{X}$ such that $\mathcal{U} \neq \lambda_* \mathcal{H}(\mathcal{U})$ for $\lambda_* \in (0, 1)$ and $\mathcal{U} \in \partial\mathcal{A}$. We suppose $\mathcal{U} \in \mathbb{X}$ such that $\mathcal{U} = \lambda_* \mathcal{H}(\mathcal{U})$ for $\lambda_* \in (0, 1)$. For $t \in [a, b]$, we have $\mathcal{U} = \lambda_* \mathcal{H}\mathcal{U}$ for some $\lambda_* \in (0, 1)$. Using same technique as Step 1, we arrive to:

$$\|\mathcal{U}\|_\infty \leq \|\mathcal{U}_0\| + \left(\theta + \kappa + C_f\right)(b - t_0)\|u\|_\infty = \mathcal{M}.$$

Define the set $\mathcal{A}$ as

$$\mathcal{A} = \{\mathcal{U} \in \mathbb{X} : \|\mathcal{U}\| \leqslant \mathcal{M} + 1\}.$$

There is no $\mathcal{U} \in \partial\mathcal{A}$ satisfying $\mathcal{U} = \lambda_* \mathcal{H}\mathcal{U}$ for some $\lambda_* \in (0, 1)$. Consequently, by the Leray-Schauder nonlinear alternative, the operator $\mathcal{H}$ has at least one fixed point in $\mathcal{A}$, which corresponds to a solution of problem (Equation 4).

The use of the Leray–Schauder nonlinear alternative in Theorem 9 allows us to establish the existence of solutions without requiring the strict contractive condition employed in Theorem 8. This is particularly useful in cases where the nonlinear term $\mathcal{F}$ does not satisfy a global Lipschitz condition, or when the combined constant $\theta + \kappa(b - t_0)$ is not strictly less than one. However, if this contractive condition *is* satisfied, then Banach's Fixed Point Theorem applies directly, ensuring both the existence and uniqueness of the solution. Therefore, the two approaches serve complementary roles: the Leray–Schauder method guarantees existence under broader assumptions, while the Banach method provides uniqueness under stronger ones.

Theorem 10. (Uniqueness via Grönwall Inequality [[42], Corollary 2.3]). Let the assumptions $\mathcal{M}_1$ and $\mathcal{M}_2$ hold, and suppose the function $\mathcal{U} \in C(\mathbb{I}, \mathbb{U})$ satisfies the integral equation (Equation 5). Then the solution to the quantum difference equation (Equation 4) is unique on the interval $\mathbb{I}$.

Proof. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two solutions of Equation 5, and define their difference $\mathcal{W}(t) = \mathcal{U}_1(t) - \mathcal{U}_2(t)$. Then:

$$\begin{aligned}\mathcal{W}(t) = \int_{t_0}^{t} &\left[\zeta(s)\mathcal{W}(s) + \mathcal{F}(s, \mathcal{U}_1(s), h(\mathcal{U}_1(s)))\right.\\ &\left. - \mathcal{F}(s, \mathcal{U}_2(s), h(\mathcal{U}_2(s)))\right] d_\beta s.\end{aligned}$$

Using the Lipschitz condition from $\mathcal{M}_2$, we estimate:

$$\|\mathcal{W}(t)\| \leq \int_{t_0}^{t} \left(|\zeta(s)| + \kappa\right) \|\mathcal{W}(s)\|\, d_\beta s.$$

Let $\phi(t) := \|\mathcal{W}(t)\|$. Then:

$$\phi(t) \leq \phi(t_0) + \int_{t_0}^{t} \left(|\zeta(s)| + \kappa\right) \phi(s)\, d_\beta s.$$

Applying the Grönwall-type inequality in the sense of quantum calculus (see Corollary 1), we obtain:

$$\phi(t) \leq \phi(t_0) \cdot \mathcal{E}_{\beta, |\zeta| + \kappa}(t, t_0).$$

Since $\mathcal{U}_1(t_0) = \mathcal{U}_2(t_0) = \mathcal{U}_0$, it follows that $\phi(t_0) = 0$. Hence:

$$\phi(t) = 0 \quad \Rightarrow \quad \mathcal{U}_1(t) = \mathcal{U}_2(t), \quad \forall t \in \mathbb{I}.$$

Therefore, the solution is unique.

4 Stability analysis

This section is devoted to the investigation of various types of stability for the nonlinear quantum difference equation (Equation 4). Throughout the analysis, the functions f, h, $\mathcal{F}$, and ζ are assumed to be continuous (see [45]).

Definition 5 (Hyers-Ulam stability). Equation 4 is said to have Hyers–Ulam stability (HUs), if there is a positive number $M > 0$, a so-called HUs constant, with the following property: For any $\epsilon > 0$, if $\mathcal{U} \in C^1(\mathbb{I}, \mathbb{U})$ is such that

$$\|\mathcal{D}_\beta \mathcal{U}(t) - \zeta(t)\mathcal{U}(t) - \mathcal{F}(t, \mathcal{U}(t), h(\mathcal{U}(t))) - f(t)\| \leq \epsilon, \quad t \in \mathbb{I}, \tag{7}$$

there exists a solution $\mathcal{U}_*$ of Equation 4 such that

$$\|\mathcal{U}(t) - \mathcal{U}_*(t)\| \leq M\epsilon, \quad t \in \mathbb{I}. \tag{8}$$

Here, $C^1(\mathbb{I}, \mathbb{U})$ is the space of all $\beta-$differentiable functions φ such that $\mathcal{D}_\beta \varphi$ is continuous.

Definition 6 (Generalized Hyers-Ulam stability). Equation 4 is said to have Generalized Hyers-Ulam stability (GHUs) if there exists a function $\phi : [0, \infty) \to [0, \infty)$, continuous at 0 with $\phi(0) = 0$, such that for any $\epsilon > 0$, if $\mathcal{U} \in C^1(\mathbb{I}, \mathbb{U})$ satisfies

$$\left\|\mathcal{D}_\beta \mathcal{U}(t) - \zeta(t)\mathcal{U}(t) - \mathcal{F}(t, \mathcal{U}(t), h(\mathcal{U}(t))) - f(t)\right\| \leq \epsilon, \quad t \in \mathbb{I}, \tag{9}$$

then there exists a solution $\mathscr{U}_*$ of Equation 4 satisfying

$$\left\|\mathscr{U}(t)-\mathscr{U}_*(t)\right\| \leq \phi(\epsilon), \quad t \in \mathbb{I}. \tag{10}$$

Theorem 11. If conditions ($\mathcal{M}_1$-$\mathcal{M}_3$), are satisfied, then the problem (Equation 4) is both Ulam-Hyers and generalized Ulam-Hyers stable.

Proof. Let $\epsilon > 0$ and $\mathscr{U} \in C^1(\mathbb{I}, \mathbb{U})$ satisfies Equation 7. Then $\mathscr{U}$ solves the approximate equation

$$\mathscr{D}_\beta \mathscr{U}(t)=\zeta(t)\mathscr{U}(t)+\mathcal{F}(t, \mathscr{U}(t), h(\mathscr{U}(t)))+f(t)+\sigma(t), \tag{11}$$

where σ is defined by

$$\sigma(t)=\mathscr{D}_\beta \mathscr{U}(t)-\zeta(t)\mathscr{U}(t)-\mathcal{F}(t, \mathscr{U}(t), h(\mathscr{U}(t)))-f(t). \tag{12}$$

In view of Equation 7, we obtain

$$\|\sigma(t)\| \leq \epsilon, \quad t \in \mathbb{I}. \tag{13}$$

Condition $\mathcal{M}_3$ implies that Equation 4 has a solution $\mathscr{U}_*$ with $\mathscr{U}_*(t_0)=\mathscr{U}_0$. By Theorem 7, the corresponding integral formulas for Equations 4, 11 are

$$\mathscr{U}_*(t)=\mathscr{U}_0+\int_{t_0}^t\left(\zeta(s)\mathscr{U}_*(s)+\mathcal{F}(s, \mathscr{U}_*(s), h(\mathscr{U}_*(s)))+f(s)\right) d_\beta s, \tag{14}$$

and

$$\mathscr{U}(t)=\mathscr{U}_0+\int_{t_0}^t\left(\zeta(s)\mathscr{U}(s)+\mathcal{F}(s, \mathscr{U}(s), h(\mathscr{U}(s)))+f(s)+\sigma(s)\right) d_\beta s. \tag{15}$$

Next, subtracting Equation 14 from Equation 15, it follows that

$$\begin{aligned}\|\mathscr{U}(t)-\mathscr{U}_*(t)\| \leq & \int_{t_0}^t\|\sigma(s)\| d_\beta s+\int_{t_0}^t(|\zeta(s)|+\kappa)\|\mathscr{U}(s) \\ & -\mathscr{U}_*(s)\| d_\beta s, \quad t \in \mathbb{I},\end{aligned} \tag{16}$$

where we have applied condition $\mathcal{M}_2$. From Equation 13, we deduce that

$$\|\mathscr{U}(t)-\mathscr{U}_*(t)\| \leq \epsilon(b-t_0)+\int_{t_0}^t(|\zeta(s)|+\kappa)\|\mathscr{U}(s)-\mathscr{U}_*(s)\| d_\beta s.$$

By Grönwall's inequality [Corollary 1]

$$\|\mathscr{U}(t)-\mathscr{U}_*(t)\| \leq \epsilon(b-t_0)\mathcal{E}_{\beta,|\zeta|+\kappa}(b, t_0), \quad t \in \mathbb{I}. \tag{17}$$

Therefore, Equation 4 has Hyers–Ulam stability with HUs constant $M=\epsilon(b-t_0)\mathcal{E}_{\beta,|\zeta|+\kappa}(b, t_0)$. Furthermore, one can notice that the generalized Ulam-Hyers stability condition also holds valid if we set $\phi(\epsilon)=\epsilon(b-t_0)\mathcal{E}_{\beta,|\zeta|+\kappa}(b, t_0)$.

We present the Hyers–Ulam–Rassias stability (HURs) of the quantum difference equation (Equation 4). For more details, see [45].

Definition 7 (Hyers–Ulam–Rassias stability). Let $\mathcal{F}$ be a family of nonnegative continuous functions defined on $\mathbb{I}$. We say that Equation 4 exhibits Hyers–Ulam–Rassias stability (HURs) of type $\mathcal{F}$ if there exists a function $\mathcal{K}: \mathcal{F} \rightarrow C(\mathbb{I},[0, \infty))$, called the $HURs_{\mathcal{F}}$ function, that satisfies the following property: for every $\varphi \in \mathcal{F}$, if $\mathscr{U} \in C^1(\mathbb{I}, \mathbb{U})$ is such that

$$\|\mathscr{D}_\beta \mathscr{U}(t)-\zeta(t)\mathscr{U}(t)-\mathcal{F}(t, \mathscr{U}(t), h(\mathscr{U}(t)))-f(t)\| \leq \varphi(t), \quad t \in \mathbb{I}, \tag{18}$$

then there exists a solution $\mathscr{U}_*$ of Equation 4 such that

$$\|\mathscr{U}(t)-\mathscr{U}_*(t)\| \leq \mathcal{K}(\varphi)(t), \quad t \in \mathbb{I}. \tag{19}$$

When $\mathcal{F}=C(\mathbb{I}, \mathbb{R}^+)$, the Equation 4 is said to have Hyers–Ulam–Rassias stability (HURs).

Theorem 12. If $\mathcal{M}_1$-$\mathcal{M}_3$ hold, then Equation 4 has Hyers–Ulam–Rassias stability with the $HURs_{\mathcal{F}}$ function given by:

$$\mathcal{K}(\varphi)(t)=(b-t_0) \cdot \mathcal{E}_{\beta,|\zeta|+\kappa}(b, t_0) \cdot\|\varphi(t)\|_{\infty}, \quad t \in \mathbb{I}.$$

Proof. Let $\varphi \in \mathcal{O}_C$, and suppose that $\mathscr{U} \in C^1(\mathbb{I}, \mathbb{U})$ satisfies Equation 18. Define σ as in Equation 12, so that $\|\sigma(t)\| \leq \varphi(t)$. Let $\mathscr{U}(t_0)=\mathscr{U}_0$. By Theorem 7, $\mathscr{U}$ satisfies the integral form:

$$\mathscr{U}(t)=\mathscr{U}_0+\int_{t_0}^t\left(\zeta(s)\mathscr{U}(s)+\mathcal{F}(s, \mathscr{U}(s), h(\mathscr{U}(s)))+f(s)+\sigma(s)\right) d_\beta s. \tag{20}$$

Let $\mathscr{U}_*$ be the solution of the exact equation with $\mathscr{U}_*(t_0)=\mathscr{U}_0$. Using the integral formulation (Theorem 7) for $\mathscr{U}_*$:

$$\mathscr{U}_*(t)=\mathscr{U}_0+\int_{t_0}^t\left[\zeta(s)\mathscr{U}_*(s)+\mathcal{F}(s, \mathscr{U}_*(s), h(\mathscr{U}_*(s)))+f(s)\right] d_\beta s.$$

Subtracting these two equations, we apply the triangle inequality:

$$\begin{aligned}\|\mathscr{U}(t)-\mathscr{U}_*(t)\| \leq & \int_{t_0}^t\Big(|\zeta(s)|\|\mathscr{U}(s)-\mathscr{U}_*(s)\| \\ & +\|\mathcal{F}(s, \mathscr{U}(s), h(\mathscr{U}(s)))-\mathcal{F}(s, \mathscr{U}_*(s), h(\mathscr{U}_*(s)))\| \\ & +\|\sigma(s)\|\Big) d_\beta s.\end{aligned}$$

By the Lipschitz condition (H2), we have:

$$\|\mathcal{F}(s, \mathscr{U}(s), h(\mathscr{U}(s)))-\mathcal{F}(s, \mathscr{U}_*(s), h(\mathscr{U}_*(s)))\| \leq \kappa\|\mathscr{U}(s)-\mathscr{U}_*(s)\|,$$

where $\kappa=L_{\mathcal{F}}(1+L_h)$. Therefore:

$$\begin{aligned}\|\mathscr{U}(t)-\mathscr{U}_*(t)\| \leq & \int_{t_0}^t(|\zeta(s)|+\kappa)\|\mathscr{U}(s)-\mathscr{U}_*(s)\| d_\beta s \\ & +\int_{t_0}^t \varphi(s) d_\beta s \\ \leq & \int_{t_0}^t(|\zeta(s)|+\kappa)\|\mathscr{U}(s)-\mathscr{U}_*(s)\| d_\beta s \\ & +(b-t_0)\|\varphi\|_{\infty}.\end{aligned}$$

Thus, we have:

$$\|\mathscr{U}(t)-\mathscr{U}_*(t)\| \leq(b-t_0)\|\varphi\|_{\infty}+\int_{t_0}^t(|\zeta(s)|+\kappa)\|\mathscr{U}(s)-\mathscr{U}_*(s)\| d_\beta s.$$

By Grönwall's inequality (Corollary 1), we conclude that:

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq (b - t_0) \cdot \mathcal{E}_{\beta,|\zeta|+\kappa}(b, t_0) \cdot \|\varphi\|_\infty, \quad \forall t \in \mathbb{I}.$$

Definition 8 (Functions of exponential order). Let $\delta > 0$ be a fixed constant. A function $u : \mathbb{I} \to \mathbb{R}$ is said to be of *exponential order* δ (in the sense of quantum calculus) if there exists a constant $C > 0$ such that

$$|u(t)| \leq C\,\mathcal{E}_{\beta,\delta}(t, t_0), \quad \forall t \in \mathbb{I},$$

where $\mathcal{E}_{\beta,\delta}(t, t_0)$ denotes the quantum exponential function with rate δ. We define the class:

$$\mathcal{O}_C := \left\{ u : \mathbb{I} \to \mathbb{R} \,\middle|\, \exists C > 0,\ \exists \delta > 0, \text{ such that } |u(t)| \leq C\mathcal{E}_{\beta,\delta}(t, t_0),\ \forall t \in \mathbb{I} \right\}.$$

Theorem 13. If $\mathcal{M}_1$-$\mathcal{M}_3$ hold, then Equation 4 has Hyers–Ulam–Rassias stability of type $\mathcal{O}_C$ with $HURs_{\mathcal{O}_C}$ function given by:

$$\mathcal{K}(\varphi)(t) := \frac{C}{\delta}\left(\mathcal{E}_{\beta,\delta}(b) - 1\right)\mathcal{E}_{\beta,|\zeta|+\kappa}(b, t_0), \quad t \in \mathbb{I}. \tag{21}$$

Proof. Let $\varphi \in \mathcal{O}_C$ and $\mathscr{U} \in C^1(\mathbb{I}, \mathbb{U})$ be such that Equation 18 holds. Define σ as in Equation 12.

We assume that $\mathscr{U}(t_0) = \mathscr{U}_0$. By Theorem 7, $\mathscr{U}$ satisfies the integral equation:

$$\mathscr{U}(t) = \mathscr{U}_0 + \int_{t_0}^{t} \left(\zeta(s)\mathscr{U}(s) + \mathcal{F}(s, \mathscr{U}(s), h(\mathscr{U}(s))) + f(s) + \sigma(s)\right) d_\beta s. \tag{22}$$

By assumption $\mathcal{M}_3$, there exists a solution $\mathscr{U}_*$ of Equation 4 such that $\mathscr{U}_*(t_0) = \mathscr{U}_0$. Again, by Theorem 7, $\mathscr{U}_*$ satisfies:

$$\mathscr{U}_*(t) = \mathscr{U}_0 + \int_{t_0}^{t} \left(\zeta(s)\mathscr{U}_*(s) + \mathcal{F}(s, \mathscr{U}_*(s), h(\mathscr{U}_*(s))) + f(s)\right) d_\beta s. \tag{23}$$

Next, subtract Equation 23 from Equation 22, yielding the following inequality for all $t \in \mathbb{I}$:

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq \int_{t_0}^{t} \left(|\zeta(s)| + \kappa\right)\|\mathscr{U}(s) - \mathscr{U}_*(s)\| d_\beta s + \int_{t_0}^{t} \|\sigma(s)\| d_\beta s, \quad t \in \mathbb{I}. \tag{24}$$

Since $\|\sigma(s)\| \leq \varphi(s)$, we can substitute this into the inequality:

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq \int_{t_0}^{t} \left(|\zeta(s)| + \kappa\right)\|\mathscr{U}(s) - \mathscr{U}_*(s)\| d_\beta s + \int_{t_0}^{t} \varphi(s) d_\beta s, \quad t \in \mathbb{I}. \tag{25}$$

This implies that:

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq \int_{t_0}^{t} \varphi(s) d_\beta s + \int_{t_0}^{t} \left(|\zeta(s)| + \kappa\right)\|\mathscr{U}(s) - \mathscr{U}_*(s)\| d_\beta s.$$

Thus, we have:

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq \frac{C}{\delta}\left(\mathcal{E}_{\beta,\delta}(t) - 1\right) + \int_{t_0}^{t} \left(|\zeta(s)| + \kappa\right)\|\mathscr{U}(s) - \mathscr{U}_*(s)\| d_\beta s.$$

Next, using the fact that the exponential function is increasing (as shown in [41]), we can simplify this further:

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq \frac{C}{\delta}\left(\mathcal{E}_{\beta,\delta}(b) - 1\right) + \int_{t_0}^{t} \left(|\zeta(s)| + \kappa\right)\|\mathscr{U}(s) - \mathscr{U}_*(s)\| d_\beta s, \quad t \in \mathbb{I}.$$

Finally, by applying Grönwall's inequality (Corollary 1), we obtain the bound:

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq \frac{C}{\delta}\left(\mathcal{E}_{\beta,\delta}(b, t_0) - 1\right)\mathcal{E}_{\beta,|\zeta|+\kappa}(t, t_0).$$

This can be further simplified as:

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq \frac{C}{\delta}\left(\mathcal{E}_{\beta,\delta}(b) - 1\right)\mathcal{E}_{\beta,|\zeta|+\kappa}(b, t_0), \quad t \in \mathbb{I}.$$

Hence, the required result follows.

Theorem 14. Let ($\mathcal{F}^* = \{\varphi \in C(\mathbb{I}, (0, \infty)) : \varphi$ is nondecreasing on $\mathbb{I}\}$). If $\mathcal{M}_1$-$\mathcal{M}_3$ hold, then Equation 4 has Hyers–Ulam–Rassias stability of type $\mathcal{F}^*$ with $HURs_{\mathcal{F}^*}$ function:

$$\mathcal{K}(\varphi)(t) := (b - t_0)\mathcal{E}_{\beta,|\zeta(s)|+L}(b, t_0)\varphi(t), \quad t \in \mathbb{I}. \tag{26}$$

Proof. Let $\varphi \in \mathcal{F}^*$ and $\mathscr{U} \in C^1(\mathbb{I}, \mathbb{U})$ be such that Equation 18 holds. Define σ as in Equation 12. Assume that $\mathscr{U}(t_0) = \mathscr{U}_0$. By Theorem 7, $\mathscr{U}$ satisfies the following integral equation:

$$\mathscr{U}(t) = \mathscr{U}_0 + \int_{t_0}^{t} \left(\zeta(s)\mathscr{U}(s) + \mathcal{F}(s, \mathscr{U}(s), h(\mathscr{U}(s))) + f(s) + \sigma(s)\right) d_\beta s. \tag{27}$$

By assumption $\mathcal{M}_3$, there exists a solution $\mathscr{U}_*$ to Equation 4 such that $\mathscr{U}_*(t_0) = \mathscr{U}_0$. Again, by Theorem 7, $\mathscr{U}_*$ satisfies the integral equation:

$$\mathscr{U}_*(t) = \mathscr{U}_0 + \int_{t_0}^{t} \left(\zeta(s)\mathscr{U}_*(s) + \mathcal{F}(s, \mathscr{U}_*(s), h(\mathscr{U}_*(s))) + f(s)\right) d_\beta s. \tag{28}$$

Next, subtract Equation 31 from Equation 32. This results in the following inequality for all $t \in \mathbb{I}$:

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq \int_{t_0}^{t} \left(|\zeta(s)| + \kappa\right)\|\mathscr{U}(s) - \mathscr{U}_*(s)\| d_\beta s + \int_{t_0}^{t} \|\sigma(s)\| d_\beta s, \quad t \in \mathbb{I}.$$

Since $\|\sigma(s)\| \leq \varphi(s)$, we substitute this into the inequality:

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq \int_{t_0}^{t} \left(|\zeta(s)| + \kappa\right)\|\mathscr{U}(s) - \mathscr{U}_*(s)\| d_\beta s + (b - t_0)\varphi(t), \quad t \in \mathbb{I}. \tag{29}$$

Now, applying Grönwall's inequality, we obtain the following bound:

$$
\begin{aligned}
\|\mathscr{U}(t)-\mathscr{U}_*(t)\| &\leq (b-t_0)\varphi(t) \\
&+\int_{t_0}^{t}\mathcal{E}_{\beta,|\zeta(s)|+\kappa}(t,\beta(s))(|\zeta(s)|+L)(b-t_0)\varphi(s)d_\beta s \\
&=(b-t_0)\varphi(t)\left(1+\int_{t_0}^{t}\mathcal{E}_{\beta,|\zeta(s)|+\kappa}(t,\beta(s))(|\zeta(s)| \right. \\
&\quad \left. +L)d_\beta s\right) \\
&\leq (b-t_0)\mathcal{E}_{\beta,|\zeta(s)|+\kappa}(b,t_0)\varphi(t), \quad t\in\mathbb{I},
\end{aligned}
$$

where we used the increasing property of the exponential function $\mathcal{E}_{\beta,|\zeta(s)|+\kappa}(t,t_0)$, as established in Theorem 6. Thus, we have established the desired bound.

Definition 9. For $p\geq 1$ and $\vartheta\geq 0$, we define the family $\mathcal{F}_p^{\vartheta}$ as follows:

$$
\mathcal{F}_p^{\vartheta}:=\left\{\varphi\in C(\mathbb{I},(0,\infty)):\int_{t_0}^{t}\varphi^p(\eta)d_\beta\eta\leq\vartheta\varphi^p(t)\quad\text{for all } t\in\mathbb{I}\right\}.
$$

Theorem 15. If assumptions $\mathcal{M}_1$-$\mathcal{M}_3$ are satisfied, then Equation 4 exhibits Hyers-Ulam-Rassias (HURs) stability of type $\mathcal{F}^*\cap\mathcal{F}_1^1$, with the corresponding HURs stability function $\mathcal{K}_{\mathcal{F}^*\cap\mathcal{F}_1^1}$ given by:

$$
\mathcal{K}(\varphi)(t):=\mathcal{E}_{\beta,|\zeta|+\kappa}(b,t_0)\varphi(t),\quad t\in\mathbb{I}. \tag{30}
$$

Proof. Let $\varphi\in\mathcal{F}^*\cap\mathcal{F}_1^1$ and assume $\mathscr{U}\in C^1(\mathbb{I},\mathbb{U})$ is a solution that satisfies Equation 18. Define σ as in Equation 12. Assume $\mathscr{U}(t_0)=\mathscr{U}_0$. By Theorem 7, $\mathscr{U}$ satisfies the integral equation:

$$
\begin{aligned}
\mathscr{U}(t) \;=\; \mathscr{U}_0+\int_{t_0}^{t}\big(&\zeta(s)\mathscr{U}(s)+\mathcal{F}(s,\mathscr{U}(s),h(\mathscr{U}(s)))+f(s) \\
&+\sigma(s)\big)\,d_\beta s.
\end{aligned} \tag{31}
$$

By assumption $\mathcal{M}_3$, there exists a solution $\mathscr{U}_*$ to Equation 4 such that $\mathscr{U}_*(t_0)=\mathscr{U}_0$. Similarly, by Theorem 7, $\mathscr{U}_*$ satisfies:

$$
\begin{aligned}
\mathscr{U}_*(t) \;=\; \mathscr{U}_0+\int_{t_0}^{t}\big(&\zeta(s)\mathscr{U}_*(s)+\mathcal{F}(s,\mathscr{U}_*(s),h(\mathscr{U}_*(s))) \\
&+f(s)\big)\,d_\beta s.
\end{aligned} \tag{32}
$$

Next, subtract Equation 31 from Equation 32. This gives the following inequality for all $t\in\mathbb{I}$:

$$
\begin{aligned}
\|\mathscr{U}(t)-\mathscr{U}_*(t)\| \;&\leq\; \int_{t_0}^{t}\big(|\zeta(s)|+\kappa\big)\,\|\mathscr{U}(s)-\mathscr{U}_*(s)\|d_\beta s \\
&+\int_{t_0}^{t}\|\sigma(s)\|d_\beta s,\quad t\in\mathbb{I}.
\end{aligned}
$$

Since $\|\sigma(s)\|\leq\varphi(s)$, we substitute this into the inequality:

$$
\begin{aligned}
\|\mathscr{U}(t)-\mathscr{U}_*(t)\| \;\leq\; &\int_{t_0}^{t}\big(|\zeta(s)|+\kappa\big)\,\|\mathscr{U}(s)-\mathscr{U}_*(s)\|d_\beta s \\
&+(b-t_0)\varphi(t),\quad t\in\mathbb{I}.
\end{aligned} \tag{33}
$$

Now, applying Grönwall's inequality (Theorem 5), we obtain the following bound:

$$
\begin{aligned}
\|\mathscr{U}(t)-\mathscr{U}_*(t)\| &\leq (b-t_0)\varphi(t) \\
&+\int_{t_0}^{t}\mathcal{E}_{\beta,|\zeta(s)|+\kappa}(t,\beta(s))(|\zeta(s)|+\kappa)(b-t_0)\varphi(s)d_\beta s \\
&=(b-t_0)\varphi(t)\left(1+\int_{t_0}^{t}\mathcal{E}_{\beta,|\zeta(s)|+\kappa}(t,\beta(s))(|\zeta(s)|+\kappa)d_\beta s\right) \\
&\leq (b-t_0)\mathcal{E}_{\beta,|\zeta(s)|+\kappa}(b,t_0)\varphi(t), \quad t\in\mathbb{I},
\end{aligned}
$$

where we used the increasing property of the exponential function $\mathcal{E}_{\beta,|\zeta(s)|+L}(t)$ and referenced Theorem 6. Thus, we have demonstrated that Equation 4 exhibits Hyers-Ulam-Rassias stability of type $\mathcal{F}^*\cap\mathcal{F}_1^1$, with the corresponding HURs stability function given by Equation 30.

Theorem 16. Assume that conditions $\mathcal{M}_1$-$\mathcal{M}_3$ hold. Then, the Equation 4 exhibits Hyers-Ulam-Rassias stability of type $\mathcal{F}_1^{\vartheta}$, with the corresponding HURs function $\mathcal{K}_{\mathcal{F}_1^{\vartheta}}$ given by:

$$
\mathcal{K}(\varphi)(t):=\big(1+\mathcal{E}_{\beta,|\zeta|+L}(b,t_0)\,(\|\zeta\|_\infty+\kappa)\big)\,\vartheta\varphi(t),\quad t\in\mathbb{I}. \tag{34}
$$

Proof. Let $\varphi\in\mathcal{F}_1^{\vartheta}$, and assume that $\mathscr{U}\in C^1(\mathbb{I},\mathbb{U})$ satisfies Equation 18. We define σ as in Equation 12. Also, let $\mathscr{U}(t_0)=\mathscr{U}_0$. By Theorem 7, the solution $\mathscr{U}$ satisfies the integral equation:

$$
\mathscr{U}(t)=\mathscr{U}_0+\int_{t_0}^{t}\big(\zeta(s)\mathscr{U}(s)+\mathcal{F}(s,\mathscr{U}(s),h(\mathscr{U}(s)))+f(s)+\sigma(s)\big)\,d_\beta s. \tag{35}
$$

By assumption $\mathcal{M}_3$, there exists a solution $\mathscr{U}_*$ to Equation 4 such that $\mathscr{U}_*(t_0)=\mathscr{U}_0$. Again, by Theorem 7, $\mathscr{U}_*$ satisfies the equation:

$$
\mathscr{U}_*(t)=\mathscr{U}_0+\int_{t_0}^{t}\big(\zeta(s)\mathscr{U}_*(s)+\mathcal{F}(s,\mathscr{U}_*(s),h(\mathscr{U}_*(s)))+f(s)\big)\,d_\beta s. \tag{36}
$$

Now, subtract Equation 35 from Equation 36. This yields the following inequality for all $t\in\mathbb{I}$:

$$
\begin{aligned}
\|\mathscr{U}(t)-\mathscr{U}_*(t)\| \;&\leq\; \int_{t_0}^{t}\big(|\zeta(s)|+L\big)\,\|\mathscr{U}(s)-\mathscr{U}_*(s)\|d_\beta s \\
&+\int_{t_0}^{t}\varphi(s)d_\beta s, t\in\mathbb{I}.
\end{aligned}
$$

Therefore, we obtain the following bound:

$$
\begin{aligned}
\|\mathscr{U}(t)-\mathscr{U}_*(t)\|\leq\vartheta\varphi(t)\;+\;&\int_{t_0}^{t}\big(|\zeta(s)|+\kappa\big)\,\|\mathscr{U}(s)-\mathscr{U}_*(s)\|d_\beta s, \\
&t\in\mathbb{I}.
\end{aligned} \tag{37}
$$

Now, applying Grönwall's inequality (Theorem 5), we get the following estimate for $t\in\mathbb{I}$:

$$
\|\mathscr{U}(t)-\mathscr{U}_*(t)\|\leq\vartheta\varphi(t)+\int_{t_0}^{t}\mathcal{E}_{\beta,|\zeta|+\kappa}(t,\beta(s))\,\big(|\zeta(s)|+\kappa\big)\,\varphi(s)d_\beta s. \tag{38}
$$

Next, we bound the integral term:

$$
\|\mathscr{U}(t)-\mathscr{U}_*(t)\|\leq\vartheta\varphi(t)+\mathcal{E}_{\beta,|\zeta|+\kappa}(b,t_0)\,(\|\zeta\|_\infty+\kappa)\int_{t_0}^{t}\varphi(s)d_\beta s.
$$

Now, since $\varphi \in \mathcal{F}_1^{\vartheta}$, we know that $\int_{t_0}^{t} \varphi(s) d_\beta s \leq \vartheta \varphi(t)$. Hence, we get:

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq \vartheta\varphi(t) + \mathcal{E}_{\beta,|\zeta|+\kappa}(b, t_0)\left(\|\zeta\|_\infty + \kappa\right)\vartheta\varphi(t).$$

Thus, we have:

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq \left(1 + \mathcal{E}_{\beta,|\zeta|+\kappa}(b, t_0)\left(\|\zeta\|_\infty + \kappa\right)\right)\vartheta\varphi(t).$$

Finally, we conclude that Equation 4 has Hyers-Ulam-Rassias stability of type $\mathcal{F}_1^{\vartheta}$, with the stability function $\mathcal{K}_{\mathcal{F}_1^{\vartheta}}(t)$ given by:

$$\mathcal{K}(\varphi)(t) := \left(1 + \mathcal{E}_{\beta,|\zeta|+\kappa}(b, t_0)\left(\|\zeta\|_\infty + \kappa\right)\right)\vartheta\varphi(t).$$

Thus, Equation 4 exhibits the desired stability.

Theorem 17 (Hyers–Ulam–Rassias Stability of Type $\mathcal{F}_p^{\vartheta}$). Let $p > 1$, and define q by $\frac{1}{p} + \frac{1}{q} = 1$. If assumptions $\mathcal{M}_1$–$\mathcal{M}_3$ hold, then Equation 4 exhibits Hyers–Ulam–Rassias stability of type $\mathcal{F}_p^{\vartheta}$.

More precisely, if $\mathscr{U} \in C^1(\mathbb{I}, \mathbb{U})$ satisfies the inequality (Equation 18) for some $\varphi \in \mathcal{F}_p^{\vartheta}$, then there exists a solution $\mathscr{U}_*$ of Equation 4 such that

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq \mathcal{K}(\varphi)(t), \quad \forall t \in \mathbb{I},$$

where the stability function $\mathcal{K}(\varphi)$ is given by:

$$\mathcal{K}(\varphi)(t) := \sqrt[p]{\vartheta}\sqrt[q]{b - t_0}\left(1 + \sqrt[q]{b - t_0} \cdot \mathcal{E}_{\beta,|\zeta|+\kappa}(b, t_0) \cdot \left(\|\zeta\|_\infty + \kappa\right)\right)\varphi(t). \tag{39}$$

Proof. Let $\varphi \in \mathcal{F}_p^{\vartheta}$, and assume that $\mathscr{U} \in C^1(\mathbb{I}, \mathbb{U})$ satisfies Equation 18. Define σ as in Equation 12. By applying the same steps as in the proof of Theorem 15, we obtain:

$$\begin{aligned}
\|\mathscr{U}(t) - \mathscr{U}_*(t)\| &\leq \int_{t_0}^{t}\left(|\zeta(s)| + \kappa\right)\|\mathscr{U}(s) - \mathscr{U}_*(s)\| d_\beta s \\
&\quad + \int_{t_0}^{t}\varphi(s) d_\beta s \\
&\leq \sqrt[q]{t - t_0}\sqrt[p]{\int_{t_0}^{t}\varphi^p(s) d_\beta s} + \int_{t_0}^{t}\left(|\zeta(s)| + \kappa\right)\|\mathscr{U}(s) \\
&\quad - \mathscr{U}_*(s)\| d_\beta s, \\
&\leq \sqrt[q]{b - t_0}\sqrt[p]{\vartheta\varphi^p(t)} + \int_{t_0}^{t}\left(|\zeta(s)| + \kappa\right)\|\mathscr{U}(s) - \mathscr{U}_*(s)\| d_\beta s, \\
&\leq \sqrt[q]{b - t_0}\sqrt[p]{\vartheta}\varphi(t) + \int_{t_0}^{t}\left(|\zeta(s)| + \kappa\right)\|\mathscr{U}(s) - \mathscr{U}_*(s)\| d_\beta s.
\end{aligned}$$

In the second step, we applied Hölder's inequality. Now, by applying Grönwall's inequality (Theorem 5), we obtain the following estimate for $t \in \mathbb{I}$:

$$\begin{aligned}
\|\mathscr{U}(t) - \mathscr{U}_*(t)\| &\leq \sqrt[q]{b - t_0}\sqrt[p]{\vartheta}\varphi(t) \\
&\quad + \int_{t_0}^{t}\mathcal{E}_{\beta,|\zeta|+\kappa}(t, \beta(s))\left(|\zeta(s)| + \kappa\right)\sqrt[p]{\vartheta}\sqrt[q]{b - t_0}\varphi(s) d_\beta s \\
&\leq \sqrt[q]{b - t_0}\sqrt[p]{\vartheta}\varphi(t) + \mathcal{E}_{\beta,|\zeta|+\kappa}(b)\left(\|\zeta\|_\infty + \kappa\right) \\
&\quad \sqrt[q]{b - t_0}\sqrt[p]{\vartheta}\int_{t_0}^{t}\varphi(s) d_\beta s \\
&\leq \sqrt[q]{b - t_0}\sqrt[p]{\vartheta}\varphi(t) + \mathcal{E}_{\beta,|\zeta|+\kappa}(b)\left(\|\zeta\|_\infty + \kappa\right) \\
&\quad \sqrt[p]{\vartheta}\sqrt[q]{b - t_0}\sqrt[q]{t - t_0}\sqrt[p]{\int_{t_0}^{t}\varphi^p(s) d_\beta s} \\
&\leq \sqrt[q]{b - t_0}\sqrt[p]{\vartheta}\varphi(t) + \mathcal{E}_{\beta,|\zeta|+\kappa}(b)\left(\|\zeta\|_\infty + \kappa\right) \\
&\quad \sqrt[p]{\vartheta}\sqrt[q]{b - t_0}\sqrt[q]{b - t_0}\sqrt[p]{\vartheta\varphi^p(t)} \\
&\leq \sqrt[q]{b - t_0}\sqrt[p]{\vartheta}\varphi(t) + \mathcal{E}_{\beta,|\zeta|+\kappa}(b)\left(\|\zeta\|_\infty + \kappa\right) \\
&\quad \sqrt[p]{\vartheta}\sqrt[q]{b - t_0}\sqrt[q]{b - t_0}\sqrt[p]{\vartheta}\varphi(t).
\end{aligned}$$

By simplifying the above expression, we arrive at the following bound:

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq \sqrt[p]{\vartheta}\sqrt[q]{b - t_0}\left(1 + \sqrt[q]{b - t_0} \cdot \mathcal{E}_{\beta,|\zeta|+\kappa}(b, t_0) \cdot \left(\|\zeta\|_\infty + \kappa\right)\right)\varphi(t).$$

Thus, we conclude that Equation 4 exhibits Hyers-Ulam-Rassias stability of type $\mathcal{F}_p^{\vartheta}$, with the corresponding stability function $\mathcal{K}(\varphi)(t)$ given by Equation 39.

Since condition $\mathcal{M}_4$ implies $\mathcal{M}_3$, we conclude that Theorems 11 and 14–17 hold when $\mathcal{M}_3$ is replaced by $\mathcal{M}_4$.

Definition 10 (Mittag-Leffler Stability). Let $\alpha \in (0, 1]$, and let $E_\alpha(z)$ denote the one-parameter Mittag-Leffler function defined by

$$E_\alpha(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k + 1)}.$$

We say that the quantum difference equation (Equation 4) is Mittag-Leffler stable of order α if there exist constants $C > 0$, $\lambda > 0$, such that for every $\epsilon > 0$, and for any function $\mathscr{U} \in C^1(\mathbb{I}, \mathbb{U})$ satisfying

$$\left\|\mathscr{D}_\beta\mathscr{U}(t) - \zeta(t)\mathscr{U}(t) - \mathcal{F}(t, \mathscr{U}(t), h(\mathscr{U}(t))) - f(t)\right\| \leq \epsilon, \quad \forall t \in \mathbb{I},$$

there exists a solution $\mathscr{U}_*$ of the exact equation such that

$$\|\mathscr{U}(t) - \mathscr{U}_*(t)\| \leq C\epsilon E_\alpha(\lambda(t - t_0)^\alpha), \quad \forall t \in \mathbb{I}.$$

Theorem 18. (Mittag–Leffler-Type Stability Under Discrete β-Integral). Let $\alpha \in (0, 1]$ and suppose that assumptions $\mathcal{M}_1$–$\mathcal{M}_3$ hold. Assume further that the backward shift operator β satisfies, for all $t \in \mathcal{I}$,

$$\beta^k(t) - \beta^{k+1}(t) = \frac{C_\alpha}{\Gamma(\alpha)}(\beta^k(t) - t_0)^{\alpha-1}, \quad k \in \mathbb{N},$$

for some constant $C_\alpha > 0$. Then Equation 4 exhibits Mittag–Leffler-type Hyers–Ulam stability. More precisely, if $\mathscr{U} \in C^1(\mathbb{I}, \mathbb{U})$

satisfies Equation 9, then there exists a unique solution $\mathscr{U}_*$ of the exact equation with the same initial value such that:

$$\|\mathscr{U}(t)-\mathscr{U}_*(t)\| \leq \epsilon(t-t_0)^{\alpha} E_{\alpha}\left(\lambda(t-t_0)^{\alpha}\right), \quad \forall t \in \mathbb{I},$$

where $\lambda = \Gamma(\alpha)(\|\zeta\|_{\infty}+\kappa)$, and $E_{\alpha}(\cdot)$ is the Mittag–Leffler function.

Proof. Let $\mathscr{U} \in C^1(\mathbb{I},\mathbb{U})$ satisfy the inequality (Equation 9)

$$\left\|\mathscr{D}_{\beta}\mathscr{U}(t)-\zeta(t)\mathscr{U}(t)-\mathcal{F}(t,\mathscr{U}(t),h(\mathscr{U}(t)))-f(t)\right\| \leq \epsilon.$$

so that $\|\sigma(t)\| \leq \epsilon$ for all $t \in \mathbb{I}$.

Let $\mathscr{U}_*$ be the exact solution of the unperturbed Equation 4 with the same initial value. Using the β-integral definition given in Equation 2, we have:

$$\int_{t_0}^{t} g(s)\, d_{\beta}s = \sum_{k=0}^{\infty}(\beta^k(t)-\beta^{k+1}(t))g(\beta^k(t)).$$

Using Theorem 7, the functions $\mathscr{U}$ and $\mathscr{U}_*$ satisfy the integral equations:

$$\begin{aligned}\mathscr{U}(t) = \mathscr{U}_0 + \sum_{k=0}^{\infty}(\beta^k(t)-\beta^{k+1}(t))\Big[&\zeta(\beta^k(t))\mathscr{U}(\beta^k(t)) \\ &+ \mathcal{F}(\beta^k(t),\mathscr{U}(\beta^k(t)),h(\mathscr{U}(\beta^k(t)))) + f(\beta^k(t)) + \sigma(\beta^k(t))\Big],\end{aligned}$$

$$\begin{aligned}\mathscr{U}_*(t) = \mathscr{U}_0 + \sum_{k=0}^{\infty}(\beta^k(t)-\beta^{k+1}(t))\Big[&\zeta(\beta^k(t))\mathscr{U}_*(\beta^k(t)) \\ &+ \mathcal{F}(\beta^k(t),\mathscr{U}_*(\beta^k(t)),h(\mathscr{U}_*(\beta^k(t)))) + f(\beta^k(t))\Big].\end{aligned}$$

Subtracting the second equation from the first and applying the triangle inequality yields:

$$\begin{aligned}\|\mathscr{U}(t)-\mathscr{U}_*(t)\| \leq \sum_{k=0}^{\infty}(\beta^k(t)-\beta^{k+1}(t))\Big[&\|\zeta(\beta^k(t))\|\cdot\|\mathscr{U}(\beta^k(t)) \\ &- \mathscr{U}_*(\beta^k(t))\| + \|\mathcal{F}(\beta^k(t),\mathscr{U}(\beta^k(t)),h(\mathscr{U}(\beta^k(t)))) \\ &- \mathcal{F}(\beta^k(t),\mathscr{U}_*(\beta^k(t)),h(\mathscr{U}_*(\beta^k(t))))\| + \|\sigma(\beta^k(t))\|\Big].\end{aligned}$$

Using the assumption $\mathcal{M}_2$ and setting $L = \|\zeta\|_{\infty}+\kappa$, we obtain:

$$\begin{aligned}\|\mathscr{U}(t)-\mathscr{U}_*(t)\| \leq \sum_{k=0}^{\infty}(\beta^k(t)-\beta^{k+1}(t))\Big[&\epsilon + L\cdot\|\mathscr{U}(\beta^k(t)) \\ &- \mathscr{U}_*(\beta^k(t))\|\Big].\end{aligned}$$

Now using the assumption:

$$\beta^k(t)-\beta^{k+1}(t) = \frac{C_{\alpha}}{\Gamma(\alpha)}(\beta^k(t)-t_0)^{\alpha-1},$$

we define:

$$y(t) := \|\mathscr{U}(t)-\mathscr{U}_*(t)\|,$$

and substitute into the inequality:

$$y(t) \leq \frac{C_{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{\infty}(\beta^k(t)-t_0)^{\alpha-1}\left[\epsilon + Ly(\beta^k(t))\right].$$

$$\begin{aligned}y(t) &\leq \frac{C_{\alpha}\epsilon}{\Gamma(\alpha)}\sum_{k=0}^{\infty}(\beta^k(t)-t_0)^{\alpha-1} \\ &\quad + \frac{C_{\alpha}L}{\Gamma(\alpha)}\sum_{k=0}^{\infty}(\beta^k(t)-t_0)^{\alpha-1}y(\beta^k(t)) \\ &= \epsilon\cdot\underbrace{\left(\frac{C_{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{\infty}(\beta^k(t)-t_0)^{\alpha-1}\right)}_{=:S_1(t)} \\ &\quad + L\cdot\underbrace{\left(\frac{C_{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{\infty}(\beta^k(t)-t_0)^{\alpha-1}y(\beta^k(t))\right)}_{=:S_2(t)}.\end{aligned}$$

We have

$$\begin{aligned}S_1(t) &= \frac{C_{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{\infty}(\beta^k(t)-t_0)^{\alpha-1} \\ &\leq \frac{C_{\alpha}}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha)}{C_{\alpha}}(t-t_0)^{\alpha} = (t-t_0)^{\alpha},\end{aligned}$$

since the series approximates the fractional power of the interval length from t_0 to t under the assumed kernel.

Moreover,

$$\begin{aligned}S_2(t) &= \frac{C_{\alpha}}{\Gamma(\alpha)}\sum_{k=0}^{\infty}(\beta^k(t)-t_0)^{\alpha-1}y(\beta^k(t)) \\ &= \sum_{k=0}^{\infty}\omega_k y(\beta^k(t)), \quad \text{where } \omega_k := \frac{C_{\alpha}}{\Gamma(\alpha)}(\beta^k(t)-t_0)^{\alpha-1}.\end{aligned}$$

Hence, the inequality becomes

$$y(t) \leq \epsilon(t-t_0)^{\alpha} + L\sum_{k=0}^{\infty}\omega_k y(\beta^k(t)).$$

This is a discrete fractional-type inequality with convolution kernel ω_k. Using the discrete fractional Grönwall inequality (Definition 4), we conclude that

$$y(t) \leq \epsilon(t-t_0)^{\alpha}E_{\alpha}\left(\lambda(t-t_0)^{\alpha}\right), \quad \lambda := \Gamma(\alpha)L.$$

5 Examples

In this section, we present illustrative examples that demonstrate the applicability of the obtained stability results under the framework of quantum difference equations.

Example 2. Let $f \in C\left([0,\frac{1}{5}],\mathbb{U}\right)$, where $C([0,\frac{1}{5}],\mathbb{U})$ denotes the space of continuous functions from the interval $[0,\frac{1}{5}]$ to the Banach space $\mathbb{U}$. Consider the quantum difference equation:

$$\mathscr{D}_{\beta}\mathscr{U}(t) = \zeta\mathscr{U}(t) + \left(\cos^3\mathscr{U}(t) + \frac{1}{1+|\mathscr{U}(t)|}\right) + f(t). \qquad (40)$$

Using the general form of Equation 4, we have the following parameters:

$$b = \frac{1}{5}, \quad t_0 = 0, \quad \mathcal{F}(t,x,y) = \cos^3 x + y,$$
$$h(x) = \frac{1}{1+|x|}, \quad \zeta(s) = \zeta \in \mathbb{R}.$$

We now apply the well-known properties of the cosine function and the function h to estimate the Lipschitz constant:

$$|\cos^3 x_1 - \cos^3 x_2| \le 3|x_1 - x_2|, \quad \left|\frac{1}{1+|x_1|} - \frac{1}{1+|x_2|}\right| \le |x_1 - x_2|.$$

Combining these results, we can bound the difference in $\mathcal{F}$:

$$|\mathcal{F}(t,x_1,h(x_1)) - \mathcal{F}(t,x_2,h(x_2))| \le 4|x_1 - x_2|.$$

Thus, we conclude that the Lipschitz constant $\kappa = 4$. Additionally, from the condition on $\zeta(s)$, we observe that $\theta = \frac{|\zeta|}{5}$. For the condition $\mathcal{M}_4$ to hold, we require $\zeta \in (-1,1)$. Therefore, Theorems 11 and 14–17 hold.

Example 3. Let $f \in C([0,1],\mathbb{U})$, and consider the quantum difference equation:

$$\mathscr{D}_{\frac{t}{3}}\mathscr{U}(t) = t\mathscr{U}(t) + \left(\arctan \mathscr{U}(t) + \sin^2(\mathscr{U}(t))\right) + f(t), \qquad (41)$$

Using the general form of Equation 4, we have the following parameters:

$$b \quad t = 1, \quad t_0 = 0, \beta(t) = \frac{t}{3}, \quad \mathcal{F}(t,x,y) = \arctan x + y,$$
$$h(x) = \sin^2(x) \quad \zeta(t) = t \in \mathbb{R}.$$

We now apply the well-known properties of the cosine function and the function h to estimate the Lipschitz constant:

$$|\arctan x_1 - \arctan x_2| \le |x_1 - x_2|, \quad \left|\sin^2(x_1) - \sin^2(x_2)\right| \le |x_1 - x_2|.$$

Combining these results, we can bound the difference in $\mathcal{F}$:

$$|\mathcal{F}(t,x_1,h(x_1)) - \mathcal{F}(t,x_2,h(x_2))| \le 2|x_1 - x_2|.$$

Thus, we conclude that the Lipschitz constant $\kappa = 2$. The constant $\theta = \sup_{t\in[0,1]} \int_{t_0}^{t} |\zeta(t)| d_\beta s = \sup_{t\in[0,1]} \frac{3t^2}{4} = \frac{3}{4}$ and hence, the condition $\mathcal{M}_4$ hold. Therefore, Theorems 11 and 14–17 hold under the condition.

Example 4. Consider the perturbed quantum difference equation:

$$\mathscr{D}_\beta \mathscr{U}(t) = \tfrac{1}{2}\mathscr{U}(t) + \left(\frac{\mathscr{U}(t)}{1+|\mathscr{U}(t)|} + \tanh(\mathscr{U}(t))\right) + \sin(t) + \sigma(t),$$

where the perturbation satisfies $|\sigma(t)| \le \epsilon$ for all $t \in \mathbb{I}$. Assume the β-integral satisfies

$$\beta^k(t) - \beta^{k+1}(t) = \frac{C_\alpha}{\Gamma(\alpha)}(\beta^k(t) - t_0)^{\alpha-1}, \quad \alpha \in (0,1).$$

The function $\mathcal{F}(t,x,h(x)) = \frac{x}{1+|x|} + h(x)$ is Lipschitz with constant $\kappa = 2$, and $\|\zeta\|_\infty = \frac{1}{2}$. Setting $L := \|\zeta\|_\infty + \kappa = \frac{5}{2}$, Theorem 18 applies and yields

$$|\mathscr{U}(t) - \mathscr{U}_*(t)| \le \epsilon(t-t_0)^\alpha E_\alpha\left(\Gamma(\alpha)L(t-t_0)^\alpha\right),$$

where $\mathscr{U}_*(t)$ is the exact solution of the unperturbed equation.

6 Conclusion

In this paper, we explored different kind of stability of first-order nonlinear quantum difference equations, offering a comprehensive analysis within the framework of quantum difference calculus. By examining the behavior of solutions to such equations under perturbations, we established the necessary conditions for stability and uniqueness. The results were derived using fixed-point theory and integral equations, offering both theoretical insights and practical applications in discrete mathematics and quantum calculus. Finally, the paper presented examples to highlight the relevance and application of these stability concepts, contributing to the broader understanding of stability in quantum difference equations. Furthermore, the methods and results derived here can be extended to higher-order nonlinear quantum difference equations, broadening the scope of applications in more complex quantum models and dynamic systems. Further studies could focus on extending these results to more complex quantum models and exploring additional boundary conditions.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

DA: Conceptualization, Formal analysis, Validation, Writing – original draft. CC: Investigation, Methodology, Writing – review & editing. SA: Conceptualization, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Methodology, Supervision, Validation.

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