



OPEN ACCESS

EDITED BY
Luis M. Nieto,
University of Valladolid, Spain

REVIEWED BY
Osman Teoman Turgut,
Boğaziçi University, Türkiye
Fabio Rinaldi,
Università degli Studi Guglielmo Marconi, Italy
Ahmad Qazza,
Zarqa University, Jordan

*CORRESPONDENCE
Yuriy Golovaty
✉ yuriy.golovaty@lnu.edu.ua

RECEIVED 21 April 2025
ACCEPTED 17 June 2025
PUBLISHED 10 July 2025

CITATION
Golovaty Y and Hryniv R (2025) On negative
eigenvalues of 1D Schrödinger operators with
 δ' -like potentials.
Front. Appl. Math. Stat. 11:1615447.
doi: 10.3389/fams.2025.1615447

COPYRIGHT
© 2025 Golovaty and Hryniv. This is an
open-access article distributed under the
terms of the [Creative Commons Attribution
License \(CC BY\)](https://creativecommons.org/licenses/by/4.0/). The use, distribution or
reproduction in other forums is permitted,
provided the original author(s) and the
copyright owner(s) are credited and that the
original publication in this journal is cited, in
accordance with accepted academic practice.
No use, distribution or reproduction is
permitted which does not comply with these
terms.

On negative eigenvalues of 1D Schrödinger operators with δ' -like potentials

Yuriy Golovaty^{1*} and Rostyslav Hryniv^{2,3}

¹Faculty of Mathematics and Mechanics, Ivan Franko National University of Lviv, Lviv, Ukraine, ²Faculty of Applied Sciences, Ukrainian Catholic University, Lviv, Ukraine, ³Faculty of Exact and Technical Sciences, University of Rzeszów, Rzeszów, Poland

In this paper, we investigate negative eigenvalues of exactly solvable quantum models, particularly one-dimensional Hamiltonians with δ' -like potentials used to represent localized dipoles. These operators arise as norm resolvent limits of Schrödinger operators with suitably regularized potentials. Although the limiting operator is bounded below, we show that the approximating operators may possess a finite but arbitrarily large number of negative eigenvalues that diverge to $-\infty$ as the regularization parameter vanishes. This phenomenon illustrates a spectral instability of Schrödinger operators with δ' -like singularities.

KEYWORDS

1D Schrödinger operator, point interaction, δ -potential, δ' -potential, exactly solvable model, discrete spectrum

1 Introduction

The aim of this study is to establish the existence and describe the asymptotic behavior of negative eigenvalues of one-dimensional Schrödinger operators that serve as regularizations of formal Hamiltonians involving δ and δ' potentials. These questions arise in the construction and analysis of exactly solvable models in quantum mechanics, a topic that continues to draw considerable attention in the literature (see Albeverio et al. [1] and Albeverio and Kurasov [2], as well as comprehensive reference lists therein, covering works up to the early 2000s).

Some point interactions (i.e., pseudopotentials supported on discrete sets) naturally lead to well-defined, exactly solvable models; others, however, exhibit essential ambiguities in defining the corresponding Hamiltonians. A notable example of this contrast is provided by the δ and δ' potentials. In the case of the δ potential, the differential equation $-y'' + \alpha\delta(x)y = \lambda y$ is well-posed in the space of distributions $\mathcal{D}'(\mathbb{R})$ and has a two-dimensional solution space. In contrast, the equation $-y'' + \alpha\delta'(x)y = \lambda y$ is ill-posed in $\mathcal{D}'(\mathbb{R})$ and admits only the trivial solution when $\alpha \neq 0$. Moreover, while every reasonable regularization of Hamiltonians involving the δ potential yields the same exactly solvable model, the δ' potential is sensitive to the regularization procedure, and different approximations may lead to different point interactions. As a result, the choice of the exactly solvable model for δ' potentials is not determined by mathematical considerations alone. However, it must reflect the specifics of the particular physical experiment—a feature that only enriches the study of exactly solvable models.

In this study, we demonstrate a further distinction between the δ and δ' potentials, this time concerning the spectral properties of their regularized Hamiltonians. Natural approximations of both the δ and δ' potentials by regular potentials yield operator families that converge in the norm resolvent topology to semi-bounded limits. However, we show that in contrast to δ -like perturbations, δ' -like perturbations lead to operator families that are not uniformly bounded from below as the regularization parameter tends to zero.

As a consequence, such regularized Hamiltonians can possess a finite (but arbitrarily large) number of low-lying eigenvalues that diverge to negative infinity. We explicitly determine the number of these eigenvalues and describe their asymptotic behavior in the singular limit.

The rest of the article is organized as follows. Section 2 gives a brief overview of studies on exactly solvable models for Hamiltonians with δ and δ' potentials. In Section 3, we derive conditions under which Schrödinger operators with δ' -like potentials possess low-lying eigenvalues that diverge to negative infinity as the regularization parameter tends to zero. Section 4 introduces methods for estimating the number of such eigenvalues and shows, in particular, that the emergence of a discrete spectrum is closely related to zero-energy resonances for the corresponding Schrödinger operators. In Section 5, we investigate how the non-trivial interaction of δ -like and δ' -like perturbations leads to the emergence of a negative eigenvalue with a finite limit as the perturbation parameter tends to zero. Finally, Section 6 contains the proofs of Theorems 3 and 4 on the asymptotic behavior of eigenvalues.

2 Short review of exactly solvable models for δ and δ' potentials

In this section, we review existing approaches to constructing exactly solvable quantum mechanical models for one-dimensional Hamiltonians with pseudopotentials involving the Dirac δ -function and its derivative δ' .

The simplest case is the formal (pseudo-)Hamiltonian

$$-\frac{d^2}{dx^2} + \alpha\delta(x), \quad \alpha \in \mathbb{R}. \quad (2.1)$$

Any reasonable method of associating a self-adjoint Hamiltonian to Equation 2.1—such as form-sum, generalized sum method, approximation by regular potentials—yields the same operator H , acting as $Hy = -y''$ on the domain

$$\text{dom } H = \{y \in W_2^2(\mathbb{R} \setminus \{0\}) : y(+0) = y(-0), \\ y'(+0) - y'(-0) = \alpha y(0)\}.$$

In other words, the distributional potential $\alpha\delta(x)$ in the one-dimensional Schrödinger operator results in the point interaction imposing the interface condition

$$\begin{pmatrix} y(+0) \\ y'(+0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} y(-0) \\ y'(-0) \end{pmatrix}. \quad (2.2)$$

Moreover, this model serves as a good approximation in the norm resolvent sense of the Schrödinger operators with integrable potentials of special form. Specifically, given a real-valued function U of compact support such that

$$\alpha = \int_{\mathbb{R}} U(x) dx,$$

the scaled potentials $U_\varepsilon(x) = \varepsilon^{-1}U(\varepsilon^{-1}x)$ converge in the space of distributions \mathcal{D}' to the distribution $\alpha\delta(x)$ as $\varepsilon \rightarrow 0$, and the corresponding operators

$$-\frac{d^2}{dx^2} + \varepsilon^{-1}U(\varepsilon^{-1}x)$$

converge to H in the norm resolvent sense [1, Theorem I.3.2.3], i.e., their resolvents converge in operator norm to the resolvent of H . Similar convergence results hold even in the presence of background potentials W , i.e., for operators of the form

$$-\frac{d^2}{dx^2} + W(x) + \alpha\delta(x).$$

One should not expect that every pseudopotential gives rise to a unique point interaction. Certain pseudopotentials are highly sensitive to the way they are approximated, and the δ' potential is one of them. In physics, the symbol δ' is often used to describe a strongly localized dipole-type potential, such as a high narrow barrier followed by a deep well. Let V be an integrable function with compact support and a finite first moment; then, the sequence $\varepsilon^{-2}V(\varepsilon^{-1}x)$ converges in \mathcal{D}' , as $\varepsilon \rightarrow 0$, if and only if $\int_{\mathbb{R}} V(x) dx = 0$, and in that case

$$\varepsilon^{-2}V(\varepsilon^{-1}x) \rightarrow \beta\delta'(x), \quad (2.3)$$

where $\beta = -\int_{\mathbb{R}} xV(x) dx$. For this reason, we refer to such families of scaled potentials as δ' -like.

The question of how to correctly define the formal Hamiltonian

$$-\frac{d^2}{dx^2} + \beta\delta'(x) \quad (2.4)$$

has a long and intricate history. As mentioned earlier, difficulties arise already at the level of interpreting the differential expression in Equation 2.4, since the equation $-y'' + \beta\delta'(x)y = \lambda y$ admits only the trivial solution in the space of distributions \mathcal{D}' . Indeed, the product $\delta'(x)\phi(x)$ is well defined in \mathcal{D}' only if ϕ is continuously differentiable, and in that case, it is equal to the distribution $\phi(0)\delta'(x) - \phi'(0)\delta(x)$. However, any non-trivial solution y of the above equation would have to be discontinuous at the origin since its second derivative $y'' = \beta\delta'(x)y - \lambda y$ would necessarily include a δ' term. In this case, the product $\delta'(x)y(x)$ is not defined in \mathcal{D}' , making the equation invalid.

Moreover, the operator in Equation 2.4 cannot be rigorously defined using standard approaches such as the form-sum or generalized sum methods, or as a relatively bounded perturbation of the free Hamiltonian. For this reason, it is natural to approach this problem via regularization: one considers families of Schrödinger operators of the form

$$H_\varepsilon = -\frac{d^2}{dx^2} + \varepsilon^{-2}V(\varepsilon^{-1}x), \quad \text{dom } H_\varepsilon = W_2^2(\mathbb{R}),$$

with δ' -like potentials $\varepsilon^{-2}V(\varepsilon^{-1}x)$ as a starting point for studying physical phenomena associated with zero-range dipoles. The construction of exactly solvable models for such dipole interactions is thus reduced to analyzing the limits of H_ε as $\varepsilon \rightarrow 0$.

It has been shown that the operator family H_ε indeed converges in the norm resolvent sense as $\varepsilon \rightarrow 0$. In a seminal paper, Šeba [3]

argued that the limiting operator is the free Hamiltonian D_0 decoupled at the origin by the Dirichlet condition, namely

$$D_0 y = -y'', \quad \text{dom } D_0 = \{y \in W_2^2(\mathbb{R} \setminus 0) : y(0) = 0\}. \quad (2.5)$$

According to this result, no meaningful definition of a Schrödinger operator with a δ' -potential would be possible since the limit D_0 is completely impenetrable to a quantum particle and is independent of the specific form of the function V . However, this conclusion contradicts the findings of Zolotaryuk et al. [4–7], who analyzed transmission probabilities through piecewise constant δ' -like potentials and observed examples of quantum tunneling. These results prompted the revision of Šeba [3]; it was later rigorously proved in Golovaty and Hryniv [8] that the operator D_0 is the norm resolvent limit of H_ε only in the so-called *non-resonant* case, while in the *resonant* case, the situation is different.

We begin by recalling the relevant definitions [9–11]. The operator $-\frac{d^2}{dx^2} + V$ is said to have a *zero-energy resonance* if the equation $v'' = Vv$ admits a non-trivial solution v that is bounded on the entire real line. Such a solution is called a *half-bound state*, and the potential V is then referred to as *resonant*. Every half-bound state v has finite, non-zero limits v_\pm at $\pm\infty$, and the ratio

$$\theta = \frac{v_+}{v_-}$$

is uniquely determined by V . As proved in [8] (see also Golovaty and Man'ko [12] and Golovaty et al. [13]), if the potential V is resonant, then the family H_ε converges in the norm resolvent sense as $\varepsilon \rightarrow 0$ to the self-adjoint operator

$$H(\theta) = -\frac{d^2}{dx^2}, \quad \text{dom } H(\theta) = \{y \in W_2^2(\mathbb{R} \setminus 0) : y(+0) = \theta y(-0), \theta y'(+0) = y'(-0)\}.$$

We call $H(\theta)$ the *Schrödinger operator with δ'_θ potential*, with θ specifying the above interface conditions.

Regardless of whether the family $\varepsilon^{-2}V(\varepsilon^{-1}x)$ converges in \mathcal{D}' as $\varepsilon \rightarrow 0$ or not, the Schrödinger operators H_ε converge in the norm resolvent sense to either $H(\theta)$ or D_0 depending on whether V is resonant or non-resonant. Moreover, there is no functional dependence between the constant β appearing in the distributional limit (Equation 2.3) and the interface parameter θ in the point interaction

$$\begin{pmatrix} y(+0) \\ y'(+0) \end{pmatrix} = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix} \begin{pmatrix} y(-0) \\ y'(-0) \end{pmatrix} \quad (2.6)$$

corresponding to $H(\theta)$. Two different resonant, zero-mean potentials V may produce the same β but different values of θ , and conversely, the same θ may arise for different β . It is worth noting that Kurasov [14, 15] was the first to establish a connection between the δ' -potential and the point interactions described by Equation 2.6.

In Golovaty [16, 17], it was proved that the approximations of pseudo-Hamiltonians $-\frac{d^2}{dx^2} + \alpha\delta(x) + \beta\delta'(x)$ by the Hamiltonians

$$-\frac{d^2}{dx^2} + \varepsilon^{-1}U(\varepsilon^{-1}x) + \varepsilon^{-2}V(\varepsilon^{-1}x)$$

with every regular functions U and V also converge in the norm resolvent topology. If V is non-resonant, the operators converge to D_0 . However, if V is resonant with a half-bound state v , then the limiting operator is associated with point interaction, producing the interface conditions

$$\begin{pmatrix} y(+0) \\ y'(+0) \end{pmatrix} = \begin{pmatrix} \theta & 0 \\ \eta & \theta^{-1} \end{pmatrix} \begin{pmatrix} y(-0) \\ y'(-0) \end{pmatrix}, \quad (2.7)$$

where

$$\theta = \frac{v_+}{v_-}, \quad \eta = \frac{1}{v_- v_+} \int_{\mathbb{R}} U v^2 dx. \quad (2.8)$$

A comprehensive study of exactly solvable models with point interactions (Equation 2.7) has been done by Gadella et al. [18–20]. Besides the approximation of pseudopotentials by regular potentials, there are other methods to construct exactly solvable models: e.g., the method of self-adjoint extensions has been used by Nizhnik [21, 22], and the distributional approach has been proposed by Lunardi et al. [23, 24].

We note that the point interactions characterized by the interface conditions

$$\begin{pmatrix} y(+0) \\ y'(+0) \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(-0) \\ y'(-0) \end{pmatrix},$$

commonly referred to as δ' -interactions, are also sometimes interpreted as models of the formal δ' potential. Exner, Neidhardt, and Zagrebnov [25] proposed a refined potential approximation of such interactions using a family of three δ -like potentials centered at the points $\pm a$ and 0, with the separation distance a tending to zero in a carefully coordinated way with the coupling constants. Further contributions in this direction include the works of Cheon and Shigehara [26], Zolotaryuk [27], and Albeverio et al. [28–30]; see also the recent publication [31]. Although the potential families in Exner et al. [25] do not converge to δ' in the sense of distributions, the term “ δ' -interactions” can be partially justified by interpreting δ' as a finite-rank perturbation; see Albeverio et al. [32] and Kuzhel and Nizhnik [33] for further discussion.

Let $\langle \cdot, \cdot \rangle$ be the dual pairing between the Sobolev spaces $W_2^{-s}(\mathbb{R})$ and $W_2^s(\mathbb{R})$. Since $\delta(x)y(x) = y(0)\delta(x) = \langle \delta, y \rangle \delta(x)$, the formal operator (Equation 2.1) can be written as

$$-\frac{d^2}{dx^2} + \alpha \langle \delta, \cdot \rangle \delta(x). \quad (2.9)$$

This shows that the δ -potential can be interpreted as a rank-one perturbation of the free Hamiltonian, and the standard theory of regular finite-rank perturbations yields the same exactly solvable model as in Equation 2.2. In the physical literature, the δ' -interaction is typically associated with rank-one perturbation of the free Hamiltonian as in Equation 2.9 but with δ' in place of δ [1, Ch. 1.4]:

$$-\frac{d^2}{dx^2} + \beta \langle \delta', \cdot \rangle \delta'(x). \quad (2.10)$$

The more general results of Albeverio and Nizhnik [34] and Albeverio et al. [32] imply that there exist regular potentials

$\phi_\varepsilon, \psi_\varepsilon \in L^2(\mathbb{R})$ converging to δ' in \mathcal{D}' such that the rank-one perturbations of the free Hamiltonian,

$$-\frac{d^2}{dx^2} + \beta \langle \phi_\varepsilon, \cdot \rangle \psi_\varepsilon(x),$$

converge to Equation 2.10 in the strong resolvent topology as $\varepsilon \rightarrow 0$. The difference, in terms of spectral effects, between the perturbation of the so-called 1D conic oscillator consisting of a mixed potential $\alpha\delta + \beta\delta'$ and the one with the δ' -interaction in Equation 2.10 was investigated in detail in Fassari et al. [35].

However, the model (Equation 2.10) is not directly related to the formal expression (Equation 2.4) with a δ' -potential. Indeed, if the product $\delta'(x)y(x)$ is well defined in the distributional sense, then

$$\delta'(x)y(x) = y(0)\delta'(x) - y'(0)\delta(x).$$

Using this identity, the formal expression (Equation 2.4) can be interpreted as a rank-two perturbation of the free Schrödinger operator:

$$-\frac{d^2}{dx^2} + \beta \langle \delta, \cdot \rangle \delta'(x) + \beta \langle \delta', \cdot \rangle \delta(x).$$

In Golovaty [36, 37], the norm resolvent convergence of the regular Hamiltonians

$$-\frac{d^2}{dx^2} + (g_\varepsilon, \cdot) f_\varepsilon + (f_\varepsilon, \cdot) g_\varepsilon + \varepsilon^{-1} U\left(\frac{x}{\varepsilon}\right).$$

was studied. Here, f_ε and g_ε are sequences of real- or complex-valued functions in $C_0^\infty(\mathbb{R})$ such that $f_\varepsilon \rightarrow \delta'$ and $g_\varepsilon \rightarrow \delta$ in the distributional sense, and (\cdot, \cdot) denotes the inner product in $L^2(\mathbb{R})$. Under suitable assumptions on f_ε , g_ε , and the potential U , such operators were shown to approximate the two-parameter family of point interactions defined by the interface conditions

$$\begin{pmatrix} y(+0) \\ y'(+0) \end{pmatrix} = \begin{pmatrix} \mu & \kappa \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} y(-0) \\ y'(-0) \end{pmatrix}.$$

Although there is no established theory of distributions on metric graphs, the notions of δ -like and δ' -like potentials can be naturally extended to this setting. The construction of exactly solvable models on quantum graphs, as well as the approximation of singular vertex couplings—including mixed $\alpha\delta' + \beta\delta$ interactions—has been explored in Cheon and Exner [38], Man'ko [39], Exner and Manko [40], and Golovaty [41].

The above results illustrate the richness of approaches to modeling point interactions and exactly solvable models in quantum mechanics. While δ -potentials admit a canonical interpretation, the situation becomes especially delicate when the formal δ' -potential is involved, as different approximations may lead to different exactly solvable models. The choice of the appropriate limit operator is, therefore, not unique and must be guided by the physical or mathematical context of the problem.

3 Existence of low-lying eigenvalues for δ' -like potentials

Let us consider the family of operators

$$\mathcal{H}_\varepsilon = -\frac{d^2}{dx^2} + W(x) + \varepsilon^{-1}U(\varepsilon^{-1}x) + \varepsilon^{-2}V(\varepsilon^{-1}x), \quad (3.1)$$

with the domain $W_2^2(\mathbb{R})$, where U , V , and W are compactly supported $L^\infty(\mathbb{R})$ -potentials. This restriction on the potentials avoids unnecessary technical complications; however, the results remain valid for a significantly broader class of potentials (cf. [13] for an example of how this constraint can be relaxed). We are interested in the emergence of negative eigenvalues in Hamiltonians due to δ -like and δ' -like perturbations. Accordingly, we assume that $W \geq 0$, so that the unperturbed operator $H_0 = -\frac{d^2}{dx^2} + W$ is non-negative and has a purely continuous spectrum.

As follows from the result of Golovaty [16], the operators \mathcal{H}_ε converge in the norm resolvent sense as $\varepsilon \rightarrow 0$. If the potential V is resonant, i.e., possesses a half-bound state v (see Section 2), then \mathcal{H}_ε converge to the operator

$$\begin{aligned} \mathcal{H} &= -\frac{d^2}{dx^2} + W, \quad \text{dom } \mathcal{H} = \{\phi \in W_2^2(\mathbb{R} \setminus \{0\}) : \\ &\phi(+0) = \theta\phi(-0), \quad \phi'(+0) = \theta^{-1}\phi'(-0) + \eta\phi(-0)\}, \end{aligned} \quad (3.2)$$

where θ and η are given by Equation 2.8. In the non-resonant case, the family converges to the operator $D_0 = -\frac{d^2}{dx^2} + W$ subject to the Dirichlet boundary condition at the origin as in Equation 2.5. Both \mathcal{H} and D_0 can be interpreted as perturbations of the operator H_0 by point interactions at the origin.

If the potential V is zero, then the family \mathcal{H}_ε is uniformly bounded from below as $\varepsilon \rightarrow 0$. This follows from the fact that the δ -like perturbation is form-bounded relative H_0 , with relative bound $a < 1$, so that there is a $b > 0$ such that

$$\forall \phi \in W_2^2(\mathbb{R}) : \quad \varepsilon^{-1}(U(\varepsilon^{-1}\cdot)\phi, \phi) \leq a(H_0\phi, \phi) + b(\phi, \phi).$$

In this case, as we show below, the operators \mathcal{H}_ε may have at most one eigenvalue, and this eigenvalue converges to a finite limit as $\varepsilon \rightarrow 0$.

In contrast, when V is not identically zero, although the limiting operators \mathcal{H} and D_0 are semibounded from below, the family \mathcal{H}_ε is generally not uniformly bounded from below. Thus, the family \mathcal{H}_ε may exhibit eigenvalues that diverge to $-\infty$ as $\varepsilon \rightarrow 0$; we refer to such eigenvalues as *low-lying eigenvalues*.

The following result characterizes precisely when such eigenvalues may occur.

Theorem 1. *Let \mathcal{H}_ε be the family of operators defined by Equation 3.1. Then, the operators \mathcal{H}_ε admit low-lying eigenvalues as $\varepsilon \rightarrow 0$ if and only if the potential V is not identically zero and*

$$\int_{\mathbb{R}} V dx \leq 0.$$

Moreover, the number of such eigenvalues is finite.

Proof. Assume that the potential V is not identically zero and that $\int_{\mathbb{R}} V dx \leq 0$. By [42, Th. XIII.110], the operator $H = -\frac{d^2}{dx^2} + V$ has then at least one negative eigenvalue $-\omega^2$, and we let u be a corresponding normalized eigenfunction. Denote by $W_\varepsilon(x) = W(x) + \varepsilon^{-1}U(\varepsilon^{-1}x) + \varepsilon^{-2}V(\varepsilon^{-1}x)$ the perturbed potential in Equation 3.1 and introduce the quadratic form

$$a_\varepsilon[\phi] = \int_{\mathbb{R}} (|\phi'(x)|^2 + W_\varepsilon(x)|\phi(x)|^2) dx.$$

Then, the scaled function $u_\varepsilon(x) = \varepsilon^{-1/2}u(\frac{x}{\varepsilon})$ belongs to $L^2(\mathbb{R})$ and has norm one. A direct computation shows that

$$\begin{aligned} \varepsilon^2 a_\varepsilon[u_\varepsilon] &= \int_{\mathbb{R}} (|u'(t)|^2 + V(t)|u(t)|^2) dt + \varepsilon \int_{\mathbb{R}} U(\frac{x}{\varepsilon}) |u_\varepsilon(x)|^2 dx \\ &\quad + \varepsilon^2 \int_{\mathbb{R}} W(x)|u_\varepsilon(x)|^2 dx = -\omega^2(1 + O(\varepsilon)), \end{aligned}$$

as $\varepsilon \rightarrow 0$. Therefore, for all sufficiently small ε , one has $a_\varepsilon[u_\varepsilon] \leq -\frac{1}{2}\omega^2\varepsilon^{-2}$, and we conclude from the minimax principle that there exists an eigenvalue λ_ε of \mathcal{H}_ε such that $\lambda_\varepsilon \leq -\frac{1}{2}\omega^2\varepsilon^{-2}$.

Assume now that $\int_{\mathbb{R}} V dx > 0$. Then, for sufficiently small $\varepsilon > 0$, the integral

$$\int_{\mathbb{R}} W_\varepsilon(x) dx = \varepsilon^{-1} \int_{\mathbb{R}} V(x) dx + \int_{\mathbb{R}} U(x) dx + \int_{\mathbb{R}} W(x) dx$$

is positive, and again by [42, Th. XIII.110] the operator \mathcal{H}_ε has no negative eigenvalues.

Let N_ε be the number of negative eigenvalues of \mathcal{H}_ε . It is known (see, e.g., [43, Th. 5.3], [44, Th. 7.5], [9, 45]) that the inequality

$$N_\varepsilon \leq 1 + \int_{\mathbb{R}} |x| |W_\varepsilon^-(x)| dx$$

holds, where $f^- = \min\{f, 0\}$ is the negative part of a function f . In view of the assumption $W \geq 0$, the negative part W_ε^- comes only from the V and U terms, and we estimate the integral above as follows:

$$\begin{aligned} \int_{\mathbb{R}} |x| |W_\varepsilon^-(x)| dx &= \varepsilon^{-2} \int_{\mathbb{R}} |x| |V^-(\frac{x}{\varepsilon})| dx + \varepsilon \int_{\mathbb{R}} |t| |U^-(t)| dt \\ &\leq \int_{\mathbb{R}} |t| |V^-(t)| dt + \varepsilon \int_{\mathbb{R}} |t| |U^-(t)| dt. \end{aligned}$$

The right-hand side remains bounded uniformly in small ε , and thus N_ε is bounded as $\varepsilon \rightarrow 0$, which completes the proof. \square

Relatively (form-) bounded symmetric perturbations preserve semi-boundedness of the perturbed operator; see [46, Th. IV.4.11, Th. VI.1.38]. However, even if a family of self-adjoint operators A_ε converges in the norm resolvent sense as $\varepsilon \rightarrow 0$ to a self-adjoint operator A that is bounded from below, the family A_ε may fail to be uniformly bounded from below. Even if each operator A_ε is individually semi-bounded, its lower bound may diverge to $-\infty$ as $\varepsilon \rightarrow 0$. A classic example due to Rellich [46, Ex. IV.4.14] gives such an operator family A_ε with a single eigenvalue tending to $-\infty$. The family of operators \mathcal{H}_ε with δ' -like perturbations provides a much stronger illustration of this effect. While \mathcal{H}_ε converges in the norm resolvent topology to a self-adjoint operator that is bounded from below, the number of eigenvalues that diverge to $-\infty$ as $\varepsilon \rightarrow 0$ can be arbitrary but finite. In the next section, we describe the procedure for counting these low-lying eigenvalues.

4 Counting the number of low-lying eigenvalues

Let us consider the Schrödinger operators

$$T_\alpha = -\frac{d^2}{dx^2} + \alpha V(x), \quad \text{dom } T_\alpha = W_2^2(\mathbb{R}), \quad (4.1)$$

with a real coupling constant α . We denote by $\mathcal{R}(V)$ the set of all values of α for which the potential αV is resonant. For each non-zero function $V \in L^\infty(\mathbb{R})$ with compact support, the set $\mathcal{R}(V)$ is a countable subset of \mathbb{R} with accumulation points at $+\infty$ and/or $-\infty$ [47].

We now recall the following definition [9]. Let A and B be self-adjoint operators, with B relatively A -compact. Suppose that (a, b) is a spectral gap of A and that $b \in \sigma_{\text{ess}}(A)$. If there exists an eigenvalue e_α of the perturbed operator $A + \alpha B$ in the gap (a, b) for all $\alpha > 0$, and if $e_\alpha \rightarrow b - 0$ as $\alpha \rightarrow 0$, then $\alpha = 0$, which is called a *coupling constant threshold*. Klaus [9] established a connection between resonant potentials and such coupling constant thresholds. Both phenomena are closely related to the emergence of negative eigenvalues in Schrödinger operators.

Suppose that a Schrödinger operator $-\frac{d^2}{dx^2} + \mathcal{V}$ has a zero-energy resonance with a corresponding half-bound state v . According to Klaus [9, Th. 3.2], if \mathcal{U} is a real-valued potential such that

$$\int_{\mathbb{R}} \mathcal{U} v^2 dx < 0,$$

then the perturbed operator $H_\kappa = -\frac{d^2}{dx^2} + \mathcal{V} + \kappa \mathcal{U}$, $\kappa > 0$, has a coupling constant threshold at $\kappa = 0$ and possesses a unique threshold eigenvalue λ_κ obeying the asymptotics $\lambda_\kappa = -a^2 \kappa^2 + O(\kappa^3)$, as $\kappa \rightarrow 0$, where the coefficient a is given by

$$a = \frac{1}{v_-^2 + v_+^2} \int_{\mathbb{R}} \mathcal{U} v^2 dx. \quad (4.2)$$

By reversing the direction of κ , we conclude that as κ increases from zero, the operator H_κ acquires a negative eigenvalue that detaches from the bottom of the continuous spectrum.

Without loss of generality, we may assume that the support of the potential V is contained in the interval $(-1, 1)$. We now consider the spectral *Regge problem* with spectral parameter ω [48]:

$$\begin{aligned} -\frac{d^2 u}{dx^2} + (V(x) + \omega^2)u &= 0, & x \in (-1, 1), \\ \frac{du}{dx}(-1) - \omega u(-1) &= 0, & \frac{du}{dx}(1) + \omega u(1) = 0. \end{aligned} \quad (4.3)$$

A complex number ω is called an eigenvalue of the Regge problem if there exists a non-trivial solution u of Equation 4.3, in which case u is a corresponding eigenfunction.

Theorem 2. *The number of low-lying eigenvalues of \mathcal{H}_ε is equal to each of the following:*

- (i) the number of negative eigenvalues of the operator $T_1 = -\frac{d^2}{dx^2} + V(x)$;
- (ii) the number of points in the set $\mathcal{R}(V)$ belonging to the interval $(0, 1)$;

(iii) the number of positive eigenvalues ω of the Regge problem (Equation 4.3).

Proof. (i) Consider the family of Schrödinger operators

$$\mathcal{S}_\varepsilon = -\frac{d^2}{dx^2} + V(x) + \varepsilon U(x) + \varepsilon^2 W(\varepsilon x)$$

with domain $W_2^2(\mathbb{R})$. This family is uniformly bounded from below and converges to the operator T_1 in the norm-resolvent sense as $\varepsilon \rightarrow 0$. Suppose that T_1 has n eigenvalues $-\omega_1^2, \dots, -\omega_n^2$. Then, for sufficiently small ε , the operators \mathcal{S}_ε have exactly n eigenvalues $-\omega_{1,\varepsilon}^2, \dots, -\omega_{n,\varepsilon}^2$ such that $\omega_{j,\varepsilon} \rightarrow \omega_j$. Since \mathcal{H}_ε is unitarily equivalent to $\varepsilon^{-2}\mathcal{S}_\varepsilon$, it follows that \mathcal{H}_ε has exactly n negative eigenvalues $-\varepsilon^{-2}\omega_{1,\varepsilon}^2, \dots, -\varepsilon^{-2}\omega_{n,\varepsilon}^2$, each diverging to $-\infty$ as $\varepsilon \rightarrow 0$.

(ii) \Leftrightarrow (i) The operator $T_0 = -\frac{d^2}{dx^2}$ has no eigenvalues. Suppose that the set $\mathcal{R}(V) \cap (0, 1)$ is non-empty, and let α_1 be its smallest element. We write

$$T_\alpha = -\frac{d^2}{dx^2} + \alpha_1 V(x) + (\alpha - \alpha_1)V(x),$$

and let v_1 be a half-bound state corresponding to the resonant potential $\alpha_1 V$. Then,

$$\int_{\mathbb{R}} V v_1^2 dx = -\frac{1}{\alpha_1} \int_{\mathbb{R}} v_1'^2 dx < 0,$$

and the operator T_α has an eigenvalue

$$\lambda_\alpha = -a_1^2(\alpha - \alpha_1)^2 + O((\alpha - \alpha_1)^3)$$

for $\alpha > \alpha_1$, where a_1 is given by Equation 4.2 with $v = v_1$ and $\mathcal{U} = V$.

As the parameter α increases, it may pass through further points in $\mathcal{R}(V) \cap (0, 1)$, and at each such crossing, the operator T_α acquires a new simple eigenvalue. Since no negative eigenvalue can get absorbed by the continuous spectrum as α increases [9], this gives a total count of the negative eigenvalues of T_1 .

(iii) \Leftrightarrow (i) Suppose $\omega > 0$ is an eigenvalue of the Regge problem with the corresponding eigenfunction u . Then, $-\omega^2$ is an eigenvalue of the operator T_1 , with the corresponding eigenfunction

$$\psi(x) = \begin{cases} u(-1)e^{\omega(x+1)}, & \text{if } x < -1, \\ u(x), & \text{if } |x| \leq 1, \\ u(1)e^{-\omega(x-1)}, & \text{if } x > 1. \end{cases} \quad (4.4)$$

Conversely, if ψ is an eigenfunction of T_1 corresponding to eigenvalue $-\omega^2$, then, since $\text{supp } V \subset (-1, 1)$, we have $\psi(x) = a_- e^{\omega x}$ for $x \leq -1$ and $\psi(x) = a_+ e^{-\omega x}$ for $x \geq 1$. This implies that ψ satisfies the boundary conditions in Equation 4.3, and its restriction to $(-1, 1)$ is an eigenfunction of the Regge problem (Equation 4.3) with eigenvalue $\omega > 0$. \square

Theorem 2 is of practical importance because solving the Regge problem on a finite interval or computing the resonance set $\mathcal{R}(V)$ is typically much easier than directly counting the eigenvalues of a Schrödinger operator on the real line. Another useful observation is

that by replacing V with cV for sufficiently large $c > 0$, we can make the number of negative eigenvalues of T_1 —and hence the number of low-lying eigenvalues of \mathcal{H}_ε —arbitrarily large.

The following theorem describes the two-term asymptotic expansion of the low-lying eigenvalues, which are constructed and justified in Section 6.

Theorem 3. Assume that the Schrödinger operator $T_1 = -\frac{d^2}{dx^2} + V$ has n eigenvalues $-\omega_1^2 < -\omega_2^2 < \dots < -\omega_n^2 < 0$ with eigenfunctions v_1, v_2, \dots, v_n . Then, the operator family \mathcal{H}_ε has n low-lying eigenvalues $\lambda_1^\varepsilon < \lambda_2^\varepsilon < \dots < \lambda_n^\varepsilon$ with asymptotics

$$\lambda_k^\varepsilon = -\varepsilon^{-2} \left(\omega_k + \varepsilon \frac{\int_{\mathbb{R}} U |v_k|^2 dx}{2\omega_k \|v_k\|^2} \right)^2 + O(1), \quad \text{as } \varepsilon \rightarrow +0. \quad (4.5)$$

The corresponding eigenfunctions $v_{k,\varepsilon}$ converge to zero in the weak topology.

We mention that one of the reasons why low-lying eigenvalues do not obstruct the norm resolvent convergence of \mathcal{H}_ε is that the corresponding eigenfunctions converge weakly to zero in $L^2(\mathbb{R})$.

5 Negative eigenvalues generated by δ -like potentials

As shown in the previous section, the emergence of low-lying eigenvalues is caused by a δ' -like perturbation, and the number of these eigenvalues is determined by the profile V of the approximating δ' -like potential. However, negative eigenvalues may also arise from δ -like perturbations, whether or not a δ' -like component is present in the operators \mathcal{H}_ε . In such cases, at most one negative eigenvalue may appear, and it always has a finite limit as $\varepsilon \rightarrow 0$.

The Schrödinger operator

$$S_\alpha = -\frac{d^2}{dx^2} + W(x) + \alpha \delta(x)$$

with a δ -potential of intensity $\alpha \in \mathbb{R}$ acts by $S_\alpha y = -y'' + W y$ on its natural domain

$$\text{dom } S_\alpha = \{y \in W_2^2(\mathbb{R} \setminus 0) : y(+0) = y(-0), y'(+0) - y'(-0) = \alpha y(0)\}.$$

So defined S_α is self-adjoint and has an absolutely continuous spectrum filling the positive half-line \mathbb{R}_+ , while its negative spectrum consists of at most one eigenvalue. We recall that the unperturbed operator $S_0 = -\frac{d^2}{dx^2} + W(x)$ is non-negative.

Lemma 1. Assume $W \in L^\infty(\mathbb{R})$ is a non-negative function of compact support. Then, there exists $\alpha_0 \in (-\infty, 0)$ such that, for all $\alpha < \alpha_0$, the operator S_α has exactly one negative eigenvalue.

Proof. If $W = 0$, then the operator S_α is non-negative for $\alpha \geq 0$, while for $\alpha < 0$, it has a unique eigenvalue $\lambda = -\frac{\alpha^2}{4}$ with the normalized eigenfunction

$$\psi_\alpha(x) = \sqrt{\frac{|\alpha|}{2}} e^{\frac{\alpha|x|}{2}},$$

see Albeverio et al. [1, Th.3.1.4]. For a generic W , we take an $\alpha < 0$ and find that

$$\begin{aligned}(S_\alpha \psi_\alpha, \psi_\alpha) &= (-\psi_\alpha'', \psi_\alpha) + (W\psi_\alpha, \psi_\alpha) \\ &= -\frac{\alpha^2}{4} + \frac{|\alpha|}{2} \int_{\mathbb{R}} W(x) e^{\alpha|x|} dx.\end{aligned}$$

Since $\int_{\mathbb{R}} W(x) e^{\alpha|x|} dx \rightarrow 0$ as $\alpha \rightarrow -\infty$ by the Lebesgue dominated convergence theorem, we conclude that the value

$$(S_\alpha \psi_\alpha, \psi_\alpha) = \frac{|\alpha|}{4} \left(\alpha + 2 \int_{\mathbb{R}} W(x) e^{\alpha|x|} dx \right)$$

becomes negative for negative α of large enough absolute value. As a result, for such α , the operator S_α has a negative eigenvalue. This eigenvalue is unique because S_α is a rank-one perturbation of the non-negative operator S_0 . \square

Lemma 1 remains valid for positive potentials W such that

$$\int_{\mathbb{R}} W(x) e^{\alpha|x|} dx < \infty, \quad \text{for all } \alpha < 0;$$

for example, for potentials with polynomial growth at infinity. Moreover, the above arguments suggest an explicit way to construct α_0 . The function

$$f(\alpha) = \alpha + 2 \int_{\mathbb{R}} W(x) e^{\alpha|x|} dx \quad (5.1)$$

is monotonically increasing in $\alpha \in (-\infty, 0]$, $f(0) > 0$ and f becomes negative as $\alpha \rightarrow -\infty$. Thus, f has a unique non-positive zero α_0 . Since $(S_\alpha \psi_\alpha, \psi_\alpha) = \frac{|\alpha|}{4} f(\alpha)$, we conclude that the operator S_α has a unique negative eigenvalue for all $\alpha < \alpha_0$.

Example 1. Consider the family of operators

$$S_\alpha = -\frac{d^2}{dx^2} + b^2(1 + \sin x) + \alpha\delta(x)$$

Since $\int_{\mathbb{R}} (1 + \sin x) e^{\alpha|x|} dx = -\frac{2}{\alpha}$ for $\alpha < 0$, the zero of f in Equation 5.1 satisfies $\alpha^2 = 4b^2$. Hence, S_α has a unique negative eigenvalue for all $\alpha < -2|b|$.

Example 2 (Cf. [49, 50]). Let S_α be the harmonic oscillator perturbed by the δ potential:

$$S_\alpha = -\frac{d^2}{dx^2} + kx^2 + \alpha\delta(x), \quad k > 0.$$

In this case, the zero of f is a negative root of $\alpha^4 = 8k$, since

$$\int_{\mathbb{R}} x^2 e^{\alpha|x|} dx = -\frac{4}{\alpha^3}, \quad \alpha < 0.$$

Therefore, the operator S_α has a unique negative eigenvalue for all $\alpha < -2^{3/4}k^{1/4}$.

When $V = 0$, the operators

$$\mathcal{H}_\varepsilon = -\frac{d^2}{dx^2} + W(x) + \varepsilon^{-1}U(\varepsilon^{-1}x), \quad (5.2)$$

are uniformly bounded from below and converge in the norm resolvent sense to S_α with $\alpha = \int_{\mathbb{R}} U dx$. This convergence, in particular, implies the convergence of negative eigenvalues; our next objective is to obtain a more precise asymptotic formula (proved in Section 6).

Theorem 4. Suppose that W and U are $L^\infty(\mathbb{R})$ -functions of compact support, and that W is non-negative. If $\int_{\mathbb{R}} U dx < \alpha_0$, where the threshold value α_0 is the root of Equation 5.1, then the operator \mathcal{H}_ε of Equation 5.2 has a unique negative eigenvalue λ_ε satisfying the asymptotics

$$\begin{aligned}\lambda_\varepsilon &= \lambda + \varepsilon \left(\frac{1}{2} \psi(0)^2 \iint_{\mathbb{R}^2} U(t)|t - \tau|U(\tau) d\tau dt \right. \\ &\quad \left. + \psi(0)(\psi'(-0) + \psi'(0)) \int_{\mathbb{R}} tU(t) dt \right) + O(\varepsilon^2), \quad \varepsilon \rightarrow 0.\end{aligned} \quad (5.3)$$

Here, ψ is a real-valued, $L^2(\mathbb{R})$ -normalized eigenfunction of S_α , with $\alpha = \int_{\mathbb{R}} U dx$, corresponds to the unique negative eigenvalue λ .

Moreover, the normalized eigenfunctions ψ_ε of S_ε can be chosen in such a way that $\psi_\varepsilon \rightarrow \psi$ in $L^2(\mathbb{R})$.

If the potential W is even, then the ground state λ_ε has asymptotics

$$\lambda_\varepsilon = \lambda + \frac{1}{2} \varepsilon \psi(0)^2 \iint_{\mathbb{R}^2} U(t)|t - \tau|U(\tau) d\tau dt + O(\varepsilon^2),$$

since the eigenfunction ψ is also even and therefore $\psi'(-0) + \psi'(0) = 0$. If $W = 0$ and $\int_{\mathbb{R}} U dx < 0$, the asymptotic formula (Equation 5.3) becomes

$$\lambda_\varepsilon = -\frac{1}{4} \left(\int_{\mathbb{R}} U(t) dt + \frac{1}{2} \varepsilon \iint_{\mathbb{R}^2} U(t)|t - \tau|U(\tau) d\tau dt \right)^2 + O(\varepsilon^2)$$

and coincides with the Abarbanel–Callan–Goldberger formula up to the factor ε^2 . The formula arises when studying the weakly coupled Hamiltonians $-\frac{d^2}{dx^2} + \gamma U$, their negative eigenvalues, and the absorption of such eigenvalues, as $\gamma \rightarrow 0$, by a continuous spectrum [51].

Now, suppose that the potential V is non-zero. If V is non-resonant, then the behavior of the negative spectrum of \mathcal{H}_ε is described by Theorem 3. However, if the shape V of the δ' -perturbation is resonant, then under certain conditions on the δ -perturbation, the operator \mathcal{H}_ε may have—in addition to low-lying eigenvalues—an extra eigenvalue that has a finite limit as $\varepsilon \rightarrow 0$. We recall that in the resonant case, the norm resolvent limit of \mathcal{H}_ε as $\varepsilon \rightarrow 0$ is the operator \mathcal{H} given by Equation 3.2, with constants θ and η determined by V and U via Equation 2.8.

Lemma 2. Let W be a non-negative function in $L^\infty(\mathbb{R})$ of compact support. If $\eta\theta < 0$ and the condition

$$\int_0^{+\infty} (W(-x) + \theta^2 W(x)) dx < \frac{|\eta\theta|}{2} \quad (5.4)$$

holds, then the operator \mathcal{H} defined by Equation 3.2 has a unique negative eigenvalue.

Proof. Assume first that $W = 0$. Integration by parts, on account of the interface conditions, yields

$$(\mathcal{H}y, y) = \|y'\|^2 + \theta\eta|y(-0)|^2,$$

thus for $\theta\eta \geq 0$, the operator \mathcal{H} is non-negative. In contrast, if $\theta\eta < 0$, then \mathcal{H} has a unique eigenvalue

$$\lambda = -\frac{\eta^2\theta^2}{(\theta^2 + 1)^2} \quad (5.5)$$

with the normalized eigenfunction

$$\Psi(x) = \frac{\sqrt{2|\eta\theta|}}{\theta^2 + 1} \cdot \begin{cases} e^{-\frac{\eta\theta}{\theta^2 + 1}x} & \text{for } x < 0, \\ \theta e^{\frac{\eta\theta}{\theta^2 + 1}x} & \text{for } x > 0, \end{cases}$$

as can be verified by straightforward calculations.

Now consider the case of arbitrary $W \in L^\infty(\mathbb{R})$, and let the constants θ and η from Equation 2.8 satisfy $\theta\eta < 0$. Using the function Ψ defined above, we find that

$$(\mathcal{H}\Psi, \Psi) = (W\Psi, \Psi) - \frac{\eta^2\theta^2}{(\theta^2 + 1)^2}.$$

Therefore, \mathcal{H} has a negative eigenvalue if

$$(W\Psi, \Psi) < \frac{\eta^2\theta^2}{(\theta^2 + 1)^2}.$$

This inequality is equivalent to

$$\int_0^{+\infty} (W(-x) + \theta^2 W(x)) e^{-\frac{2|\eta\theta|}{\theta^2 + 1}x} dx < \frac{|\eta\theta|}{2},$$

which is guaranteed under condition (Equation 5.4). \square

Theorem 5. Assume that V is resonant with a half-bound state v , and that the potentials W and U satisfy the conditions $W \geq 0$ and

$$\int_0^{+\infty} (v_-^2 W(-x) + v_+^2 W(x)) dx < -\frac{1}{2} \int_{\mathbb{R}} Uv^2 dx. \quad (5.6)$$

Then, for ε small enough, the operator \mathcal{H}_ε has a negative eigenvalue λ_ε converging, as $\varepsilon \rightarrow 0$, to the negative eigenvalue of the operator \mathcal{H} defined by Equation 3.2, where the parameters θ and η are given by Equation 2.8.

If $W = 0$ and $\int_{\mathbb{R}} Uv^2 dx < 0$, then this eigenvalue λ_ε has asymptotics

$$\lambda_\varepsilon = -\frac{1}{(v_-^2 + v_+^2)^2} \left(\int_{\mathbb{R}} Uv^2 dx \right)^2 + O(\varepsilon), \quad \varepsilon \rightarrow 0. \quad (5.7)$$

Proof. Inequality (Equation 5.6) and asymptotic formula (Equation 5.7) are equivalent forms of Equations 5.4, 5.5 when evaluated for the specific values θ and η . In addition, inequality (Equation 5.6) ensures that $\eta\theta < 0$. Indeed, it implies that $\int_{\mathbb{R}} Uv^2 dx < 0$, and since

$$\eta\theta = \frac{1}{v_-^2} \int_{\mathbb{R}} Uv^2 dx,$$

we conclude that $\eta\theta < 0$. The convergence $\mathcal{H}_\varepsilon \rightarrow \mathcal{H}$ in the norm resolvent sense as $\varepsilon \rightarrow 0$ then guarantees that \mathcal{H}_ε has a negative eigenvalue λ_ε approaching the unique negative eigenvalue of \mathcal{H} . \square

6 Asymptotic expansions of eigenvalues

In this section, we derive asymptotic formulas (Equations 4.5, 5.3) by constructing and justifying formal asymptotic expansions of the eigenvalues. For the sake of definiteness, we assume that the supports of U and V are contained in $(-1, 1)$.

6.1 Formal asymptotics

We start with asymptotics (Equation 5.3). The equation

$$-\frac{d^2 y_\varepsilon}{dx^2} + (W(x) + \varepsilon^{-1} U(\varepsilon^{-1}x)) y_\varepsilon = \lambda_\varepsilon y_\varepsilon$$

on $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$ reads

$$-\frac{d^2 y_\varepsilon}{dx^2} + W(x)y_\varepsilon = \lambda_\varepsilon y_\varepsilon, \quad (6.1)$$

while after rescaling $(-\varepsilon, \varepsilon)$ to $(-1, 1)$ and introducing $w_\varepsilon(t) = y_\varepsilon(\varepsilon t)$, one gets

$$-\frac{d^2 w_\varepsilon}{dt^2} + \varepsilon^2 W(\varepsilon t)w_\varepsilon + \varepsilon U(t)w_\varepsilon = \varepsilon^2 \lambda_\varepsilon w_\varepsilon \quad (6.2)$$

on $(-1, 1)$. In addition, the components y_ε and w_ε must satisfy the matching conditions

$$w_\varepsilon(\pm 1) = y_\varepsilon(\pm \varepsilon), \quad w'_\varepsilon(\pm 1) = \varepsilon y'_\varepsilon(\pm \varepsilon).$$

We look for approximations of eigenvalues and eigenfunctions of the form

$$\lambda_\varepsilon \sim \lambda_0 + \varepsilon \lambda_1, \quad (6.3)$$

$$y_\varepsilon(x) \sim Y_\varepsilon(x) = \begin{cases} y_0(x) + \varepsilon y_1(x), & \text{if } |x| > \varepsilon, \\ w_0(\varepsilon^{-1}x) + \varepsilon w_1(\varepsilon^{-1}x) \\ + \varepsilon^2 w_2(\varepsilon^{-1}x), & \text{if } |x| < \varepsilon, \end{cases} \quad (6.4)$$

where $y_0 \neq 0$. Substituting the approximations into Equations 6.1, 6.2, we find that y_0 and y_1 satisfy the equations

$$-\frac{d^2 y_0}{dx^2} + W(x)y_0 = \lambda_0 y_0, \quad -\frac{d^2 y_1}{dx^2} + W(x)y_1 = \lambda_0 y_1 + \lambda_1 y_0$$

on $\mathbb{R} \setminus \{0\}$, while the fast-variable components w_0 , w_1 , and w_2 are solutions to the boundary value problems

$$\frac{d^2 w_0}{dt^2} = 0, \quad \frac{dw_0}{dt}(-1) = 0, \quad \frac{dw_0}{dt}(1) = 0; \quad (6.5)$$

$$\frac{d^2 w_1}{dt^2} = U(t)w_0, \quad \frac{dw_1}{dt}(-1) = y'_0(-0), \quad \frac{dw_1}{dt}(1) = y'_0(+0); \quad (6.6)$$

$$\frac{d^2 w_2}{dt^2} = U(t)w_1 + (W(0) - \lambda_0)w_0,$$

$$\frac{dw_2}{dt}(-1) = y'_1(-0) - y''_0(-0), \quad (6.7)$$

$$\frac{dw_2}{dt}(1) = y'_1(+0) + y''_0(+0).$$

Furthermore, the equalities

$$w_0(-1) = y_0(-0), \quad w_0(1) = y_0(+0), \quad (6.8)$$

$$w_1(-1) = y_1(-0) - y'_0(-0), \quad w_1(1) = y_1(+0) + y'_0(+0) \quad (6.9)$$

hold. In view of Equations 6.5, 6.8, w_0 is a constant function and therefore $y_0(+0) = y_0(-0)$. Set $w_0(t) = y_0(0)$. Problem (Equation 6.6) can be solved if and only if

$$\frac{dw_1}{dt}(1) - \frac{dw_1}{dt}(-1) = y_0(0) \int_{-1}^1 U(\tau) d\tau,$$

which yields the second interface condition for y_0 :

$$y'_0(+0) - y'_0(-0) = \alpha y_0(0), \quad \alpha = \int_{\mathbb{R}} U(\tau) d\tau. \quad (6.10)$$

Therefore, the leading terms y_0 and λ_0 of Equations 6.3, 6.4 solve the problem

$$-\frac{d^2 y_0}{dx^2} + W(x)y_0 = \lambda_0 y_0 \quad \text{on } \mathbb{R} \setminus \{0\}, \\ y_0(+0) = y_0(-0), \quad y'_0(+0) - y'_0(-0) = \alpha y_0(0).$$

Since y_0 must be a non-trivial solution, we conclude that λ_0 is an eigenvalue of S_α and y_0 is the corresponding (real-valued) eigenfunction; we denote it by ψ and normalize by $\|\psi\| = 1$. Moreover, problem (Equation 6.6) is solvable now, and the solution w_1 is defined up to a constant.

Integrating twice the equation for w_1 and using the relations $w'_1(-1) = \psi'(-0)$ and $w_1(-1) = y_1(-0) - \psi'(-0)$, we arrive at the formula

$$w_1(t) = \psi(0) \int_{-1}^t (t - \tau) U(\tau) d\tau + \psi'(-0)t + y_1(-1), \quad (6.11)$$

which on account of $w_1(1) = y_1(+0) + \psi'(+0)$ yields

$$y_1(+0) + \psi'(+0) = \psi(0) \int_{-1}^1 (1 - \tau) U(\tau) d\tau + \psi'(-0) + y_1(-0).$$

Set $\alpha_1 = \int_{\mathbb{R}} \tau U(\tau) d\tau$. Then, $y_1(+0) - y_1(-0) = -\alpha_1 \psi(0)$, by Equation 6.10.

To get the second interface relation for y_1 , we integrate the equation for w_2 and find that

$$w'_2(1) - w'_2(-1) = \int_{-1}^1 U(t)w_1(t) dt + 2(W(0) - \lambda_0)\psi(0).$$

We assume that W is continuous in the vicinity of $x = 0$, and this implies that

$$\psi''(+0) = \psi''(-0) = (W(0) - \lambda_0)\psi(0).$$

Combining this with the boundary conditions in Equations 6.7, 6.11, we obtain

$$y'_1(+0) - y'_1(-0) = \alpha y_1(-0) + \alpha_1 \psi'(-0) + \gamma \psi(0),$$

where

$$\gamma = \int_{\mathbb{R}} \int_{-\infty}^t U(t)(t - \tau) U(\tau) d\tau dt.$$

So, we get the boundary value problem for y_1 :

$$-\frac{d^2 y_1}{dx^2} + W(x)y_1 = \lambda_0 y_1 + \lambda_1 \psi \quad \text{on } \mathbb{R} \setminus \{0\}, \quad (6.12)$$

$$y_1(+0) - y_1(-0) = -\alpha_1 \psi(0), \quad (6.13)$$

$$y'_1(+0) - y'_1(-0) - \alpha y_1(-0) = \alpha_1 \psi'(-0) + \gamma \psi(0). \quad (6.14)$$

Observe that the solution y_1 , if it exists, is determined up to adding a multiple of ψ ; therefore, by the Fredholm alternative, the above non-homogeneous problem is solvable only when some extra conditions are met. To derive them, we multiply Equation 6.12 by the eigenfunction ψ and then integrate by parts twice to get

$$\begin{aligned} \psi(0)(y'_1(+0) - y'_1(-0) - \alpha y_1(-0)) - \psi'(+0)(y_1(+0) - y_1(-0)) \\ = \lambda_1 \int_{\mathbb{R}} \psi^2(x) dx. \end{aligned}$$

Relations (Equations 6.13, 6.14) result in the expression

$$\lambda_1 = \gamma \psi^2(0) + \alpha_1 \psi(0)(\psi'(-0) + \psi'(+0))$$

for the second term in asymptotics (Equation 5.3).

We seek an approximation of the low-lying eigenvalues and the corresponding eigenfunctions of the form

$$\sqrt{-\lambda_\varepsilon} \sim \omega_\varepsilon = \varepsilon^{-1}(\omega + \varepsilon \kappa), \quad (6.15)$$

$$y_\varepsilon(x) \sim Y_\varepsilon(x) = \begin{cases} e^{\frac{(\omega + \varepsilon \kappa)x}{\varepsilon}}, & \text{if } x < -\varepsilon, \\ u(\varepsilon^{-1}x) + \varepsilon w(\varepsilon^{-1}x), & \text{if } |x| < \varepsilon, \\ (a + b\varepsilon)e^{-\frac{(\omega + \varepsilon \kappa)x}{\varepsilon}}, & \text{if } x > \varepsilon, \end{cases} \quad (6.16)$$

where $\omega > 0$. On the region $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$, the function Y_ε satisfies (Equation 6.1) up to terms with a small norm in $L^2(\mathbb{R})$. For instance, only the term $W(x)e^{\frac{(\omega + \varepsilon \kappa)x}{\varepsilon}}$ remains if $x < -\varepsilon$, and its norm in $L^2(-\infty, 0)$ is of order $O(\varepsilon^{1/2})$. By substituting Y_ε into Equation 6.2 and matching the terms at $x = \pm\varepsilon$, we obtain

$$\begin{aligned} -\frac{d^2 u}{dt^2} + (V(t) + \omega^2)u &= 0, \\ -\frac{d^2 w}{dt^2} + (V(t) + \omega^2)w &= -(2\omega\kappa + U(t))u, \\ u(-1) &= e^{-\omega}, \quad \frac{du}{dt}(-1) = \omega e^{-\omega}, \\ u(1) &= a e^{-\omega}, \quad \frac{du}{dt}(1) = -a\omega e^{-\omega}, \\ w(-1) &= -\kappa e^{-\omega}, \quad \frac{dw}{dt}(-1) = \kappa(1 - \omega)e^{-\omega}, \\ w(1) &= (b - a\kappa)e^{-\omega}, \quad \frac{dw}{dt}(1) = (a\kappa(\omega - 1) - b\omega)e^{-\omega}. \end{aligned} \quad (6.17)$$

The relations for u can be written as

$$-\frac{d^2 u}{dt^2} + (V(t) + \omega^2)u = 0, \quad t \in (-1, 1), \quad (6.18)$$

$$\frac{du}{dt}(-1) - \omega u(-1) = 0, \quad \frac{du}{dt}(1) + \omega u(1) = 0, \quad (6.19)$$

which is the Regge problem (Equation 4.3). We assume that ω is an eigenvalue of the problem with real-valued eigenfunction u .

Recall that $-\omega^2$ is an eigenvalue of the operator T_1 of Equation 4.1 with the eigenfunction ψ given by Equation 4.4. We also have $a = u(1)/u(-1)$.

From Equation 6.17, we similarly obtain the problem for w :

$$-\frac{d^2 w}{dt^2} + (V(t) + \omega^2)w = -(2\omega\kappa + U(t))u, \quad t \in (-1, 1), \quad (6.20)$$

$$\frac{dw}{dt}(-1) - \omega w(-1) = \kappa u(-1), \quad \frac{dw}{dt}(1) + \omega w(1) = -\kappa u(1). \quad (6.21)$$

The problem is generally unsolvable because ω is an eigenvalue of the homogeneous problem (Equations 6.18, 6.19). In this situation, however, the free parameter κ can be chosen so that the problem admits solutions.

Solvability condition of Equations 6.20, 6.21 has the form

$$\kappa = -\frac{\int_{-1}^1 U u^2 dt}{u^2(-1) + u^2(1) + 2\omega \|u\|^2}.$$

When $\omega > 0$, then the denominator can be written as $2\omega \|\psi\|^2$, with the eigenstate ψ of Equation 4.4 resulting in

$$\kappa = -\frac{\int_{\mathbb{R}} U |\psi|^2(x) dx}{2\omega \|\psi\|^2}.$$

With κ as above, there exists a solution w of Equations 6.20, 6.21 defined up to the additive term cu . Finally, we can calculate

$$b = \frac{u(-1)w(1) - u(1)w(-1)}{u^2(-1)}.$$

Observe that the right hand side of the latter expression is independent of the chosen partial solution w . Hence, we have formally obtained asymptotics (Equation 4.5).

6.2 Justification of asymptotics

We now justify the asymptotic representations for λ_ε and y_ε by constructing a so-called *quasimode* for the operator \mathcal{H}_ε . Let A be a self-adjoint operator in a Hilbert space L . A pair $(\mu, \phi) \in \mathbb{R} \times \text{dom } A$ is called a *quasimode* of A with accuracy ϵ if $\|\phi\|_L = 1$ and $\|(A - \mu I)\phi\|_L \leq \epsilon$.

Lemma 3 ([52, p.139]). Assume (μ, ϕ) is a quasimode of A with accuracy $\epsilon > 0$ and that the spectrum of A in the interval $[\mu - \epsilon, \mu + \epsilon]$ is discrete. Then there exists an eigenvalue λ of A such that $|\lambda - \mu| \leq \epsilon$.

Moreover, if the interval $[\mu - \Delta, \mu + \Delta]$ contains precisely one simple eigenvalue λ with normalized eigenvector u , then

$$\|\phi - e^{ia}u\| \leq 2\epsilon\Delta^{-1} \quad (6.22)$$

for some real number a .

A quasimode of \mathcal{H}_ε can be constructed based on the approximation Y_ε . Note, however, that the function Y_ε defined

by Equation 6.4 is not smooth enough to belong to the domain $\text{dom } \mathcal{H}_\varepsilon$, as it has jump discontinuities at the points $x = \pm\varepsilon$. Nevertheless, all these jumps are small due to the construction; namely,

$$|[Y_\varepsilon]_{-\varepsilon}| + |[Y_\varepsilon]_\varepsilon| + |[Y'_\varepsilon]_{-\varepsilon}| + |[Y'_\varepsilon]_\varepsilon| \leq c\varepsilon^2, \quad (6.23)$$

where $[\cdot]_x$ denotes the jump of a function at the point x .

Suppose the functions ζ and η are smooth outside the origin, have compact supports contained in $[0, \infty)$, and $\zeta(+0) = 1$, $\zeta'(+0) = 0$, $\eta(+0) = 0$, $\eta'(+0) = 1$. We introduce the function

$$r_\varepsilon(x) = [Y_\varepsilon]_{-\varepsilon} \zeta(-x - \varepsilon) - [Y'_\varepsilon]_{-\varepsilon} \eta(-x - \varepsilon) - [Y_\varepsilon]_\varepsilon \zeta(x - \varepsilon) - [Y'_\varepsilon]_\varepsilon \eta(x - \varepsilon),$$

which has the jumps at $\pm\varepsilon$ that are negative of those of Y_ε . Therefore, the function $\hat{y}_\varepsilon = Y_\varepsilon + r_\varepsilon$ is continuous on \mathbb{R} along with its derivative and consequently belongs to $W_2^2(\mathbb{R})$. Moreover, $\|\hat{y}_\varepsilon\| = 1 + O(\varepsilon)$ as $\varepsilon \rightarrow 0$, because the main term $y_0 = \psi$ is normalized in $L^2(\mathbb{R})$. The corrector function r_ε is small, because r_ε is identically zero on $(-\varepsilon, \varepsilon)$, and Equation 6.23 makes it obvious that

$$\max_{|x| \geq \varepsilon} |r_\varepsilon^{(k)}(x)| \leq c\varepsilon^2, \quad k = 0, 1, 2.$$

A straightforward computation shows that a pair $(\lambda_0 + \lambda_1\varepsilon, \hat{y}_\varepsilon)$ is a quasimode of \mathcal{H}_ε with accuracy of order $O(\varepsilon^2)$, i.e., $\|\mathcal{H}_\varepsilon \hat{y}_\varepsilon - (\lambda_0 + \lambda_1\varepsilon)\hat{y}_\varepsilon\| \leq c\varepsilon^2 \|\hat{y}_\varepsilon\|$. Hence,

$$|\lambda_\varepsilon - (\lambda_0 + \lambda_1\varepsilon)| \leq C\varepsilon^2,$$

where λ_ε is an eigenvalue of \mathcal{H}_ε . Since λ_ε is a simple eigenvalue, the corresponding eigenfunction y_ε can be chosen so that $y_\varepsilon \rightarrow y_0$ in $L^2(\mathbb{R})$, by Equation 6.22. Theorem 4 is proved.

The proof of Theorem 3 is similar, with one key difference. The approximation given by Equations 6.15, 6.16 is not sufficient to construct a quasimode of \mathcal{H}_ε with sufficiently small accuracy. It is, therefore, necessary to refine the approximation as follows:

$$\sqrt{-\lambda_\varepsilon} \sim \omega_\varepsilon = \varepsilon^{-1}(\omega + \varepsilon\kappa_1 + \varepsilon^2\kappa_2 + \varepsilon^3\kappa_3),$$

$$y_\varepsilon(x) \sim Y_\varepsilon(x) = \begin{cases} e^{\omega_\varepsilon x}, & \text{if } x < -\varepsilon, \\ u\left(\frac{x}{\varepsilon}\right) + \varepsilon u_1\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(\frac{x}{\varepsilon}\right) + \varepsilon^3 u_3\left(\frac{x}{\varepsilon}\right), & \text{if } |x| < \varepsilon, \\ (a + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3)e^{-\omega_\varepsilon x}, & \text{if } x > \varepsilon, \end{cases}$$

after which the construction proceeds as in the case of Theorem 4.

7 Concluding remarks

Exactly solvable models of quantum mechanics with point interactions provide useful approximations for short-range interactions between particles. In particular, δ - and δ' -type potentials serve as mathematical abstractions representing idealized phenomena such as sharply localized charges or dipoles.

However, while these models facilitate rigorous quantitative analysis, they do not always preserve the qualitative behavior of the corresponding regular systems.

For example, Hamiltonians with δ -potentials capture well the spectral and qualitative behavior of regular Hamiltonians with sharply localized attractive wells; in particular, the unique bound state of the regular system persists in the limit model. In contrast, Hamiltonians with localized dipoles may possess arbitrarily many negative eigenvalues, while the corresponding exactly solvable model with δ'_θ -potential has none. We stress that, in both settings, the exactly solvable models arise as norm resolvent limits of families of regular Schrödinger operators as the regularization parameter tends to zero.

Our results show that despite norm resolvent convergence to a bounded below operator with δ'_θ -potential, the family of approximating Hamiltonians with localized dipoles is not uniformly bounded below: their low-lying eigenvalues diverge to $-\infty$ as the regularization parameter tends to zero. This demonstrates that even the strongest form of convergence of Hamiltonians does not ensure that the spectral or qualitative properties of the real physical models are reflected in the idealized limit. Therefore, while exactly solvable models offer powerful tools for quantitative analysis, caution is needed when interpreting their qualitative features as representative of the physical systems they are meant to approximate.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

References

- Albeverio S, Gesztesy F, Høegh-Krohn R, Holden H. *Solvable Models in Quantum Mechanics*. Providence, RI: AMS Chelsea Publishing (2005). p. 488. doi: 10.1090/chel/350
- Albeverio S, Kurasov P. *Singular Perturbations of Differential Operators. Solvable Schrödinger Type Operators*, (London Mathematical Society Lecture Note Series vol 271) Cambridge: Cambridge University Press (2000). p. 429. doi: 10.1017/CBO9780511758904
- Šeba P. Some remarks on the δ' -interaction in one dimension. *Rep Mathem Phys.* (1986) 24:111–20. doi: 10.1016/0034-4877(86)90045-5
- Christiansen PL, Arnbak HC, Zolotaryuk AV, Ermakov VN, Gaididei YB. On the existence of resonances in the transmission probability for interactions arising from derivatives of Dirac δ function. *J Phys A Math Gen.* (2003) 36:7589–600. doi: 10.1088/0305-4470/36/27/311
- Zolotaryuk AV. Two-parametric resonant tunneling across the $\delta'(x)$ potential. *Adv Sci Lett.* (2008) 1:187–91. doi: 10.1166/asl.2008.019
- Zolotaryuk AV. Point interactions of the dipole type defined through a three-parametric power regularization. *J Phys A.* (2010) 43:105302. doi: 10.1088/1751-8113/43/10/105302
- Zolotaryuk AV. Boundary conditions for the states with resonant tunnelling across the δ' -potential. *Phys Lett A.* (2010) 374, 1636–1641. doi: 10.1016/j.physleta.2010.02.005
- Golovaty YD, Hryniv RO. On norm resolvent convergence of Schrödinger operators with δ' -like potentials. *J Phys A.* (2010) 43:155204. doi: 10.1088/1751-8113/43/15/155204
- Klaus M. Some applications of the Birman-Schwinger principle. *Helv Phys Acta* (1982/83) 55:49–68.
- Hinton DB, Klaus M, Shaw JK. Embedded half-bound states for potentials of Wigner-von Neumann type. *Proc London Mathem Soc.* (1991) 62:607–646. doi: 10.1112/plms/s3-62.3.607
- Hinton DB, Klaus M, Shaw JK. Half-bound states and Levinson's theorem for discrete systems. *SIAM J Mathem Anal.* (1991) 22:754–68. doi: 10.1137/0522045
- Golovaty M, Man'ko SS. Solvable models for the Schrödinger operators with δ' -like potentials. *Ukr Math Bull.* (2009) 6:169–203. doi: 10.48550/arXiv.0909.1034
- Golovaty M, Yu D, Hryniv RO. Norm resolvent convergence of singularly scaled Schrödinger operators and δ' -potentials. *Proc R Soc Edinburgh.* (2013) 143:791–816. doi: 10.1017/S0308210512000194
- Kurasov P, Elander N. *On the delta'-interactions in one dimension*. Technical report, MSI, Stockholm (1993).
- Kurasov P. Distribution theory for discontinuous test functions and differential operators with generalized coefficients. *J Math Anal Appl.* (1996) 201:297–323. doi: 10.1006/jmaa.1996.0256
- Golovaty Y. Schrödinger operators with $(\alpha\delta' + \beta\delta)$ -like potentials: norm resolvent convergence and solvable models. *Methods Funct Anal Topol.* (2012) 18:243–55. Available online at: <http://mfat.imath.kiev.ua/article/?id=633>
- Golovaty Y. 1D Schrödinger operators with short range interactions: two-scale regularization of distributional potentials. *Integr Equat Operator Theory.* (2012) 75:341–62. doi: 10.1007/s00020-012-2027-z

Author contributions

YG: Investigation, Conceptualization, Writing – original draft.
RH: Investigation, Writing – original draft, Conceptualization.

Funding

The author(s) declare that no financial support was received for the research and/or publication of this article.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Generative AI statement

The author(s) declare that no Gen AI was used in the creation of this manuscript.

Publisher's note

All claims expressed in this article are solely those of the authors and do not necessarily represent those of their affiliated organizations, or those of the publisher, the editors and the reviewers. Any product that may be evaluated in this article, or claim that may be made by its manufacturer, is not guaranteed or endorsed by the publisher.

18. Gadella M, Heras FJH, Negro J, Nieto LM. A delta well with a mass jump. *J Phys A*. (2009) 42:465207. doi: 10.1088/1751-8113/42/46/465207
19. Gadella M, Negro J, Nieto LM. Bound states and scattering coefficients of the potential. *Phys Lett A*. (2009) 373:1310–3. doi: 10.1016/j.physleta.2009.02.025
20. Gadella M, Glasser ML, Nieto LM. One dimensional models with a singular potential of the type $-\alpha\delta'(x) + \beta\delta(x)$. *Int J Theor Phys*. (2010) 50:2144–52. doi: 10.1007/s10773-010-0641-6
21. Nizhnik LP. A Schrödinger operator with δ' -interaction. *Funct Anal Applic*. (2003) 37:72–74. doi: 10.1023/A:1022932229094
22. Nizhnik LP. A one-dimensional Schrödinger operator with point interactions on Sobolev spaces. *Funct Anal Applic*. (2006) 40:143–7. doi: 10.1007/s10688-006-0022-3
23. CalÁada M, Lunardi JT, Manzoni LA, Monteiro W. Distributional approach to point interactions in one-dimensional quantum mechanics. *Front Phys*. (2014) 2:23. doi: 10.3389/fphy.2014.00023
24. Lee MA, Lunardi JT, Manzoni LA, Nyquist EA. Double general point interactions: symmetry and tunneling times. *Front Phys*. (2016) 4:10. doi: 10.3389/fphy.2016.00010
25. Exner P, Neidhardt H, Zagrebnov VA. Potential approximations to δ' : an inverse Klauder phenomenon with norm-resolvent convergence. *Commun Math Phys*. (2001) 224:593–612. doi: 10.1007/s002200100567
26. Cheon T, Shigehara T. Realizing discontinuous wave functions with renormalized short-range potentials. *Phys Lett A*. (1998) 243:111–6. doi: 10.1016/S0375-9601(98)00188-1
27. Zolotaryuk AV. Families of one-point interactions resulting from the squeezing limit of the sum of two- and three-delta-like potentials. *J Phys A*. (2017) 50:225303. doi: 10.1088/1751-8121/aa6dc2
28. Fassari S, Rinaldi F. On the spectrum of the Schrödinger Hamiltonian with a particular configuration of three one-dimensional point interactions. *Rep Math Phys*. (2009) 64:367–93. doi: 10.1016/S0034-4877(10)00004-2
29. Albeverio S, Fassari S, Rinaldi F. A remarkable spectral feature of the Schrödinger Hamiltonian of the harmonic oscillator perturbed by an attractive δ' -interaction centred at the origin: double degeneracy and level crossing. *J Phys A*. (2013) 46:385305. doi: 10.1088/1751-8113/46/38/385305
30. Albeverio S, Fassari S, Rinaldi F. The Hamiltonian of the harmonic oscillator with an attractive δ' -interaction centred at the origin as approximated by the one with a triple of attractive δ -interactions. *J Phys A*. (2015) 49:025302. doi: 10.1088/1751-8113/49/2/025302
31. Fassari S, Gadella M, Nieto LM, Rinaldi F. Analysis of a one-dimensional Hamiltonian with a singular double well consisting of two nonlocal δ' interactions. *Eur Phys J Plus*. (2024) 139:132. doi: 10.1140/epjp/s13360-024-04923-8
32. Albeverio S, Koshmanenko V, Kurasov P, Nizhnik L. On approximations of rank one \mathcal{H}_{-2} -perturbations. *Proc Am Math Soc*. (2002) 131:1443–1452. doi: 10.1090/S0002-9939-02-06694-7
33. Kuzhel S, Nizhnik L. Finite rank self-adjoint perturbations. *Methods Funct Anal Topol*. (2006) 12:243–53. <http://mfat.imath.kiev.ua/article/?id=375>
34. Albeverio S, Nizhnik L. Approximation of general zero-range potentials. *Ukrainian Mathem J*. (2000) 52:664–72. doi: 10.1007/BF02487279
35. Fassari S, Gadella M, Glasser ML, Nieto LM. Spectroscopy of a one-dimensional V-shaped quantum well with a point impurity. *Ann Phys*. (2018) 389:48–62. doi: 10.1016/j.aop.2017.12.006
36. Golovaty Y. Two-parametric δ' -interactions: approximation by Schrödinger operators with localized rank-two perturbations. *J Phys A*. (2018) 51:255202. doi: 10.1088/1751-8121/aac110
37. Golovaty Y. Schrödinger operators with singular rank-two perturbations and point interactions. *Integ Equat Oper Theory*. (2018) 90:57. doi: 10.1007/s00020-018-2482-2
38. Cheon T, Exner P. An approximation to couplings on graphs. *J Phys A Math Gen*. (2004) 37:L329–35. doi: 10.1088/0305-4470/37/29/L01
39. Man'ko SS. On δ' -like potential scattering on star graphs. *J Phys A*. (2010) 43:445304. doi: 10.1088/1751-8113/43/44/445304
40. Exner P, Manko SS. Approximations of quantum-graph vertex couplings by singularly scaled potentials. *J Phys A*. (2013) 46:345202. doi: 10.1088/1751-8113/46/34/345202
41. Golovaty Y. Quantum graphs: coulomb-type potentials and exactly solvable models. *Ann Henri Poincaré*. (2023) 24:2557–85. doi: 10.1007/s00023-023-01270-9
42. Reed M, Simon B. *Methods of Modern Mathematical Physics*. Vol. 4: Analysis of Operators. Academic Press, New York (1978).
43. Berezin FA, Shubin MA. *The Schrödinger Equation*. Netherlands: Springer. (1991). doi: 10.1007/978-94-011-3154-4
44. Simon B. *Trace Ideals and Their Applications*. 2nd ed. Providence, RI: American Mathematical Society (2005).
45. Klaus M. On the bound state of Schrödinger operators in one dimension. *Ann Phys*. (1977) 108:288–300. doi: 10.1016/0003-4916(77)90015-X
46. Kato T. *Perturbation Theory for Linear Operators*. New York: Springer Science Business Media (2013).
47. Golovaty Y. Some remarks on 1D Schrödinger operators with localized magnetic and electric potentials. *Front Phys*. (2019) 7:70. doi: 10.3389/fphy.2019.00070
48. Shkalikov AA. Spectral analysis of the Redge problem. *J Mathem Sci*. (2007) 144:4292–300. doi: 10.1007/s10958-007-0272-5
49. Avakian MP, Pogosyan GS, Sissakian AN, Ter-Antonyan VM. Spectroscopy of a singular linear oscillator. *Phys Lett A*. (1987) 124:233–236. doi: 10.1016/0375-9601(87)90627-X
50. Fassari S, Inglese G. On the spectrum of the harmonic oscillator with a δ -type perturbation. *Helv. Phys Acta*. (1994) 67:650–9.
51. Simon B. The bound state of weakly coupled Schrödinger operators in one and two dimensions. *Ann Phys*. (1976) 97:279–88. doi: 10.1016/0003-4916(76)90038-5
52. Lazutkin VF. Semiclassical asymptotics of eigenfunctions. *Part Differ Equat V*. (1999) 133–71. doi: 10.1007/978-3-642-58423-7_4