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Bootstrap confidence intervals of process capability indices C_{py} and C_{Npmk} using different methods of estimation for Frechet distribution

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Process capability analysis is the statistical evaluation of process capability to examine how well it meets or exceeds the customers satisfaction. In present work, we are intended to evaluate two critical metrics for process quality assessment, C_{py} and C_{Npmk} for asymmetric Frechet distribution using different classical estimation methods namely: maximum likelihood, least squares, weighted least squares, Cramer-von-Mises, Anderson-Darling, right-tail Anderson-Darling, along Bayesian estimation using reference and Jeffreys priors. Furthermore, Monte Carlo simulations are conducted for comparative analysis of aforementioned estimation methods, using mean squared error and width of bootstrap confidence intervals. Comparative analysis demonstrates that Bayesian estimation using reference prior consistently producing smaller coefficient of mean squared errors and reduced width of bootstrap intervals for small to moderately large sample sizes. Moreover, the real data analysis is also validating the advantages of Bayesian estimations for process capability analysis.

KEYWORDS

process capability indices, Bayesian estimators, classical estimators, bootstrap confidence intervals, mean squared errors

1 Introduction

Statistical process control (SPC) is crucial for quality control in the highly competitive manufacturing industry. Process capability analysis is a vital component of SPC quality improvement programs. An essential part of SPC quality improvement initiatives is process capability analysis. Manufacturing process capabilities can be evaluated quantitatively and intuitively with the help of process capability indices (PCIs). These indices help identify areas for improvement and show how well a process meets customer needs. Over capability analysis, a process is evaluated using statistical methods to see whether it fulfills a set of customers preset specifications or not. Suppliers and manufacturers employ such measures to ensure that their finished products are of excellent quality and have the least amount of discrepancies. Capability indices are the most commonly used techniques for capability analysis. A PCI quantifies how well the process meets customer expectations.

Several PCIs have been suggested to quantify process performance. The readers may refer to Kane [1], addressed two often used indices: C_p and C_{pk}, Chan et al. [2] and Pearn et al. [3] devised two more advanced indices: C_{pm} and C_{pmk}. Choi and Owen [4] provided thorough comparisons of C_p, C_{pk}, C_{pm}, and C_{pmk}. Boyles [5] introduced PCI with asymmetric tolerance for C_{pk}, C_{pm}, and C_{pmk}. Many statisticians and quality researchers, including Kane [1], Chan et al. [2], Chou et al. [6], Pearn et al. [3], and Pearn and Chen [7], have discussed and analyzed point estimation and interval estimation of various indices.

The efficacy of these indicators is dependent on the process following a normal distribution [8–10]. However, the presence of various noise elements might occasionally result in abnormal manufacturing processes [11]. If the assumption is not met, these basic indices are not reliable [12–16]. For non-normal processes, the normality-based classical PCI may produce unreliable and misleading interpretations [17]. Pearn and Chen [18], Tong and Chen [19], and Senvar and Kahraman [17] all studied non-normal distributions. Still, we were unable to depend on one method that functions efficiently in all situations [11]. Furthermore, for various non-normal distributions, it is also claimed that each method performed differently [10]. Thus, for statisticians and practitioners, evaluating various PCIs under non-normal distributions is a crucial research topic. The widespread usage of PCIs in manufacturing requires a thorough analysis of proposed structures of PCIs in the literature using different estimating techniques and distributions.
A PCI evaluates the amount of variation process experiences in relation to specification limitations. Point and interval estimations are often emphasized while analyzing PCIs. However, there is a possibility that point estimate will not conclude a realistic assessment of a process' capabilities and confidence intervals (CIs) are considered more useful in determining the variability of an estimate [20]. Hsiang and Taguchi [21] initiated the first move toward constructing CIs for the PCI. Since then, various methods for constructing CIs have been developed. One of these methods is bootstraping, which is unconstrained by distributional assumptions and makes complex inferences in simplified manners. In reality, insufficient or limited data is evaluated using a basic re-sampling method owing to time and budgetary restrictions. Franklin and Wasserman [22] initially used the bootstrap approach to evaluate the PCI-C_{pk}. Recently, significant advances have been made in the literature about bootstrap CIs and how they behave for processes in absence of normality assumption. Some studies in this regard include Kashif et al. [23], Rao et al. [24, 25], Peng [26], Dey and Saha [27, 28], Dey et al., [29], Saha et al. [30, 31], and Weber et al. [32].

Most statistical studies rely on one of two estimating categories: Bayesian or frequentist. PCIs for both normal and non-normal processes were developed using Bayesian and classical perspectives. Several statisticians, including Chan et al. [2], Cheng and Spiring [33], and Shiau et al. [34, 35], have justified and emphasized the advantages of the Bayesian method. Ramos et al. [36] provided a brief comparison of conventional and Bayesian estimation methods.

In present work, we are intended to develop PCIs named C_{py} and C_{Npmk} for Frechet distribution (FD) utilizing different classical methods provided in Ramos et al. [36] listed as, maximum likelihood, least square, Cramer-von-Mises and Anderson-Darling. Additionally, we are using weighted least square, right-tail Anderson-Darling estimators and Bayesian estimators using reference prior along Jeffrey prior for evaluation of PCIs.
The FD is a special case of the generalized extreme value distribution, used to model the maximum of large numbers of independent positive random variables. FD is equivalent to the inverse exponential distribution for the scale parameter $\beta = 1$; it is the inverse gamma distribution when $\beta < 1$ and for $\beta = 2$; it is the inverse Rayleigh distribution. The FD may be utilized in reliability analysis, especially when examining the quality characteristic of fatigue lifespan (the marketing life expectancy of pharmaceutical, automobile radiators, textile fatigue, and electron tubes, among other devices and variables). It provides flexible models for non-normal and asymmetric quality characteristics. Most suited to right-skewed characteristics where small values are most likely but rare extreme values exist. The probability density function (PDF), cumulative distribution function (CDF) and quantile function (QF) are given by

$$f(x \mid \alpha, \beta) = \left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{x}\right)^{\alpha+1} \exp\left[-(\beta/x)^{\alpha}\right];\ x,\ \alpha,\ \beta > 0 \tag{1.1}$$

$$F(x \mid \alpha, \beta) = \exp\left[-(\beta/x)^{\alpha}\right] \tag{1.2}$$

$$Q(\gamma_i \mid \alpha, \beta) = \beta\left[-(\log(\gamma_i))^{-\frac{1}{\alpha}}\right] \tag{1.3}$$

Kanwal and Abbas [37] evaluated PCIs S_{pmk} and C_s for FD. Kanwal and Abbas [38] provided comparative analysis of quantile based control charts for FD. It is important to note that while the quality characteristic X specifically follows FD, no effort has been taken to investigate PCIs particularly C_{py} and C_{Npmk}. The research aims to bridge the gap. Our objective is to provide recommendations for selecting the most effective PCI estimation technique. It is hoped this study will benefit quality engineers, management professionals, and statisticians. In continuation to the introductory section, the remainder of the article is structured as follows. Section 2 comprises of introduction to PCI C_{py} and PCI C_{Npmk}. In Section 3 sensitivity analysis is performed, Section 4 presents several classical estimation methods for C_{py} and C_{Npmk} whereas bootstrap confidence intervals (BCIs) are covered in Section 5. Section 6 includes a simulation study to evaluate the estimators' effectiveness in estimating C_{py} and C_{Npmk} in terms of mean squared error (MSE), coefficient of skewness ($\sqrt{\beta_1}$), coefficient of kurtosis (β_2), and BCIs width. Section 7 analyzes the real-life data set as an illustration. Finally, the findings of the present study are shown in Section 8.

2 PCIs C_{py} and C_{Npmk}

The detail of PCIs C_{py} and C_{Npmk} is given below.

2.1 PCI C_{py}

A generalized PCI C_{py} has been proposed by Maiti et al. [39]. It includes continuous and discrete random variables, both normal and non-normal, and is either directly or indirectly associated with most PCIs in literature. It can be defined as:

$$C_{py} = \frac{F(USL) - F(LSL)}{F(UDL) - F(LDL)} = \frac{p}{p_0} \tag{2.1}$$

where p is the process output, p_0 is the desired output, LDL and UDL denotes the lower and upper desirable limits, and USL and LSL are upper and lower specifications respectively. Where, $LDL = \mu - 3\sigma$ and $UDL = \mu + 3\sigma$. PCI C_{py} may be expressed as $(p/0.9973)$ for process following a normal distribution. The C_{py} for FD quality characteristic can be expressed as

$$C_{py} = \frac{1}{p_0}\left[\exp\left[-(\beta/USL)^{\alpha}\right] - \exp\left[-(\beta/LSL)^{\alpha}\right]\right] \tag{2.2}$$

2.2 PCI C_{Npmk}

Vannman [40] combined these four indices C_p, C_{pk}, C_{pm} and C_{pmk} into one super-structure, defined as

$$C_p(u,v) = \frac{d - u|\mu - m|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}, \quad u, v \geq 0$$

where $d = (USL - LSL)/2$ and $m = (USL + LSL)/2$. In particular, we have, $C_p(0,0) = C_p, C_p(1,0) = C_{pk}, C_p(0,1) = C_{pm}$ and $C_p(1,1) = C_{pmk}$. These indices have been developed when underlying distribution is normal. Later, Chen and Pearn [16] generalized Vannman [40] work and developed a new quantile-based PCI index CNp (u, v) for any underlying distribution, defined as

$$C_{Np}(u,v) = \frac{d - u|M - m|}{3\sqrt{\left(\frac{F_{99.865} - F_{0.135}}{6}\right)^2 + v(M - T)^2}},$$

where F_{α} is the α-th percentile, M is the median. Whereas, median M for is a more robust measure skewed distributions than the process mean μ. Setting $(u,v)= (1,1)$ leads to C_{Npmk}, for any distribution, given as

$$C_{Npmk} = \min\left\{C_{NpmkU}, C_{NpmkL}\right\}$$

where

$$C_{NpmkU} = \frac{USL - M}{3\sqrt{\left(\frac{Q(\gamma_3|\alpha,\beta) - Q(\gamma_1|\alpha,\beta)}{6}\right)^2 + \left(Q\left(\gamma_2 \mid \alpha,\beta\right) - T\right)^2}}$$

and

$$C_{NpmkL} = \frac{M - LSL}{3\sqrt{\left(\frac{Q(\gamma_3|\alpha,\beta) - Q(\gamma_1|\alpha,\beta)}{6}\right)^2 + \left(Q\left(\gamma_2 \mid \alpha,\beta\right) - T\right)^2}}.$$

where, $\gamma_i - th$ are quantiles of the two parameter FD with parameters (α, β). The C_{Npmk} for FD quality characteristic can be written as

$$C_{Npmk} = \frac{\min(USL - M, M - LSL)}{3\sqrt{\left(\frac{\beta\left(-\log\left(\frac{3}{4}\right)^{-\left(\frac{1}{\alpha}\right)} - \log\left(\frac{1}{4}\right)^{-\left(\frac{1}{\alpha}\right)}\right)}{6}\right)^2 + (M - T)^2}} \tag{2.3}$$

TABLE 1 Sensitivity analysis for FD.

Distribution	$f(U)$	$f(L)$	NS (dpm)
Frechet (Shape = 2.1, Scale = 1.6)	0.09986	0.19780	−103096.5
Weibul (Shape = 2.1, Scale = 1.6)	0.003810	0.53915	−563515.3
Log-logistic (Shape = 2.1, Scale = 1.6)	0.058361	0.41536	−375785.9

3 Sensitivity analysis

For a given PCI, the net sensitivity (NS) analysis for distribution function is proposed by Maiti et al. [39], and is defined as

$$NS = \frac{1}{p_0}\lim_{\epsilon \to 0}\left[\frac{\{F(U) - F(L)\} - \{F(U - \epsilon) - F(L - \epsilon)\}}{\epsilon}\right] = \frac{f(U) - f(L)}{p_0},$$

If the NS values are low (in absolute) with respect to the PCI, the distribution is less sensitive and more robust. However, if the NS values are positive, the considered distribution is more sensitive (or less robust) at USL than at LSL; if the NS values are negative, the converse is true. The unit of measurement for the NS values is defectives per million (dpm). The NS values for the following distributions are presented in Table 1. It is apparent from Table 1 that overall FD is more robust as compare to Weibul and log-logistic distribution and Figure 1 shows that Weibull, log-logistic and FD all are less sensitive (or more robust) at USL for the considered values of $(L; U) = (1; 4)$ and $p_0 = 0.95$, respectively.

4 Different methods of estimation for C_{py} and C_{Npmk}

Here, we describe eight different estimators, namely, maximum likelihood estimator (MLE), ordinary least square estimator(LSE) and weighted least squares estimator (WLSE), Cramer-von-Mises estimator (CVME), Anderson-Darling estimator (ADE), and right tail Anderson-Darling estimator (RTADE), Bayesian estimator under reference prior (BERP), and Jeffrey prior (BEJP) to find out the estimates of the parameters α and β for FD and the corresponding estimators of C_{py} and C_{Npmk} respectively. The non-linear equations can be solved by using nleqslv package which is available in the R-software [41].

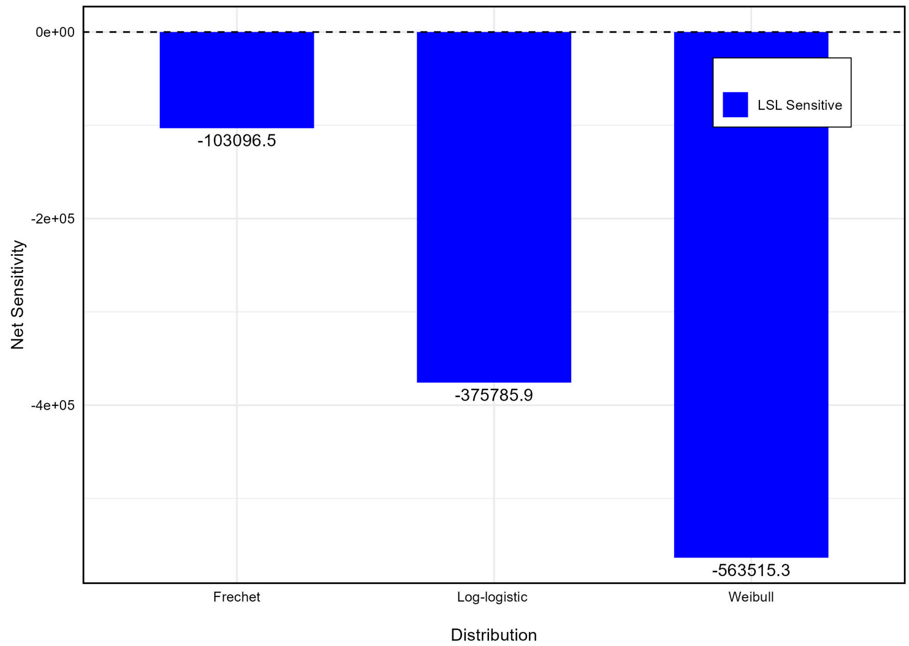

FIGURE 1
Sensitivity for FD.

4.1 Maximum likelihood estimator

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n taken from Equation 1.1. Then the log-likelihood function is

$$l = n\log(\alpha) - n\alpha\log(\beta) - (\alpha+1)\sum_{i=1}^{n}\log\left(x_i\right) - \sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\alpha}$$

The partial derivatives are

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + n\log(\beta) - \sum_{i=1}^{n}\log\left(x_i\right) - \sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\alpha}\log\left(\frac{\beta}{x_i}\right) = 0 \tag{4.1}$$

$$\frac{\partial l}{\partial \beta} = \frac{n\alpha}{\beta} - \left(\frac{\alpha}{\beta}\right)\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\right)^{\alpha} = 0 \tag{4.2}$$

The solutions to these equations do not result in closed form for MLEs. However, these equations are solved simultaneously using a numerical approach. Substituting the MLEs in Equations 2.2, 2.3, we can get the estimators of C_{py} and C_{Npmk} as

$$\hat{C}_{py}^{MLE} = \frac{1}{p_0}\left[\exp\{-(\hat{\beta}_{MLE}/USL)^{\hat{\alpha}_{MLE}}\} - \exp\{-(\hat{\beta}_{MLE}/LSL)^{\hat{\alpha}_{MLE}}\}\right] \tag{4.3}$$

$$\hat{C}_{Npmk}^{MLE} = \frac{\min(USL-M, M-LSL)}{3\sqrt{\left(\frac{\hat{\beta}_{MLE}\left(-\log\left(\frac{3}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{MLE}}\right)} - \log\left(\frac{1}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{MLE}}\right)}\right)}{6}\right)^2 + (M-T)^2}}. \tag{4.4}$$

4.1.1 Ordinary and weighted least squares estimator

Swain et al. [42] proposed a least squares approach to minimize the distance in Johnson's translation system. Suppose $F(x_{(i:n)}|\alpha,\beta)$ denotes the CDF of ordered random variables $x_1 \leq x_2 \leq \ldots \leq x_n$ of size n from a distribution function $F(\cdot|\alpha,\beta)$. The least squares estimators of $\hat{\alpha}_{LSE}$ and $\hat{\beta}_{LSE}$ can be obtained by minimizing the function

$$S(\alpha,\beta) = \sum_{i=1}^{n}\left[F\left(x_{(i:n)}|\alpha,\beta\right) - \frac{i}{n+1}\right]^2, \tag{4.5}$$

differentiating with respect to α and β. The estimators can be obtained by solving the following

$$\sum_{i=1}^{n}\left[F\left(x_{(i)}|\alpha,\beta\right) - \frac{i}{n+1}\right]\xi_1(x_i|\alpha,\beta) = 0, \tag{4.6}$$

$$\sum_{i=1}^{n}\left[F\left(x_{(i)}|\alpha,\beta\right) - \frac{i}{n+1}\right]\xi_2(x_i|\alpha,\beta) = 0, \tag{4.7}$$

where

$$\xi_1(x_i|\alpha,\beta) = -\exp\left\{-\left(\frac{\beta}{x_i}\right)\right\}^{\alpha}\left(\frac{\beta}{x_i}\right)^{\alpha}\log\left(\frac{\beta}{x_i}\right) = 0, \quad (4.8)$$

and

$$\xi_2(x_i|\alpha,\beta) = -\frac{\exp\left\{-\left(\frac{\beta}{x_i}\right)\right\}^{\alpha}\alpha\left(\frac{\beta}{x_i}\right)^{\alpha-1}}{x_i} = 0. \quad (4.9)$$

Similarly, the weighted least squares estimator of $\hat{\alpha}_{WLSE}$ and $\hat{\beta}_{WLSE}$ can be obtained by minimizing the function by Erto and Genesis [43].

$$W(\alpha,\beta) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{(i:n)}|\alpha,\beta\right) - \frac{i}{n+1}\right]^2 = 0, \quad (4.10)$$

WLSEs can be obtained by solving the following

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{(i:n)}|\alpha,\beta\right) - \frac{i}{n+1}\right]\xi_1(x_i|\alpha,\beta) = 0, \quad (4.11)$$

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(x_{(i:n)}|\alpha,\beta\right) - \frac{i}{n+1}\right]\xi_2(x_i|\alpha,\beta) = 0. \quad (4.12)$$

Where $\xi_1(x_i|\alpha,\beta)$ and $\xi_2(x_i|\alpha,\beta)$ are shown in Equations 4.8, 4.9. The least squares and weighted least squares estimators of PCI-C_{py} for FD are

$$\hat{C}_{py}^{LSE} = \frac{1}{p_0}\left[\exp\{-(\hat{\beta}_{LSE}/USL)^{\hat{\alpha}_{LSE}}\} - \exp\{-(\hat{\beta}_{LSE}/LSL)^{\hat{\alpha}_{LSE}}\}\right] \quad (4.13)$$

$$\hat{C}_{py}^{WLSE} = \frac{1}{p_0}\left[\exp\{-(\hat{\beta}_{WLSE}/USL)^{\hat{\alpha}_{WLSE}}\} - \exp\{-(\hat{\beta}_{WLSE}/LSL)^{\hat{\alpha}_{WLSE}}\}\right] \quad (4.14)$$

Similarly, the least squares and weighted least squares estimators of PCI-C_{Npmk} for FD are

$$\hat{C}_{Npmk}^{LSE} = \frac{\min(USL-M, M-LSL)}{3\sqrt{\left(\frac{\hat{\beta}_{LSE}\left(-\log\left(\frac{3}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{LSE}}\right)} - \log\left(\frac{1}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{LSE}}\right)}\right)}{6}\right)^2 + (M-T)^2}} \quad (4.15)$$

$$\hat{C}_{Npmk}^{WLSE} = \frac{\min(USL-M, M-LSL)}{3\sqrt{\left(\frac{\hat{\beta}_{WLSE}\left(-\log\left(\frac{3}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{WLSE}}\right)} - \log\left(\frac{1}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{WLSE}}\right)}\right)}{6}\right)^2 + (M-T)^2}}. \quad (4.16)$$

4.1.2 Cramer-von-Mises estimator

The Cramer-von-Mises test was proposed by Cramer [44] and von-Mises [45], which measures the distance between empirical distribution function and CDF of assumed model. Moreover, Boos [46] studied its properties, which can be expressed as

$$CVM(\alpha,\beta) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F\left(x_{(i:n)}|\alpha,\beta\right) - \frac{2i-1}{2n}\right]^2, \quad (4.17)$$

The Cramer-von-Mises estimator of $\hat{\alpha}_{CVME}$ and $\hat{\beta}_{CVME}$ can be incurred by solving the following

$$\sum_{i=1}^{n}\left[F\left(x_{(i:n)}|\alpha,\beta\right) - \frac{2i-1}{2n}\right]\xi_1(x_i|\alpha,\beta) = 0, \quad (4.18)$$

$$\sum_{i=1}^{n}\left[F\left(x_{(i:n)}|\alpha,\beta\right) - \frac{2i-1}{2n}\right]\xi_2(x_i|\alpha,\beta) = 0. \quad (4.19)$$

Where $\xi_1(x_i|\alpha,\beta)$ and $\xi_2(x_i|\alpha,\beta)$ are mentioned in Equations 4.8, 4.9. The corresponding estimators of PCIs-C_{py} and C_{Npmk} are

$$\hat{C}_{py}^{CVME} = \frac{1}{p_0}\left[\exp\{-(\hat{\beta}_{CVME}/USL)^{\hat{\alpha}_{CVME}}\} - \exp\{-(\hat{\beta}_{CVME}/LSL)^{\hat{\alpha}_{CVME}}\}\right] \quad (4.20)$$

$$\hat{C}_{Npmk}^{CVME} = \frac{\min(USL-M, M-LSL)}{3\sqrt{\left(\frac{\hat{\beta}_{CVME}\left(-\log\left(\frac{3}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{CVME}}\right)} - \log\left(\frac{1}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{CVME}}\right)}\right)}{6}\right)^2 + (M-T)^2}}. \quad (4.21)$$

4.1.3 Anderson-Darling estimator

Anderson-Darling [47] statistic belong to the class of quadratic empirical distribution function [48]. The Anderson-Darling test (ADT) converges rapidly toward the asymptote [49], which is

$$ADT(\alpha,\beta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log(F\left(x_{(i)}|\alpha,\beta\right)) + \log(1-(F\left(x_{(1-i+n)}|\alpha,\beta\right)))] \quad (4.22)$$

The Anderson-Darling estimator (ADE), $\hat{\alpha}_{ADE}$ and $\hat{\beta}_{ADE}$ can also be obtained by solving the following non-linear equations

$$\sum_{i=1}^{n}(2i-1)\left[\frac{-\exp\left\{-\left(\frac{\beta}{x_i}\right)^{\alpha}\right\}\log\left(\frac{\beta}{x_{(i)}}\right)\left(\frac{\beta}{x_{(i)}}\right)^{\alpha} + \frac{\exp\left\{-\left(\frac{\beta}{x_{(1-i+n)}}\right)^{\alpha}\right\}\log\left(\frac{\beta}{(1-i+n)}\right)\left(\frac{\beta}{x_{(1-i+n)}}\right)^{\alpha}}{1-\exp\left\{-\left(\frac{\beta}{x_{(1-i+n)}}\right)^{\alpha}\right\}}}{\exp\left\{-\left(\frac{\beta}{x_i}\right)^{\alpha}\right\} + \log\left[1-\exp\left\{-\left(\frac{\beta}{x_{(1-i+n)}}\right)^{\alpha}\right\}\right]}\right] \quad (4.23)$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{-\frac{\exp\left\{-\left(\frac{\beta}{x_i}\right)^{\alpha}\right\}\alpha\left(\frac{\beta}{x_{(i)}}\right)^{-1+\alpha}}{x_i}+\frac{\exp\left\{-\left(\frac{\beta}{x_{(1-i+n)}}\right)^{\alpha}\right\}\alpha\left(\frac{\beta}{x_{(1-i+n)}}\right)^{-1+\alpha}}{\left(1-\exp\left\{-\left(\frac{\beta}{x_{(1-i+n)}}\right)^{\alpha}\right\}\right)x_{(1-i+n)}}}{\exp\left\{-\left(\frac{\beta}{x_i}\right)^{\alpha}\right\}+\log\left(1-\exp\left\{-\left(\frac{\beta}{x_{(1-i+n)}}\right)^{\alpha}\right\}\right)}\right] \tag{4.24}$$

The right tail Anderson-Darling estimator, $\hat{\alpha}_{RTADE}$ and $\hat{\beta}_{RTADE}$ are obtained by minimizing with respect to α and β, the function

$$RTADE(\alpha,\beta) = \frac{n}{2}-2\sum_{i=1}^{n}F\left(x_{(i)}|\alpha,\beta\right)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log(1-(F\left(x_{(1-i+n)}|\alpha,\beta\right))), \tag{4.25}$$

These estimators can also be obtained by solving the following non-linear equations

$$-2\sum_{i=1}^{n}\frac{\exp\left\{-\left(\frac{\beta}{x_{(i)}}\right)^{\alpha}\right\}\log\left(\frac{\beta}{x_{(i)}}\right)\left(\frac{\beta}{x_{(i)}}\right)^{\alpha}}{1-\exp\left\{-\left(\frac{\beta}{x_{(i)}}\right)^{\alpha}\right\}} -\frac{1}{n}\sum_{i=1}^{n}\frac{\exp\left\{-\left(\frac{\beta}{x_{(i)}}\right)^{\alpha}-\left(\frac{\beta}{x_{(1-i+n)}}\right)^{\alpha}\right\}\log\left(\frac{\beta}{x_{(1-i+n)}}\right)\left(\frac{\beta}{x_{(1-i+n)}}\right)^{\alpha}}{\left(1-\exp\left\{-\left(\frac{\beta}{x_{(i)}}\right)^{\alpha}\right\}\right)^{2}} = 0, \tag{4.26}$$

$$-2\sum_{i=1}^{n}\frac{\exp\left\{-\left(\frac{\beta}{x_{(i)}}\right)^{\alpha}\right\}\alpha\left(\frac{\beta}{x_{(i)}}\right)^{-1+\alpha}}{\left(1-\exp\left\{-\left(\frac{\beta}{x_{(i)}}\right)^{\alpha}\right\}\right)x_{(i)}} -\frac{1}{n}\sum_{i=1}^{n}\frac{\exp\left\{-\left(\frac{\beta}{x_{(i)}}\right)^{\alpha}-\left(\frac{\beta}{x_{(1-i+n)}}\right)^{\alpha}\right\}\alpha\left(\frac{\beta}{x_{(1-i+n)}}\right)^{-1+\alpha}}{\left(1-\exp\left\{-\left(\frac{\beta}{x_{(i)}}\right)^{\alpha}\right\}\right)^{2}x_{(1-i+n)}} = 0. \tag{4.27}$$

The ADE and RTADE of PCI C_{py} for FD are

$$\hat{C}_{py}^{ADE} = \frac{1}{p_0}\left[\exp\{-(\hat{\beta}_{ADE}/USL)^{\hat{\alpha}_{ADE}}\} - \exp\{-(\hat{\beta}_{ADE}/LSL)^{\hat{\alpha}_{ADE}}\}\right] \tag{4.28}$$

$$\hat{C}_{py}^{RTADE} = \frac{1}{p_0}\left[\exp\{-(\hat{\beta}_{RTADE}/USL)^{\hat{\alpha}_{RTADE}}\} - \exp\{-(\hat{\beta}_{RTADE}/LSL)^{\hat{\alpha}_{RTADE}}\}\right] \tag{4.29}$$

Similarly, the corresponding ADE and RTADE of PCI C_{Npmk} for FD are

$$\hat{C}_{Npmk}^{ADE} = \frac{\min(USL-M,M-LSL)}{3\sqrt{\left(\frac{\hat{\beta}_{ADE}\left(-\log\left(\frac{3}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{ADE}}\right)}-\log\left(\frac{1}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{ADE}}\right)}\right)}{6}\right)^{2}+(M-T)^{2}}} \tag{4.30}$$

$$\hat{C}_{Npmk}^{RTADE} = \frac{\min(USL-M,M-LSL)}{3\sqrt{\left(\frac{\hat{\beta}_{RTADE}\left(-\log\left(\frac{3}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{RTADE}}\right)}-\log\left(\frac{1}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{RTADE}}\right)}\right)}{6}\right)^{2}+(M-T)^{2}}}. \tag{4.31}$$

4.2 Bayesian estimator

This section considers the BERP and BEJP for estimation of unknown parameters of the FD. Abbas and Tang [50] derived Jeffrey's and reference priors for FD, which are as follows:

$$\pi_J(\alpha,\beta) \propto \sqrt{|I(\alpha,\beta)|} \propto \frac{1}{\beta}$$
$$\pi_R(\alpha,\beta) \propto \sqrt{|I(\alpha,\beta)|} \propto \frac{1}{\alpha\beta}$$

The joint posterior distributions using Jeffrey's and reference prior can be written as

$$\pi_J(\alpha,\beta\mid \boldsymbol{x}) = \frac{\alpha^n\beta^{n\alpha-1}\prod_{i=1}^{n}x_i^{-(\alpha+1)}\exp[-\sum_{i=1}^{n}(\beta/x_i)^{\alpha}]}{\int_0^{\infty}\int_0^{\infty}\alpha^n\beta^{n\alpha-1}\prod_{i=1}^{n}x_i^{-(\alpha+1)}\exp[-\sum_{i=1}^{n}(\beta/x_i)^{\alpha}]d\alpha d\beta}. \tag{4.32}$$

$$\pi_R(\alpha,\beta\mid \boldsymbol{x}) = \frac{\alpha^{n-1}\beta^{n\alpha-1}\prod_{i=1}^{n}x_i^{-(\alpha+1)}\exp[-\sum_{i=1}^{n}(\beta/x_i)^{\alpha}]}{\int_0^{\infty}\int_0^{\infty}\alpha^{n-1}\beta^{n\alpha-1}\prod_{i=1}^{n}x_i^{-(\alpha+1)}\exp[-\sum_{i=1}^{n}(\beta/x_i)^{\alpha}]d\alpha d\beta}. \tag{4.33}$$

The Bayesian estimators of model parameters employing squared error loss function are obtained using Laplace approximation. Which is accessible through the LearnBayes package [51]. The BERP and BEJP of PCI-C_{py} are given as

TABLE 2 Estimates of C_{py} when $\alpha = 1.2$ and $\beta = 2.2$.

n	Method	$C_{py} \downarrow$	$\hat{C}_{py}$	$\sqrt{\beta}_1$	β_2	BCI	Width	MSE
10	MLE	0.56605	0.57012	0.05505	2.9867	(0.53870, 0.60170)	0.06300	2.751e-04
	LSE		0.56371	0.17548	4.2374	(0.49364, 0.63604)	0.14240	1.188e-03
	WLSE		0.57643	−0.03607	3.0715	(0.39799, 0.74646)	0.34847	7.639e-03
	CVME		0.58262	0.65055	4.0189	(0.54003, 0.64051)	0.10048	8.963e-04
	ADE		1.03800	−2.97280	9.9895	(0.94149, 1.05040)	0.11087	2.239e-01
	RTADE		0.49594	−0.58147	3.1611	(0.35001, 059320)	0.24319	8.936e-03
	BEJP		0.57407	0.11641	3.0179	(0.54335, 0.60675)	0.06340	3.271e-04
	BERP		0.56837	0.13821	3.0136	(0.53857, 0.60074)	0.06217	2.533e-04
15	MLE	0.56605	0.57032	0.03828	2.9638	(0.54483, 0.59622)	0.05139	1.910e-04
	LSE		0.56379	0.16613	3.9124	(0.50983, 0.62016)	0.11033	7.429e-04
	WLSE		0.57640	-0.07305	3.0157	(0.43520, 0.71360)	0.27840	5.141e-03
	CVME		0.58255	0.54148	3.7550	(0.54752, 0.62795)	0.08043	6.814e-04
	ADE		1.03890	-2.43410	8.5448	(0.97456, 1.05220)	0.07765	2.242e-01
	RTADE		0.49726	-0.51604	3.1970	(0.38269, 0.58303)	0.20035	7.358e-03
	BEJP		0.57421	0.11159	3.0300	(0.54916, 0.60052)	0.05136	2.389e-04
	BERP		0.56854	0.10858	3.0362	(0.54421, 0.59437)	0.05016	1.721e-04
20	MLE	0.56605	0.57018	0.13914	3.0398	(0.54822, 0.59340)	0.04518	1.484e-04
	LSE		0.56346	0.10367	3.5655	(0.51661, 0.61272)	0.09610	5.718e-04
	WLSE		0.57662	−0.05736	2.9682	(0.45567,0.69306)	0.23739	3.835e-03
	CVME		0.58254	0.47585	3.3830	(0.55135, 0.62085)	0.06950	5.850e-04
	ADE		1.03800	−2.02890	6.7161	(0.99031, 1.05210)	0.06176	2.233e-01
	RTADE		0.49773	−0.42974	3.0861	(0.40093, 0.57569)	0.17476	6.668e-03
	BEJP		0.57402	0.06770	2.9974	(0.54668, 0.59058)	0.04390	1.899e-04
	BERP		0.56826	0.07117	3.0529	(0.55250, 0.59630)	0.04380	1.310e-04
25	MLE	0.56605	0.57027	0.05150	2.9647	(0.55042, 0.59060)	0.04018	1.240e-04
	LSE		0.56347	0.05371	3.7666	(0.52062, 0.60881)	0.08819	4.688e-04
	WLSE		0.57577	−0.03274	2.9772	(0.46858, 0.68117)	0.21259	3.042e-03
	CVME		0.58259	0.42236	3.5440	(0.55479, 0.61695)	0.06215	5.217e-04
	ADE		1.03830	−1.91420	6.5133	(0.96746, 1.05200)	0.08451	2.235e-01
	RTADE		0.49810	−0.42952	3.2787	(0.41468, 0.56691)	0.15223	6.169e-03
	BEJP		0.57406	0.09228	3.0006	(0.54885, 0.58821)	0.03936	1.642e-04
	BERP		0.56850	0.04138	2.9985	(0.55522, 0.59375)	0.03853	1.069e-04
30	MLE	0.56605	0.57035	0.06506	2.9802	(0.55250, 0.58902)	0.03652	1.058e-04
	LSE		0.56306	0.04006	3.4653	(0.52458, 0.60266)	0.07809	3.872e-04
	WLSE		0.57631	-0.05145	3.0012	(0.48017, 0.67325)	0.19308	2.551e-03
	CVME		0.58250	0.37966	3.3320	(0.55668, 0.61267)	0.05599	4.756e-04
	ADE		1.03840	−1.74070	5.9311	(0.98076, 1.05180)	0.07108	2.235e-01
	RTADE		0.49719	−0.36023	3.1541	(0.41987, 0.56143)	0.14156	6.064e-03
	BEJP		0.57399	0.04908	3.0142	(0.55628, 0.59201)	0.03573	1.466e-04
	BERP		0.56831	0.05177	2.9213	(0.55074, 0.58646)	0.03572	8.828e-05
50	MLE	0.56605	0.57034	0.06676	3.0517	(0.55645, 0.58479)	0.02834	7.065e-05
	LSE		0.56339	0.05829	3.3026	(0.53315, 0.59361)	0.06047	2.393e-04
	WLSE		0.57583	−0.06247	3.0646	(0.49931, 0.65055)	0.15124	1.593e-03
	CVME		0.58243	0.33132	3.2584	(0.56253, 0.60581)	0.04328	3.897e-04
	ADE		1.03850	−1.34450	4.7413	(1.00518, 1.05160)	0.04638	2.234e-01
	RTADE		0.49749	−0.27802	3.0563	(0.43892, 0.54865)	0.10974	5.495e-03
	BEJP		0.57389	0.07766	3.0868	(0.55443, 0.58262)	0.02819	1.119e-04
	BERP		0.56841	0.07173	3.0823	(0.56011, 0.58774)	0.02764	5.596e-05

TABLE 3 Estimates of C_{py} when $\alpha = 1$ and $\beta = 3.5$.

n	Method	$C_{py}\downarrow$	$\hat{C}_{PY}$	$\sqrt{\beta}_1$	β_2	BCI	Width	MSE
10	MLE	0.40702	0.40530	0.09179	2.9434	(0.37231, 0.43978)	0.06748	3.047e-04
	LSE		0.25497	0.11220	3.0198	(0.11768, 0.40687)	0.28920	2.887e-02
	WLSE		0.32178	0.05405	3.1500	(0.17873, 0.46602)	0.28728	1.262e-02
	CVME		0.28388	0.18455	3.0487	(0.13357, 0.44327)	0.30969	2.142e-02
	ADE		0.97504	−1.06205	4.1864	(0.82950, 1.04720)	0.21773	3.264e-01
	RTADE		0.15256	0.25335	2.9100	(0.03756, 0.29251)	0.25494	6.926e-02
	BEJP		0.41245	0.13347	3.0518	(0.37974, 0.44734)	0.06760	3.241e-04
	BERP		0.40897	0.07903	2.9902	(0.37598, 0.44328)	0.06729	2.945e-04
15	MLE	0.40702	0.40521	0.13431	2.9619	(0.37804, 0.43426)	0.05622	2.052e-04
	LSE		0.25507	0.09782	2.9428	(0.13955, 0.37868)	0.23913	2.694e-02
	WLSE		0.32151	0.07469	3.0600	(0.20734, 0.44310)	0.23575	1.087e-02
	CVME		0.28366	0.07301	2.9725	(0.15744, 0.41294)	0.25550	1.945e-02
	ADE		0.97533	−0.89276	3.9387	(0.85502, 1.04380)	0.18882	3.256e-01
	RTADE		0.15225	0.18680	2.8800	(0.05417, 0.26454)	0.21036	6.788e-02
	BEJP		0.41234	0.08246	3.0949	(0.38490, 0.44060)	0.05570	2.275e-04
	BERP		0.40855	0.08261	2.9546	(0.38198, 0.43645)	0.05447	1.967e-04
20	MLE	0.40702	0.40576	0.05108	3.0039	(0.38212, 0.42997)	0.04785	1.499e-04
	LSE		0.25520	0.04696	2.9397	(0.15094, 0.36219)	0.21125	2.601e-02
	WLSE		0.32273	0.07678	3.1424	(0.22413, 0.42797)	0.20384	9.752e-03
	CVME		0.28288	0.05092	2.9926	(0.17317, 0.39429)	0.22112	1.864e-02
	ADE		0.97583	−0.70435	3.4445	(0.87686, 1.04130)	0.16446	3.254e-01
	RTADE		0.15168	0.18961	2.9300	(0.06528, 0.24658)	0.18129	6.739e-02
	BEJP		0.41213	0.05399	3.0860	(0.38828, 0.43639)	0.04810	1.732e-04
	BERP		0.40839	0.08971	2.9578	(0.38561, 0.43199)	0.04638	1.428e-04
25	MLE	0.40702	0.40553	0.07135	2.9954	(0.38493, 0.42681)	0.04188	1.192e-04
	LSE		0.25569	0.07415	2.9542	(0.16253, 0.35166)	0.18913	2.522e-02
	WLSE		0.32271	0.04750	3.0077	(0.23407, 0.41455)	0.18047	9.199e-03
	CVME		0.28334	0.07169	3.0002	(0.18825, 0.38419)	0.19594	1.783e-02
	ADE		0.97504	-0.65186	3.3719	(0.88733, 1.03650)	0.14920	3.242e-01
	RTADE		0.15215	0.11416	2.9100	(0.07299, 0.23595)	0.16295	6.673e-02
	BEJP		0.41201	0.07108	2.9755	(0.39134, 0.43362)	0.04282	1.401e-04
	BERP		0.40850	0.07053	2.9804	(0.38831, 0.43016)	0.04186	1.157e-04
30	MLE	0.40702	0.40571	0.07454	2.9292	(0.38654, 0.42562)	0.03908	1.017e-04
	LSE		0.25631	0.03988	2.9900	(0.17071, 0.34454)	0.17383	2.471e-02
	WLSE		0.32137	0.07787	3.0556	(0.24143, 0.40638)	0.16495	9.092e-03
	CVME		0.28245	0.02977	3.0077	(0.19400, 0.37411)	0.18011	1.763e-02
	ADE		0.97525	−0.57235	3.3980	(0.89689, 1.0342)	0.13728	3.242e-01
	RTADE		0.15208	0.12479	3.0000	(0.08043, 0.22961)	0.14918	6.646e-02
	BEJP		0.41239	0.04366	2.9108	(0.38942, 0.42815)	0.03873	1.266e-04
	BERP		0.40843	0.08236	3.0145	(0.39319, 0.43186)	0.03867	9.867e-05
50	MLE	0.40702	0.40536	0.07219	2.9969	(0.39027, 0.42082)	0.03055	6.365e-05
	LSE		0.25586	0.06688	2.9600	(0.18972, 0.32464)	0.13492	2.403e-02
	WLSE		0.32239	0.06065	2.9593	(0.25938, 0.38662)	0.12724	8.217e-03
	CVME		0.28327	0.04923	2.9766	(0.21283, 0.35567)	0.14284	1.663e-02
	ADE		0.97452	−0.45220	3.2079	(0.91546, 1.02230)	0.10685	3.228e-01
	RTADE		0.15261	0.09032	3.0010	(0.09588, 0.21205)	0.11617	6.561e-02
	BEJP		0.41239	0.05742	3.0480	(0.39410, 0.42406)	0.02996	8.781e-05
	BERP		0.40884	0.10077	3.0134	(0.39768, 0.42743)	0.02975	6.117e-05

TABLE 4 Estimates of C_{py} when $\alpha = 1$ and $\beta = 2$.

n	Method	C_{py} ↓	$\hat{C}_{py}$	$\sqrt{\beta}_1$	β_2	BCI	Width	MSE
10	MLE	0.49602	0.49701	−0.03060	2.8877	(0.46787, 0.52574)	0.05787	2.222e-04
	LSE		0.56196	0.08620	3.0879	(0.40218, 0.72906)	0.32688	1.132e-02
	WLSE		0.52948	0.18783	3.0937	(0.39098, 0.67552)	0.28454	6.473e-03
	CVME		0.53739	−0.41261	3.8640	(0.34077, 0.69931)	0.35854	9.703e-03
	ADE		1.01710	−1.86390	6.5701	(0.85332, 1.05210)	0.19877	2.750e-01
	RTADE		0.34039	−0.19153	2.8225	(0.19262, 0.47794)	0.28532	3.004e-02
	BEJP		0.49976	−0.02051	3.0222	(0.47126, 0.52805)	0.05679	2.224e-04
	BERP		0.49484	-0.05949	2.8928	(0.46590, 0.52240)	0.05649	2.115e-04
15	MLE	0.49602	0.49683	0.00178	2.9213	(0.47292, 0.52046)	0.04754	1.475e-04
	LSE		0.56262	0.07329	3.0148	(0.43180, 0.69814)	0.26634	9.076e-03
	WLSE		0.53050	0.17292	3.1391	(0.41711, 0.65022)	0.23311	4.756e-03
	CVME		0.53787	−0.32895	3.5243	(0.37901, 0.67590)	0.29689	7.224e-03
	ADE		1.01680	−1.39545	4.7228	(0.90012, 1.05190)	0.15175	2.734e-01
	RTADE		0.34021	−0.18954	2.9669	(0.21063, 0.45521)	0.24458	2.822e-02
	BEJP		0.49976	−0.01053	3.0071	(0.47633, 0.52348)	0.04716	1.580e-04
	BERP		0.49445	-0.01627	2.9659	(0.47141, 0.51684)	0.04543	1.400e-04
20	MLE	0.49602	0.49732	−0.01473	2.9575	(0.47715, 0.51762)	0.04047	1.099e-04
	LSE		0.56165	0.05548	3.0004	(0.44911, 0.67787)	0.22876	7.721e-03
	WLSE		0.52884	0.15966	3.1078	(0.43344, 0.63187)	0.19844	3.670e-03
	CVME		0.53905	−0.30669	3.2706	(0.40199, 0.65576)	0.25377	5.884e-03
	ADE		1.01630	−1.25788	4.6636	(0.92397, 1.05160)	0.12764	2.724e-01
	RTADE		0.33932	−0.18655	2.9635	(0.22767, 0.44118)	0.21351	2.754e-02
	BEJP		0.49975	-0.04226	2.9773	(0.47433, 0.51491)	0.04058	1.203e-04
	BERP		0.49448	-0.00757	2.9855	(0.47956, 0.51967)	0.04012	1.079e-04
25	MLE	0.49602	0.49702	−0.02031	2.9575	(0.47907, 0.51506)	0.03600	8.613e-05
	LSE		0.56165	0.04774	3.0970	(0.46030, 0.66525)	0.20495	7.016e-03
	WLSE		0.52922	0.15166	3.0231	(0.44195, 0.62382)	0.18187	3.245e-03
	CVME		0.53798	−0.25070	3.3359	(0.41721, 0.64190)	0.22468	4.987e-03
	ADE		1.01650	−1.14438	4.3247	(0.92265, 1.05140)	0.12880	2.723e-01
	RTADE		0.33914	−0.14414	2.9208	(0.24134, 0.42927)	0.18793	2.693e-02
	BEJP		0.49967	0.00180	3.0083	(0.48170, 0.51739)	0.03568	9.649e-05
	BERP		0.49445	−0.01019	2.9687	(0.47672, 0.51209)	0.03537	8.436e-05
30	MLE	0.49602	0.49706	−0.01300	2.9251	(0.48046, 0.51350)	0.03304	7.340e-05
	LSE		0.56140	0.03890	3.0190	(0.46892, 0.65733)	0.18841	6.607e-03
	WLSE		0.52865	0.15259	3.0194	(0.45009, 0.61419)	0.16410	2.803e-03
	CVME		0.53882	-0.25250	3.3717	(0.42905, 0.63682)	0.20777	4.573e-03
	ADE		1.01640	−0.99069	3.7646	(0.93525, 1.05130)	0.11603	2.720e-01
	RTADE		0.33980	−0.12655	2.9507	(0.25281, 0.42237)	0.16956	2.632e-02
	BEJP		0.49982	−0.05797	2.9793	(0.47822, 0.51101)	0.03279	8.300e-05
	BERP		0.49445	0.02200	3.0249	(0.48354, 0.51569)	0.03216	7.249e-05
50	MLE	0.49602	0.49696	0.02456	2.9154	(0.48419, 0.50987)	0.02568	4.442e-05
	LSE		0.56150	0.04580	3.0235	(0.49037, 0.63516)	0.14479	5.663e-03
	WLSE		0.52988	0.10008	3.1144	(0.46792, 0.59535)	0.12743	2.202e-03
	CVME		0.53858	−0.23460	3.1342	(0.45456, 0.61347)	0.15892	3.456e-03
	ADE		1.01620	−0.78818	3.6470	(0.95791, 1.05080)	0.09287	2.713e-01
	RTADE		0.33990	−0.12557	2.9999	(0.27112, 0.40491)	0.13379	2.555e-02
	BEJP		0.49989	−0.00213	3.0138	(0.48715, 0.51239)	0.02525	5.765e-05
	BERP		0.49471	0.02875	3.0059	(0.48220, 0.50739)	0.02519	4.322e-05

TABLE 5 Estimates for C_{py} when $\alpha = 1$ and $\beta = 4$.

n	Method	$C_{py} \downarrow$	$\hat{C}_{py}$	$\sqrt{\beta}_1$	β_2	BCI	Width	MSE
10	MLE	0.36796	0.36568	0.11034	2.9586	(0.33273, 0.40048)	0.06775	3.063e-04
10	LSE	0.36796	0.19080	0.17514	2.9241	(0.07213, 0.32634)	0.25421	3.591e-02
10	WLSE	0.36796	0.25480	−0.19327	2.9046	(0.13651, 0.36179)	0.22528	1.620e-02
10	CVME	0.36796	0.20682	0.03828	2.7957	(0.08103, 0.33429)	0.25326	3.017e-02
10	ADE	0.36796	0.99052	−1.37873	4.8897	(0.84687, 1.04620)	0.19937	3.906e-01
10	RTADE	0.36796	0.09641	0.46132	3.1155	(0.00000, 0.21365)	0.21365	7.679e-02
10	BEJP	0.36796	0.37330	0.13874	3.0160	(0.34056, 0.40848)	0.06792	3.260e-04
10	BERP	0.36796	0.37055	0.09407	2.9998	(0.33802, 0.40461)	0.06659	2.993e-04
15	MLE	0.36796	0.36562	0.13610	2.9744	(0.33872, 0.39428)	0.05556	2.059e-04
15	LSE	0.36796	0.19137	0.15969	3.0533	(0.08399, 0.30325)	0.21926	3.424e-02
15	WLSE	0.36796	0.25555	−0.13407	2.8802	(0.16126, 0.34356)	0.18231	1.487e-02
15	CVME	0.36796	0.20533	0.05215	2.8837	(0.10439, 0.30876)	0.20437	2.929e-02
15	ADE	0.36796	0.99049	−1.19568	4.6797	(0.88114, 1.04380)	0.16265	3.896e-01
15	RTADE	0.36796	0.09675	0.32440	2.9335	(0.019291, 0.19226)	0.17297	7.558e-02
15	BEJP	0.36796	0.37317	0.10219	3.1256	(0.34625, 0.40150)	0.05526	2.270e-04
15	BERP	0.36796	0.37017	0.10855	2.9891	(0.34366, 0.39853)	0.05487	2.012e-02
20	MLE	0.36796	0.36613	0.05804	2.9748	(0.34253, 0.39057)	0.04804	1.514e-04
20	LSE	0.36796	0.19096	0.12178	3.0157	(0.09858, 0.28651)	0.18792	3.361e-02
20	WLSE	0.36796	0.25608	-0.13123	2.9617	(0.17517, 0.33128)	0.15611	1.415e-02
20	CVME	0.36796	0.20567	0.02509	2.9131	(0.11638, 0.29785)	0.18147	2.853e-02
20	ADE	0.36796	0.99109	−0.97555	3.9295	(0.90026, 1.04090)	0.14065	3.898e-01
20	RTADE	0.36796	0.09682	0.27134	2.9437	(0.02343, 0.17752)	0.15409	7.503e-02
20	BEJP	0.36796	0.37292	0.07705	3.0909	(0.34919, 0.39725)	0.04806	1.721e-04
20	BERP	0.36796	0.36997	0.09578	2.9808	(0.34707, 0.39352)	0.04645	1.450e-04
25	MLE	0.36796	0.36595	0.07873	2.9938	(0.34534, 0.38737)	0.04203	1.212e-04
25	LSE	0.36796	0.19174	0.13076	2.9465	(0.11190, 0.27817)	0.16626	3.286e-02
25	WLSE	0.36796	0.25662	−0.13167	2.9892	(0.18289, 0.32521)	0.14232	1.372e-02
25	CVME	0.36796	0.20572	0.02878	2.8879	(0.12642, 0.28764)	0.16122	2.804e-02
25	ADE	0.36796	0.99102	−0.85147	3.6568	(0.91095, 1.03850)	0.12751	3.894e-01
25	RTADE	0.36796	0.09614	0.25230	2.9488	(0.03261, 0.16856)	0.13595	7.511e-02
25	BEJP	0.36796	0.37281	0.07700	2.9670	(0.35213, 0.39436)	0.04223	1.396e-04
25	BERP	0.36796	0.37011	0.08075	2.9726	(0.34987, 0.39172)	0.04185	1.188e-04
30	MLE	0.36796	0.36614	0.08563	2.9404	(0.34702, 0.38620)	0.03919	1.035e-04
30	LSE	0.36796	0.19195	0.10708	3.0941	(0.116963, 0.27152)	0.15456	3.251e-02
30	WLSE	0.36796	0.25564	−0.10124	3.0373	(0.18900, 0.31931)	0.13031	1.371e-02
30	CVME	0.36796	0.20520	0.00810	2.9982	(0.13087, 0.27997)	0.14910	2.795e-02
30	ADE	0.36796	0.99068	−0.79111	3.7523	(0.91983, 1.03660)	0.11677	3.888e-01
30	RTADE	0.36796	0.09630	0.20794	3.0644	(0.03863, 0.15966)	0.12102	7.479e-02
30	BEJP	0.36796	0.37322	0.05016	2.9017	(0.35378, 0.39289)	0.03911	1.269e-04
30	BERP	0.36796	0.37002	0.08875	3.0109	(0.35099, 0.38996)	0.03897	1.013e-04
50	MLE	0.36796	0.36576	0.07466	3.0422	(0.35092, 0.38151)	0.03058	6.590e-05
50	LSE	0.36796	0.19182	0.06632	3.0537	(0.133439, 0.2521)	0.11866	3.195e-02
50	WLSE	0.36796	0.25594	-0.09146	2.9899	(0.20423, 0.30540)	0.10116	1.321e-02
50	CVME	0.36796	0.20511	0.04087	2.9279	(0.14840, 0.26474)	0.11635	2.741e-02
50	ADE	0.36796	0.99113	−0.65469	3.4911	(0.93607, 1.03040)	0.09428	3.889e-01
50	RTADE	0.36796	0.09678	0.19481	2.9655	(0.05066, 0.14786)	0.09720	7.416e-02
50	BEJP	0.36796	0.37320	0.06430	3.0493	(0.35844, 0.38832)	0.02989	8.706e-05
50	BERP	0.36796	0.37042	0.10585	3.0258	(0.35582, 0.38566)	0.02984	6.430e-05

TABLE 6 Estimates of C_{Npmk} when $\alpha = 1$ and $\beta = 2.5$.

n	Method	$C_{Npmk} \downarrow$	$\hat{C}_{Npmk}$	$\sqrt{\beta}_1$	β_2	BCI	Width	MSE
10	MLE	0.09455	0.13953	0.36470	3.1193	(0.01376, 0.28910)	0.27533	0.00704
	LSE		0.15226	0.78429	3.9585	(0.01193, 0.36018)	0.34825	0.01102
	WLSE		0.27343	1.66631	4.2491	(0.01456, 0.90664)	0.89208	0.09107
	CVME		0.14713	0.33556	3.0434	(0.01209, 0.30671)	0.29461	0.00855
	ADE		0.66038	1.03112	4.0391	(0.04223, 1.84680)	1.80454	0.56854
	RTADE		0.16432	1.91069	5.6468	(0.01631, 0.50793)	0.49162	0.02020
	BEJP		0.14120	0.38805	3.0782	(0.01599, 0.29734)	0.28134	0.00734
	BERP		0.13859	0.36549	3.1273	(0.01479, 0.28694)	0.27215	0.00688
15	MLE	0.09455	0.13932	0.28261	3.0187	(0.03607, 0.26155)	0.22548	0.00531
	LSE		0.15161	0.74879	4.2557	(0.03215, 0.31622)	0.28407	0.00859
	WLSE		0.27543	1.29449	4.9068	(0.04192, 0.75112)	0.70920	0.07149
	CVME		0.14754	0.25259	2.9692	(0.03438, 0.27495)	0.24057	0.00656
	ADE		0.66514	0.85420	3.7272	(0.08776, 1.60610)	1.51831	0.49357
	RTADE		0.16235	1.61139	5.4367	(0.03254, 0.41462)	0.38208	0.01472
	BEJP		0.14183	0.27228	3.0392	(0.03531, 0.26153)	0.22621	0.00567
	BERP		0.13853	0.30617	3.0872	(0.03499, 0.25871)	0.22371	0.00523
20	MLE	0.09455	0.13880	0.23714	2.9623	(0.04723, 0.24261)	0.19538	0.00450
	LSE		0.15107	0.63144	3.8170	(0.04592, 0.28949)	0.24358	0.00711
	WLSE		0.27609	1.08803	4.2963	(0.05741, 0.68140)	0.62399	0.06191
	CVME		0.14654	0.24878	3.0479	(0.04862, 0.25746)	0.20884	0.00552
	ADE		0.66295	0.68044	3.3094	(0.12926, 1.45640)	1.32714	0.44743
	RTADE		0.16334	1.32521	5.2102	(0.04839, 0.37206)	0.32368	0.01223
	BEJP		0.14088	0.23669	2.9742	(0.04881, 0.24612)	0.19731	0.00469
	BERP		0.13850	0.25072	3.0102	(0.04769, 0.24126)	0.19357	0.00440
25	MLE	0.09455	0.13879	0.24068	3.1166	(0.05692, 0.22977)	0.17385	0.00395
	LSE		0.15205	0.53035	3.5591	(0.05452, 0.27410)	0.21958	0.00645
	WLSE		0.27727	1.03943	4.1911	(0.07113, 0.64561)	0.57448	0.05733
	CVME		0.14680	0.19582	2.9780	(0.05844, 0.24370)	0.18526	0.00500
	ADE		0.66551	0.63101	3.4219	(0.16243, 1.34770)	1.18527	0.42458
	RTADE		0.16445	1.25068	5.0149	(0.05621, 0.36619)	0.30997	0.01126
	BEJP		0.14176	0.30045	3.0557	(0.05955, 0.23802)	0.17847	0.00431
	BERP		0.13891	0.21879	3.0615	(0.05681, 0.23020)	0.17340	0.00393
30	MLE	0.09455	0.13882	0.21285	3.1236	(0.06233, 0.22116)	0.15883	0.00364
	LSE		0.15187	0.45674	3.3196	(0.06296, 0.26255)	0.19959	0.00592
	WLSE		0.27621	0.93131	4.0583	(0.07769, 0.60792)	0.53023	0.05281
	CVME		0.14719	0.22773	2.9944	(0.06673, 0.23637)	0.16964	0.00468
	ADE		0.66206	0.54354	3.2394	(0.19099, 1.29500)	1.10398	0.40351
	RTADE		0.16383	1.10622	4.5156	(0.06379, 0.34576)	0.28197	0.00987
	BEJP		0.14191	0.22424	3.0324	(0.06602, 0.22748)	0.16147	0.00394
	BERP		0.13852	0.21154	3.0490	(0.06357, 0.22116)	0.15759	0.00357
50	MLE	0.09455	0.13856	0.17490	2.9501	(0.07919, 0.20378)	0.12459	0.00297
	LSE		0.15174	0.36482	3.2641	(0.08096, 0.23697)	0.15601	0.00484
	WLSE		0.27535	0.74077	3.6606	(0.10917, 0.51851)	0.40934	0.04429
	CVME		0.14728	0.17589	2.9332	(0.08456, 0.21594)	0.13139	0.00391
	ADE		0.66504	0.42852	3.1434	(0.27343, 1.14640)	0.87302	0.37583
	RTADE		0.16320	0.84069	3.7693	(0.08159, 0.29251)	0.21091	0.00777
	BEJP		0.14196	0.14256	3.0218	(0.08166, 0.20587)	0.12421	0.00327
	BERP		0.13900	0.15142	3.0044	(0.07997, 0.20281)	0.12283	0.00297

TABLE 7 Estimates of C_{Npmk} when $\alpha = 1.6$ and $\beta = 1.7$.

n	Method	$C_{Npmk} \downarrow$	$\hat{C}_{Npmk}$	$\sqrt{\beta}_1$	β_2	BCI	Width	MSE
10	MLE	0.06594	0.70153	0.04900	2.8845	(0.53020, 0.87813)	0.34793	0.41186
	LSE		0.76866	2.10380	6.1934	(0.52897, 1.37259)	0.84362	0.54022
	WLSE		1.18990	1.92985	5.0223	(0.55272, 3.10470)	2.55200	1.82390
	CVME		0.88657	2.94090	5.3842	(0.54136, 2.52560)	1.98425	0.98903
	ADE		2.11950	0.63642	3.2982	(0.80043, 4.01100)	3.21050	4.92480
	RTADE		0.80604	1.55667	5.3036	(0.54629, 1.32340)	0.77706	0.58650
	BEJP		0.70932	0.06076	2.9270	(0.53447, 0.89010)	0.35563	0.42212
	BERP		0.69954	0.06605	2.9406	(0.53110, 0.87279)	0.34170	0.40911
15	MLE	0.06594	0.70155	0.07823	3.0091	(0.56182, 0.84667)	0.28485	0.40923
	LSE		0.77023	1.71376	6.5295	(0.56242, 1.21072)	0.64831	0.52730
	WLSE		1.18630	1.53920	5.5949	(0.59654, 2.76760)	2.17100	1.62250
	CVME		0.88474	2.50740	4.5150	(0.57800, 1.96300)	1.38504	0.88311
	ADE		2.13660	0.57467	3.3212	(0.97284, 3.69920)	2.72640	4.77420
	RTADE		0.80764	1.17270	4.7734	(0.58485, 1.19430)	0.60946	0.57517
	BEJP		0.70788	0.03705	2.9200	(0.56611, 0.85217)	0.28606	0.41748
	BERP		0.69916	0.04418	2.9470	(0.56104, 0.84057)	0.27954	0.40608
20	MLE	0.06594	0.70235	0.02737	2.9104	(0.58063, 0.82588)	0.24525	0.40895
	LSE		0.76874	1.52761	5.8188	(0.58247, 1.13963)	0.55716	0.51734
	WLSE		1.19090	1.32374	4.7077	(0.62395, 2.54830)	1.92440	1.54620
	CVME		0.88485	2.08890	4.2080	(0.59518, 1.68810)	1.09290	0.82612
	ADE		2.12760	0.48419	3.1855	(1.10305, 3.44750)	2.34440	4.60790
	RTADE		0.80761	1.00244	4.4125	(0.60046, 1.13680)	0.53632	0.56917
	BEJP		0.70929	0.05036	2.9331	(0.58521, 0.83369)	0.24849	0.41788
	BERP		0.69966	0.04989	2.9509	(0.57950, 0.82272)	0.24322	0.40546
25	MLE	0.06594	0.70261	0.06433	3.0153	(0.59606, 0.81409)	0.21803	0.40846
	LSE		0.76715	1.40514	5.5511	(0.59732, 1.10873)	0.51141	0.51002
	WLSE		1.18760	1.19803	4.4456	(0.64949, 2.35030)	1.70080	1.48190
	CVME		0.88257	1.91560	4.5144	(0.60868, 2.04250)	1.43384	0.79120
	ADTE		2.12010	0.43055	3.1391	(1.19456, 3.26940)	2.07490	4.50210
	RTADE		0.80806	0.91228	4.0493	(0.62462, 1.10080)	0.47618	0.56576
	BEJP		0.70824	0.00779	3.0144	(0.59604, 0.82048)	0.22443	0.41578
	BERP		0.69908	0.04375	2.9620	(0.59201, 0.80796)	0.21595	0.40392
30	MLE	0.06594	0.70260	0.00100	3.0736	(0.60123, 0.80618)	0.20495	0.40800
	LSE		0.77008	1.21663	4.8143	(0.60656, 1.08910)	0.48255	0.51133
	WLSE		1.19920	1.09693	4.2720	(0.66685, 2.24140)	1.57450	1.47740
	CVME		0.87902	1.74380	3.9542	(0.61608, 1.84880)	1.23268	0.76143
	ADE		2.12450	0.38145	3.1384	(1.25815, 3.18710)	1.92890	4.47830
	RTADE		0.80702	0.91331	4.2439	(0.63572, 1.07120)	0.43552	0.56199
	BEJP		0.70799	0.06095	2.9868	(0.60941, 0.81023)	0.20083	0.41488
	BERP		0.69952	0.04044	2.9881	(0.60175, 0.79953)	0.19779	0.40396
50	MLE	0.06594	0.70260	0.00375	2.9379	(0.62288, 0.78021)	0.15733	0.40693
	LSE		0.76966	0.92481	3.9304	(0.63230, 0.99682)	0.36452	0.50438
	WLSE		1.19470	0.82282	3.6438	(0.72147, 1.98080)	1.25940	1.38740
	CVME		0.88226	1.32040	4.5029	(0.64110, 1.45050)	0.80938	0.72926
	ADE		2.12650	0.29756	3.1288	(1.43097, 2.92490)	1.49400	4.39110
	RTADE		0.80974	0.67315	3.7464	(0.66469, 1.00980)	0.34512	0.56114
	BEJP		0.70827	0.03773	2.9884	(0.63086, 0.78671)	0.15586	0.41418
	BERP		0.69893	0.02658	2.9961	(0.62219, 0.77623)	0.15404	0.40222

TABLE 8 Estimates of C_{Npmk} when $\alpha = 1.6$ and $\beta = 2.1$.

n	Method	$C_{Npmk} \downarrow$	$\hat{C}_{Npmk}$	$\sqrt{\beta}_1$	β_2	BCI	Width	MSE
10	MLE	0.06447	0.76520	−0.05003	2.9246	(0.57345, 0.95087)	0.37742	0.50045
	LSE		0.81850	0.89867	4.2260	(0.56583, 1.21000)	0.64418	0.59582
	WLSE		1.13010	2.18072	6.4322	(0.59444, 2.83610)	2.24160	0.50530
	CVME		0.80843	0.28310	3.1917	(0.59317, 1.05497)	0.46179	0.56711
	ADE		2.65660	0.96581	4.1162	(1.12640, 5.25460)	4.12830	0.86150
	RTADE		0.77954	0.04965	2.9953	(0.57525, 0.98999)	0.41473	0.52244
	BEJP		0.77160	-0.02646	2.9416	(0.57862, 0.96259)	0.38397	0.50966
	BERP		0.76141	−0.01763	2.8989	(0.57168, 0.94891)	0.37722	0.49506
15	MLE	0.06447	0.76685	−0.04684	2.9497	(0.60518, 0.91555)	0.31037	0.49955
	LSE		0.81620	0.74592	3.9361	(0.60310, 1.11770)	0.51459	0.58268
	WLSE		1.11940	1.73378	6.3472	(0.64183, 2.31550)	1.67370	0.54150
	CVME		0.81051	0.18594	3.1284	(0.63405, 1.00302)	0.36897	0.56553
	ADE		2.64490	0.71706	3.5806	(1.31630, 4.59160)	3.27530	0.49640
	RTADE		0.78041	0.06998	2.8889	(0.61646, 0.95028)	0.33382	0.52005
	BEJP		0.77384	−0.02554	3.0099	(0.61491, 0.92996)	0.31506	0.50969
	BERP		0.76110	−0.01516	2.9562	(0.60984, 0.91730)	0.30747	0.49156
20	MLE	0.06447	0.76474	−0.01665	2.9672	(0.62627, 0.90196)	0.27570	0.49536
	LSE		0.81837	0.56310	3.4250	(0.62534, 1.07250)	0.44713	0.58137
	WLSE		1.12390	1.47282	5.3546	(0.66532, 2.16600)	1.50070	0.49820
	CVME		0.80825	0.20891	3.1193	(0.65435, 0.97936)	0.32501	0.55991
	ADE		2.65500	0.72309	3.6753	(1.45220, 4.39540)	2.94320	0.49570
	RTADE		0.77960	0.03223	2.9538	(0.63365, 0.92695)	0.29330	0.51703
	BEJP		0.77350	-0.03561	2.9413	(0.63708, 0.90752)	0.27044	0.50759
	BERP		0.76109	−0.01433	2.9773	(0.62683, 0.89429)	0.26746	0.48994
25	MLE	0.06447	0.76595	−0.04300	2.9863	(0.64311, 0.88684)	0.24373	0.49519
	LSE		0.81662	0.54491	3.3650	(0.64402, 1.04300)	0.39898	0.57617
	WLSE		1.12120	1.28389	4.7477	(0.68549, 2.06630)	1.38080	0.49310
	CVME		0.80927	0.14946	3.0232	(0.66974, 0.95898)	0.28924	0.54017
	ADE		2.65860	0.62683	3.5024	(1.54360, 4.18580)	2.64220	0.49030
	RTADE		0.78030	0.08275	3.0305	(0.65132, 0.91397)	0.26265	0.51693
	BEJP		0.77202	−0.02624	2.9754	(0.64868, 0.89473)	0.24604	0.50450
	BERP		0.76147	−0.00103	2.9677	(0.64232, 0.88090)	0.23858	0.48951
30	MLE	0.06447	0.76601	−0.01087	2.9115	(0.65454, 0.87549)	0.22095	0.49541
	LSE		0.81842	0.51678	3.4372	(0.65736, 1.02710)	0.36976	0.57732
	WLSE		1.12170	1.22006	4.6331	(0.70872, 1.96350)	1.25480	0.48970
	CVME		0.80873	0.13459	2.9765	(0.68134, 0.94577)	0.26443	0.55842
	ADE		2.65750	0.53615	3.2677	(1.62500, 4.00850)	2.38350	0.48940
	RTADE		0.78037	0.07493	3.1438	(0.66205, 0.90231)	0.24025	0.51629
	BEJP		0.77432	−0.01194	3.0289	(0.66404, 0.88545)	0.22141	0.50711
	BERP		0.76121	−0.01693	2.9651	(0.65142, 0.86941)	0.21799	0.48854
50	MLE	0.06447	0.76573	0.03407	2.9812	(0.68060, 0.85301)	0.17241	0.49367
	LSE		0.81800	0.41427	3.3362	(0.68925, 0.97190)	0.28265	0.57318
	WLSE		1.12640	0.97227	4.0802	(0.75837, 1.77420)	1.01580	0.48901
	CVME		0.80859	0.13744	3.0367	(0.71136, 0.91330)	0.20195	0.55638
	ADE		2.65370	0.43342	3.2279	(1.82660, 3.66830)	1.84170	0.48940
	RTADE		0.77986	0.06125	2.9552	(0.68746, 0.87480)	0.18734	0.51404
	BEJP		0.77307	−0.05154	3.0901	(0.68593, 0.85740)	0.17146	0.50405
	BERP		0.76142	-0.02131	2.9737	(0.67605, 0.84538)	0.16933	0.48761

TABLE 9 Estimates of C_{Npmk} when $\alpha = 2.1$ and $\beta = 1.6$.

n	Method	$C_{Npmk} \downarrow$	$\hat{C}_{Npmk}$	$\sqrt{\beta}_1$	β_2	BCI	Width	MSE
10	MLE	0.47655	0.52159	0.41323	3.2662	(0.38109, 0.69146)	0.31038	0.23090
	LSE		0.52237	0.43211	3.3533	(0.38299, 0.69148)	0.30848	0.23159
	WLSE		0.73258	3.20300	3.8283	(0.39077, 2.6616)	2.27087	0.89789
	CVME		0.52955	0.39481	3.2154	(0.38489, 0.70508)	0.32018	0.23883
	ADE		0.81543	2.58380	6.2601	(0.39641, 2.3826)	1.98622	0.91452
	RTADE		0.53082	0.48359	3.3701	(0.38389, 0.7166)	0.33271	0.24066
	BEJP		0.52417	0.39720	3.1479	(0.3834, 0.69498)	0.31158	0.23348
	BERP		0.52103	0.40900	3.1885	(0.38383, 0.69076)	0.30693	0.23025
15	MLE	0.47655	0.52163	0.36142	3.1844	(0.4056, 0.65967)	0.25407	0.22883
	LSE		0.52445	0.36119	3.1353	(0.40878, 0.66411)	0.25533	0.23156
	WLSE		0.74171	2.51340	6.5179	(0.41653, 1.986)	1.56946	0.77982
	CVME		0.52867	0.38010	3.2944	(0.41127, 0.66843)	0.25716	0.23569
	ADE		0.81296	2.18060	8.3435	(0.43139, 1.9793)	1.54787	0.80397
	RTADE		0.53097	0.40413	3.1628	(0.40792, 0.68294)	0.27502	0.23856
	BEJP		0.52362	0.35439	3.1544	(0.40812, 0.66204)	0.25392	0.23082
	BERP		0.52140	0.32376	3.1292	(0.40521, 0.65774)	0.25254	0.22860
20	MLE	0.47655	0.52329	0.30568	3.1316	(0.42129, 0.64088)	0.21959	0.22948
	LSE		0.52395	0.32337	3.2239	(0.42157, 0.64472)	0.14108	0.23011
	WLSE		0.73482	2.14570	6.4165	(0.4334, 1.6568)	1.22340	0.68507
	CVME		0.52997	0.31707	3.1633	(0.42624, 0.65139)	0.22515	0.23587
	ADE		0.81264	1.79040	6.3876	(0.45231, 1.7421)	1.28976	0.74580
	RTADE		0.53137	0.36075	3.3029	(0.42238, 0.66036)	0.23798	0.23770
	BEJP		0.52447	0.29892	3.0931	(0.42105, 0.6422)	0.22114	0.23053
	BERP		0.52122	0.28793	3.1155	(0.42022, 0.63795)	0.21773	0.22736
25	MLE	0.47655	0.52211	0.23235	3.0287	(0.42975, 0.62682)	0.19708	0.22764
	LSE		0.52414	0.30262	3.2097	(0.43216, 0.6303)	0.22314	0.22960
	WLSE		0.73886	1.91220	6.6826	(0.44341, 2.1052)	1.66176	0.65213
	CVME		0.52959	0.24340	3.0745	(0.43585, 0.63508)	0.19923	0.23486
	ADE		0.80790	1.70400	6.3697	(0.46904, 1.8337)	1.36469	0.70554
	RTADE		0.53154	0.32597	3.0502	(0.43608, 0.64457)	0.20849	0.23706
	BEJP		0.52451	0.21781	3.0358	(0.42853, 0.62985)	0.20132	0.23000
	BERP		0.52084	0.26573	3.1086	(0.43039, 0.62451)	0.19412	0.22636
30	MLE	0.47655	0.52222	0.23819	3.0753	(0.43771, 0.61769)	0.17998	0.22732
	LSE		0.52343	0.24792	3.0193	(0.43993, 0.61942)	0.19814	0.22847
	WLSE		0.73980	1.75380	5.9867	(0.45367, 1.9053)	1.45165	0.62549
	CVME		0.52952	0.20622	3.0336	(0.44321, 0.62435)	0.18114	0.23436
	ADE		0.81393	1.50990	5.6278	(0.47725, 1.7014)	1.22411	0.69531
	RTADE		0.53137	0.27704	3.0533	(0.44112, 0.63541)	0.19429	0.23645
	BEJP		0.52330	0.21565	3.0883	(0.43894, 0.61768)	0.17873	0.22831
	BERP		0.52107	0.24305	3.0738	(0.43795, 0.61535)	0.17741	0.22617
50	MLE	0.47655	0.52254	0.13895	2.9262	(0.45583, 0.59356)	0.13772	0.22679
	LSE		0.52413	0.22964	3.2282	(0.45667, 0.59775)	0.17949	0.22830
	WLSE		0.73327	1.41750	5.1888	(0.47236, 1.3974)	0.92502	0.55478
	CVME		0.52948	0.19328	3.0662	(0.46251, 0.6029)	0.14039	0.23346
	ADE		0.81536	1.18070	4.6248	(0.51149, 1.4187)	0.90719	0.65512
	RTADE		0.53153	0.20504	2.9924	(0.4602, 0.60997)	0.14977	0.23559
	BEJP		0.52374	0.20226	3.1459	(0.45686, 0.59659)	0.13973	0.22792
	BERP		0.52074	0.19038	3.0383	(0.45519, 0.59291)	0.13772	0.22505

TABLE 10 Lifetime (in months) for failure of 20 electric carts.

0.90	1.50	2.30	3.20	3.90	5.00	6.20	7.50	8.30	10.40
11.10	12.60	15.00	16.30	19.30	22.60	24.80	31.50	38.10	53.00

TABLE 11 The goodness-of-fit test for FD.

Goodness-of-fit criteria	Statistic	p-value
KS test	D = 0.13296	0.8264
CVM test	W^2 =0.08647	0.6604
AD test	A^2 = 0.55865	0.6858

$$\hat{C}_{py}^{BERP} = \frac{1}{p_0}\left[\exp\{-(\hat{\beta}_{BERP}/USL)^{\hat{\alpha}}_{BERP}\} - \exp\{-(\hat{\beta}_{BERP}/LSL)^{\hat{\alpha}_{BERP}}\}\right] \tag{4.34}$$

$$\hat{C}_{py}^{BEJP} = \frac{1}{p_0}\left[\exp\{-(\hat{\beta}_{BEJP}/USL)^{\hat{\alpha}}_{BEJP}\} - \exp\{-(\hat{\beta}_{BEJP}/LSL)^{\hat{\alpha}_{BEJP}}\}\right] \tag{4.35}$$

Similarly, the corresponding BERP and BEJP of PCI C_{Npmk} for FD are

$$\hat{C}_{Npmk}^{BERP} = \frac{\min(USL-M, M-LSL)}{3\sqrt{\left(\frac{\hat{\beta}_{BERP}\left(-\log\left(\frac{3}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{BERP}}\right)} - \log\left(\frac{1}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{BERP}}\right)}\right)}{6}\right)^2 + (M-T)^2}} \tag{4.36}$$

$$\hat{C}_{Npmk}^{BEJP} = \frac{\min(USL-M, M-LSL)}{3\sqrt{\left(\frac{\hat{\beta}_{BEJP}\left(-\log\left(\frac{3}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{BEJP}}\right)} - \log\left(\frac{1}{4}\right)^{-\left(\frac{1}{\hat{\alpha}_{BEJP}}\right)}\right)}{6}\right)^2 + (M-T)^2}}. \tag{4.37}$$

5 BCIs for C_{py} and C_{Npmk}

Franklin and Wasserman [22] developed the bootstrap approach, which is typically employed to obtain the empirical distribution of different PCIs. We evaluated standard BCIs for C_{py} and C_{Npmk}. The basic steps for bootstrapping are as follows:

- A random sample $(x_1, x_2, \ldots, x_n)$ of size n, is drawn from FD(α, β), followed by n bootstrap samples (with replacement) taken from the original sample with a mass of $1/n$ at each. Classical and Bayesian estimates of $(\hat{\alpha}, \hat{\beta})$ are obtained and then R-th bootstrap estimator are calculated for C_{py} & C_{Npmk}, i.e.,

$$\hat{C}_{py}^{(R)} = \hat{C}_{py}(x_1, x_2, \cdots, x_n) \;\&$$
$$\hat{C}_{Npmk}^{(R)} = \hat{C}_{Npmk}(x_1, x_2, \cdots, x_n)$$

- A total of n^n re-samples exist, and for each sample, $\hat{C}_{py}^{(R)}$ is obtained, resulting in a full bootstrapped distribution $\left\{\hat{C}_{py}^{*(\mathcal{J})}; \mathcal{J} = 1(1)\mathcal{B}\right\}$ and is denoted by $\hat{C}_{py}^{*(1)} \leq \hat{C}_{py}^{*(2)} \leq \cdots \leq \hat{C}_{py}^{*(\mathcal{B})}$, and same process is done for $\hat{C}_{Npmk}^{(R)}$.

6 Simulation study

The present section deals with the simulation experiment for performance assessments of the PCIs namely C_{py} and C_{Npmk} for FD based on aforementioned estimation methods. Different parametric combinations (α, β) = [(1, 3.5)(1, 2)(1, 4)(1.2, 2.2)] for C_{py} and (1, 2.5), (2.1, 1.6), [1.6, 2.1, (1.6,1.7)] for C_{Npmk} against sample size $n = 10, 15, 20, 25, 30, 50$ are evaluated and the LSL and USL are set as 1 and 4, respectively whereas, target value is 2.5 and $p_0 = 0.95$. For each design, 95% BCIs were evaluated, along the difference of both the upper and lower confidence bounds, referred as the average width, $(\sqrt{\beta_1})$, (β_2) , and MSE, to compare the results shown in Tables 2–5 for PCI C_{py} and Tables 6–9 for PCI C_{Npmk} with 10,000 replications using the R-software. Results are evaluated based on indices with reduced average width and minimum MSE.

The results presented in Tables 2–9, all demonstrate that with increase in sample size, both average width and MSE are reduced for all PCIs against all estimation methods, suggesting that a small sample size, $n < 25$ and moderately large (30, 50) sample produces better results.

If we look more closely, we can find that the BERP has minimum MSEs for small to moderate sample sizes when compared to MLE, LSE, WLSE, CVME, ADE, RTADE, and BEJP. Moreover, the MSE are least for BERP for both PCIs C_{py} and C_{Npmk}. Furthermore, in this investigation BERP outperformed other estimating methods by producing the least width of estimates for the configurations under consideration.

7 Real data illustration

This section examines the actual dataset comprising of the failure timings of 20 electric carts, as presented by Rao et al. [24], listed in Table 10. To evaluate the BCIs of PCIs the estimation methods ML, LS, WLS, CVM, AD, RTAD, BEJP, and BERP are

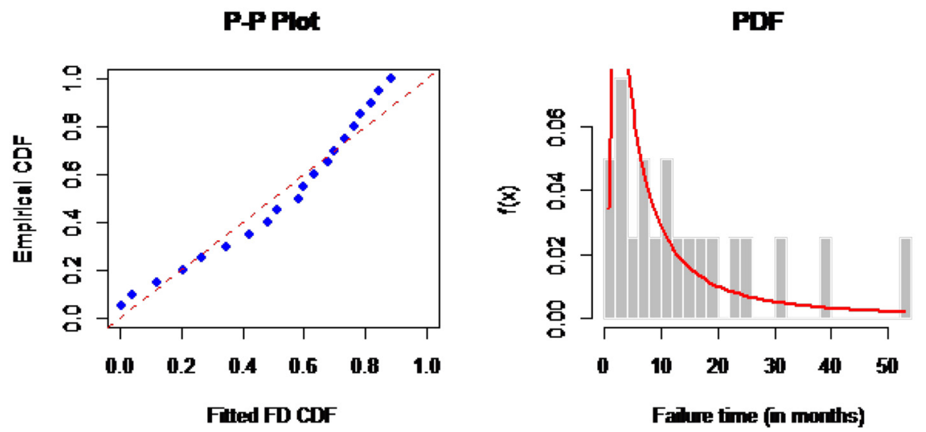

FIGURE 2
PP and density plots of FD.

TABLE 12 Estimates of PCIs.

Method	$\hat{C}_{py}$	BCI	Width	$\sqrt{\beta_1}$	β_2
C_{py}					
MLE	0.92078	(0.89945, 0.93904)	0.03959	−0.13548	2.77005
LSE	0.91166	(0.63039, 1.05263)	0.42224	−0.74486	2.98523
WLSE	0.89512	(0.68421, 1.05263)	0.36842	−0.52125	3.09433
CVME	0.87545	(0.62719, 1.05059)	0.42340	−0.85247	3.48811
ADE	1.02326	(0.99105, 1.04822)	0.05717	−0.13970	2.50095
RTADE	0.71724	(0.52361, 0.90628)	0.38267	−0.13970	3.05548
BEJP	0.92516	(0.90316, 0.93872)	0.03556	−0.57699	3.08739
BERP	0.91571	(0.89684, 0.93173)	0.03489	−0.34767	2.34959
C_{Npmk}					
MLE	0.23669	(0.33183, 0.17723)	0.15461	0.78127	3.63986
LSE	0.24974	(0.37872, 0.17924)	0.19948	0.99275	3.88410
WLSE	0.25439	(0.38106, 0.17974)	0.20131	0.87826	3.51100
CVME	0.25045	(0.37489, 0.17886)	0.19604	0.97198	4.09305
ADE	0.25037	(0.37252, 0.17733)	0.19518	0.96169	4.09897
RTADE	0.25016	(0.36858, 0.18292)	0.18566	0.89554	3.61039
BEJP	0.23583	(0.32502, 0.17629)	0.14873	0.75081	3.55995
BERP	0.23581	(0.32371, 0.17853)	0.14518	0.85753	3.94472

utilized. Initially, we evaluate if the analyzed data set originates from a FD by examining the p-value of the Kolmogorov-Smirnov (KS) test, Anderson-Darling (AD) and Cramer-von Mises (CVM) test. The point estimates for the data set are $\alpha = 0.9070176$, $\beta = 5.2836813$. The statistic along with p-values of corresponding goodness of fit test are summarized in Table 11 and complemented the analysis with graphical diagnostics, density and PP plots in Figure 2. Which are evident that this data set follows two parameter FD quite well. To estimate the PCIs, the USL and LSL, were set to 53.0 and 0.90 respectively and $p_0 = 0.95$. The mean of the data set, $T = 26.95$, is set as the target value. Table 12 shows the BCI widths of the PCIs C_{py} and C_{Npmk} based on the aforementioned estimation techniques for FD. It is clear from Table 12 that BERP has a narrow BCI than its counterparts. BERP is a more effective method for employing PCI interval estimations. For all PCIs, BERP outperformed MLE and BEJP and all aforementioned methods in terms of BCI width. The distribution shape of PCIs is approximately normal. This study suggests that BERP is more effective for examining observed PCIs.

8 Conclusion

This article examines two PCIs: C_{py} and C_{Npmk}, using various estimating approaches for FD. In general, we used the MLE, LSE, WLSE, CVME, ADE, RTADE, BERP, and BEJP to produce estimates and BCIs for the aforementioned PCIs. Extensive simulation studies were conducted to compare these approaches with various small to moderately large sample sizes and parametric values. The average estimates for these PCIs, along with their coefficient of skewness, kurtosis, BCIs, and corresponding MSEs, have been acquired. The MSE of C_{Npmk} is significantly less than that of C_{py}. Furthermore, simulation findings indicated that BERP outperformed the other estimators for MSEs and the width of PCIs. The examination of the real data set validated this finding. This study aims to help engineering professionals and applied statisticians.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

TK: Formal analysis, Methodology, Software, Writing – original draft, Conceptualization. KA: Project administration, Supervision, Software, Writing – review & editing. MZ: Resources, Writing – review & editing. ZH: Writing – review & editing, Resources, Methodology.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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