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Weak equivalence principle in quantum space

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Owing to the development of String Theory and Quantum Gravity, studies of quantized spaces described by deformed commutation relations for operators of coordinates and operators of momenta have received much attention. In this paper, the implementation of the weak equivalence principle is examined in the quantized spaces described by different types of deformed algebras, among them the noncommutative algebra of canonical type, Lie type, and the nonlinear deformed algebra with an arbitrary function of deformation depending on momenta. It is shown that the deformation of commutation relations leads to the mass-dependence of motion of a particle (a composite system) in a gravitational field, and, hence, to violation of the weak equivalence principle. We conclude that this principle is recovered in quantized spaces if one considers the parameters of the deformed algebras to be different for different particles (bodies) and to be determined by their masses.

KEYWORDS

minimal length, quantized space, deformed Heisenberg algebra, weak equivalence principle, parameters of deformed algebras

1 Introduction

String Theory and Quantum Gravity predict the existence of a minimal length [see, for instance, (Gross and Mende, 1988; Maggiore, 1993)]. This, one of the most important suggestions of these theories, follows from the generalized uncertainty principle (GUP)

$$\Delta X \geq \frac{\hbar}{2}\left(\frac{1}{\Delta P} + \beta \Delta P\right), \tag{1}$$

where β is a constant which is called the parameter of deformation. Notations ΔX, ΔP are used for position and momentum uncertainties. The inequality Eq. 1 leads to the existence of a minimal value of ΔX which is determined by the parameter β and reads $\Delta X_{min} = \hbar\sqrt{\beta}$.

One can obtain the generalized uncertainty principle (1) by considering a quadratic deformation of the commutation relations for the operator of coordinate and the operator of momentum

$$[X, P] = i\hbar\left(1 + \beta P^2\right). \tag{2}$$

Relation (2) can be generalized as

$$[X, P] = i\hbar F\left(\sqrt{\beta}\,|P|\right), \tag{3}$$

where $F(\sqrt{\beta}|P|)$ is a function, which is called the deformation function, β is a constant, $\beta \geq 0$, $F(0) = 1$.

For the invariance of the deformed commutation relation (3) upon reflection ($X \to -X, P \to -P$) and for preserving of the time-reversal symmetry the deformation function has to be even, so that $F = F(\sqrt{\beta}|P|)$. Also from Eq. 3 it follows that the deformation function has to be dimensionless, therefore a dependence of F on the dimensionless product $\sqrt{\beta}|P|$ is considered. In (Masłowski et al., 2012) the results of studies of the minimal length in the context of the deformed algebra (3) are presented and the answer to the question regarding what functions $F(\sqrt{\beta}|P|)$ lead to the minimal length is found.

Historically the first deformed algebra was proposed by Snyder in 1947 (Snyder, 1947). The algebra is well studied [see, for instance, (Romero and Zamora, 2008; Mignemi, 2011; Lu and Stern, 2012; Gnatenko Kh. P. and Tkachuk V. M., 2019b)]. In the nonrelativistic case the Snyder algebra reads

$$[X_i, X_j] = i\hbar\beta(X_iP_j - X_jP_i), \tag{4}$$

$$[X_i, P_j] = i\hbar(\delta_{ij} + \beta P_iP_j), \tag{5}$$

$$[P_i, P_j] = 0. \tag{6}$$

Also, a three-dimensional algebra leading to the minimal length was proposed by Kempf [see, for instance, (Kempf et al., 1995; Kempf, 1997; Sandor et al., 2002; Menculini et al., 2013; Gnatenko Kh. and Tkachuk V. M., 2019a)]

$$[X_i, X_j] = i\hbar\frac{(2\beta - \beta') + (2\beta + \beta')\beta P^2}{1 + \beta P^2}(P_iX_j - P_jX_i), \tag{7}$$

$$[X_i, P_j] = i\hbar(\delta_{ij}(1 + \beta P^2) + \beta' P_iP_j), \tag{8}$$

$$[P_i, P_j] = 0, \tag{9}$$

where β, β' are constants. Here the minimal length is determined by the parameters of deformation, it reads $\Delta X_{\min} = \hbar\sqrt{\beta + \beta'}$.

The most simple algebras leading to space quantization (i. e. the existence of a minimal length and minimal area), are noncommutative algebras of canonical type. In this algebras the commutators for coordinates and momenta are equal to constants

$$[X_i, X_j] = i\hbar\theta_{ij}, \tag{10}$$

$$[X_i, P_j] = i\hbar(\delta_{ij} + \sigma_{ij}), \tag{11}$$

$$[P_i, P_j] = i\hbar\eta_{ij}, \tag{12}$$

where $\theta_{ij}, \sigma_{ij}, \eta_{ij}$ are elements of constant antisymmetric matrixes [see, for example, (Djemai and Smail, 2004; Alavi, 2007; Bastos and Bertolami, 2008; Bertolami and Queiroz, 2011)]. Modification of the commutation relations in the form (10)-(12) leads to both a minimal length and a minimal momentum [see, for instance, (Gnatenko Kh. P. and Tkachuk V. M., 2018b)].

Another type of deformed algebra describing features of the spatial structure at the Planck scale is the noncommutative algebra of Lie type. It is characterized by the following commutation relations

$$[X_i, X_j] = i\hbar\theta^k_{ij}X_k. \tag{13}$$

Here θ^k_{ij} are the parameters of noncommutativity which are constants (see, for instance, (Lukierski and Woronowicz, 2006; Daszkiewicz and Walczyk, 2008; Lukierski et al., 2018)).

So, different deformed algebras, which describe features of the spatial structure at the Planck scale were proposed. These algebras can be divided into algebras of canonical type, noncommutative algebras of Lie type, and nonlinear deformed algebras (commutators for coordinates and momenta that are equal to a nonlinear function of coordinates and momenta). We would like to note that the status of the minimal length in the frame of all the algebras is the same. The minimal length indicates the min linear range in which a particle can be localized.

It is important to mention that a modification of the commutation relations for coordinates and momenta leads to violations of the fundamental laws and principles of physics, among them the weak equivalence principle. This principle is also known as the Galilean equivalence principle or universality of free fall, and is a restatement of the equality of gravitational and inertial mass. According to the weak equivalence principle, the kinematic characteristics, such as the velocity and position of a point mass in a gravitational field do not depend on its mass, composition and structure and are determined only by its initial position and velocity.

The equivalence principle was examined in the context of a noncommutative algebra of canonical type in (Bastos et al., 2011; Gnatenko, 2013; Saha, 2014; Bertolami and Leal, 2015; Gnatenko Kh. and Tkachuk V., 2017b, Gnatenko Kh. and Tkachuk, V. 2018a). The weak equivalence principle in noncommutative phase space was studied in (Bastos et al., 2011; Bertolami and Leal, 2015; Gnatenko Kh. and Tkachuk V., 2017b, Gnatenko Kh. and Tkachuk, V. 2018a). The authors of (Bertolami and Leal, 2015) concluded that the equivalence principle holds in the quantized space in the sense that an accelerated frame of reference is locally equivalent to a gravitational field, unless the parameters of noncommutativity are anisotropic ($\eta_{xy} \neq \eta_{xz}$). In the paper (Lake et al., 2019) generalized uncertainty relations that do not lead to the violation of the equivalence principle were presented. GUP models that do not require modified commutation relations, have also been proposed in (Bishop et al., 2021).

In this paper we study the weak equivalence principle in the context of different deformed algebras leading to space quantization. We show that the motion of a particle (a body) in a gravitational field in quantized space depends on its mass and composition. The weak equivalence principle is violated in quantized space. It is important that space quantization leads to a great violation of the weak equivalence principle if one considers the parameters of the deformed algebras to be the same for different particles (bodies). We conclude that in the context of different algebras (algebras with arbitrary deformation function depending on momentum, noncommutative algebras of canonical type, and noncommutative algebras of Lie type) the weak equivalence principle is recovered in the case when the parameters of deformation are different for different particles and are determined by their masses.

The paper is organized as follows. In Section 2 the weak equivalence principle is studied in the space with GUP. It is shown that the deformation of the commutation relations leads to a great violation of the weak equivalence principle. We find a condition on the parameter of deformation in which the weak equivalence principle is preserved. Section 3 is devoted to studies of the motion of a particle in a gravitational field in a noncommutative phase space of canonical type. The way to recover the weak equivalence principle in the space is proposed. Section 4 is devoted to studying a quantized space with Lie algebraic noncommutativity. It is shown that the weak equivalence principle is recovered due to the relation of the parameters of the noncommutative algebra with mass. Conclusions are presented in Section 5.

2 The weak equivalence principle in a quantized space with a nonlinear deformed algebra and the parameters of deformation

Let us examine the motion of a particle in a gravitational field in one-dimensional space characterized by a deformed algebra (1D) with an arbitrary function of deformation dependent on momenta (3). We study the following Hamiltonian

$$H = \frac{P^2}{2m} + mV(X), \tag{14}$$

where m is the mass of the particle, $V(X)$ corresponds to the gravitational potential. Note that in Eq. 14 we consider the inertial mass (mass in the first term) to be equal to the gravitational mass (mass in the second term).

In the classical limit $\hbar \to 0$ on the basis of Eq. 3 we find the deformed Poisson brackets

$$\{X, P\} = F\left(\sqrt{\beta}|P|\right). \tag{15}$$

The definition of the brackets reads

$$\{f, g\} = F\left(\sqrt{\beta}|P|\right)\left(\frac{\partial f}{\partial X}\frac{\partial g}{\partial P} - \frac{\partial f}{\partial P}\frac{\partial g}{\partial X}\right). \tag{16}$$

So, using Eq. 16, one can write the equations of motion of a particle in the gravitational field in the deformed space as follows

$$\dot{X} = \{X, H\} = \frac{P}{m}F\left(\sqrt{\beta}|P|\right), \tag{17}$$

$$\dot{P} = \{P, H\} = -m\frac{\partial V(X)}{\partial X}F\left(\sqrt{\beta}|P|\right). \tag{18}$$

From Eqs. 17, 18 it follows that the motion of a particle in a gravitational field in the space (3) depends on its mass. So, the deformed relation (3) leads to violation of the weak equivalence principle. Moreover, the GUP (3) causes a great violation of the weak equivalence principle (the value of the Eötvös parameter is many orders larger then that obtained experimentally). Let us show this by considering the motion of two particles in a uniform gravitational field $V(X) = -gX$, where g is the gravitational acceleration. On the basis of Eqs. 17, 18 one can write

$$\dot{X}^{(b)} = \frac{P^{(b)}}{m_b}F\left(\sqrt{\beta}|P^{(b)}|\right), \tag{19}$$

$$\dot{P} = m_b g F\left(\sqrt{\beta}|P^{(b)}|\right). \tag{20}$$

So, up to the first order in β we find

$$\ddot{X}^{(b)} = g + 3F'(0)g\sqrt{\beta}m_b|v| + \left(2F''(0) - (F'(0))^2\right)g\beta m_b^2 v^2, \tag{21}$$

where m_b is the mass of a particle labeled by index b ($b = (1, 2)$), $F'(x) = dF/dx$, $F''(x) = d^2F/dx^2$. The notation v is used for the velocity of motion in the gravitational field $V(X) = -gX$ in the ordinary space (i. e. the space with $\beta = 0$).

So, up to the first order in β the Eötvös parameter for particles with masses m_1, m_2 reads

$$\begin{aligned}\frac{\Delta a}{a} &= \frac{2\left(\ddot{X}^{(1)} - \ddot{X}^{(2)}\right)}{\ddot{X}^{(1)} + \ddot{X}^{(2)}}\\ &= 3F'(0)|v|\sqrt{\beta}(m_1 - m_2) + \left(2F''(0) - (F'(0))^2\right)v^2\beta\\ &\quad \times (m_1^2 - m_2^2).\end{aligned} \tag{22}$$

To estimate the value of Eq. 22 let us put $\hbar\sqrt{\beta} = l_P = \sqrt{\hbar G}/\sqrt{c^3}$ (l_P is the Planck length, c is the speed of light, G is the gravitational constant). So, we have

$$\begin{aligned}\frac{\Delta a}{a} &= 3F'(0)\frac{|v|}{c}\frac{(m_1 - m_2)}{m_P} + \left(2F''(0) - (F'(0))^2\right)\frac{v^2}{c^2}\\ &\quad \times \frac{(m_1^2 - m_2^2)}{m_P^2},\end{aligned} \tag{23}$$

with $m_P = \sqrt{\hbar c}/\sqrt{G}$ being the Planck mass (Gnatenko and Tkachuk, 2020).

Note that for $m_1 = 1$ kg, $m_2 = 0.1$ kg in the case of deformation function $F(\sqrt{\beta}|P|) = 1 + \beta P^2$ form Eq. 23 we obtain great violation of the weak equivalence principle $\Delta a/a \approx 0.1$ which has not been seen experimentally (Gnatenko and Tkachuk, 2020). From the tests of the weak equivalence principle it follows that this principle holds with high accuracy. For instance, on the basis of the Lunar Laser Ranging experiment it is known that the equivalence principle holds with precision $\Delta a/a = (-0.8 \pm 1.3) \cdot 10^{-13}$ (Williams et al., 2012). Similar results were obtained from the laboratory torsion-balance tests of the weak equivalence principle for Be and Ti in which $\Delta a/a = (0.3 \pm 1.8) \cdot 10^{-13}$ and $\Delta a/a = (-0.7 \pm 1.3) \cdot 10^{-13}$ for Be and Ti or Al (Wagner et al., 2012). The MICROSCOPE space mission aims to test the principle with accuracy 10^{-15} (Touboul et al., 2017).

It is important to mention that we have obtained a great violation of the weak equivalence principle in a space with GUP assuming that parameter of deformation β is the same for different particles. Let us consider a more general case in which the parameter of deformation is different for different particles. We use notation β_b for the parameter of deformation corresponding to a particle with index b. The weak equivalence principle can be recovered in a space with GUP, if we assume that β_b is determined by the mass of a particle as

$$\sqrt{\beta_b} m_b = \gamma = \text{const}, \tag{24}$$

where the constant γ is the same for different particles and does not depend on mass (Quesne and Tkachuk, 2010; Tkachuk, 2012; Gnatenko and Tkachuk, 2020).

Taking into account Eq. 24, we find that the Eötvös parameter written up to the first order in β is equal to zero

$$\frac{\Delta a}{a} = 3F'(0)|v|\left(\sqrt{\beta_1}\, m_1 - \sqrt{\beta_2}\, m_2\right) + \left(2F''(0) - (F'(0))^2\right)v^2 \times \left(\beta_1 m_1^2 - \beta_2 m_2^2\right) = 0. \tag{25}$$

Also, considering the parameter of deformation to be dependent on mass according to

$$\beta = \frac{\gamma^2}{m^2}, \tag{26}$$

(this expression follows from Eq. 24), the equations of motion of a particle in a gravitational field Eqs. 17, 18 read

$$\dot{X} = \frac{P}{m} F\left(\gamma \frac{|P|}{m}\right), \tag{27}$$

$$\frac{\dot{P}}{m} = -\frac{\partial V(X)}{\partial X} F\left(\gamma \frac{|P|}{m}\right). \tag{28}$$

On the basis of Eqs. 27, 28 we have that the equations for X and P/m do not contain mass. Therefore, the solutions $X(t)$, $P(t)/m$ of these equations also do not depend on mass. So, the weak equivalence principle is recovered due to the assumption Eq. 24 (Tkachuk, 2012; Gnatenko and Tkachuk, 2020).

Let us also study the weak equivalence principle in the more general three-dimensional case of deformed (3D) algebras. Namely, let us consider the following commutation relations

$$\left[X_i, X_j\right] = G(P^2)\left(X_i P_j - X_j P_i\right), \tag{29}$$

$$\left[X_i, P_j\right] = f(P^2)\delta_{ij} + F(P^2) P_i P_j, \tag{30}$$

$$\left[P_i, P_j\right] = 0. \tag{31}$$

The algebra Eqs. 29–31 is a generalization of the well known Snyder Eqs. 4–6 and Kempf Eqs. 7–9 algebras. The functions $G(P^2)$, $F(P^2)$ and $f(P^2)$ in Eqs. 29–31 have to satisfy the relation

$$f(F - G) - 2f'(f + FP^2) = 0, \tag{32}$$

(here $f' = \partial f/\partial P^2$) which follows from the Jacobi identity (Frydryszak and Tkachuk, 2003).

From the classical limit of Eqs. 29–31 we obtain the following Poisson brackets

$$\left\{X_i, X_j\right\} = G(P^2)\left(X_i P_j - X_j P_i\right), \tag{33}$$

$$\left\{X_i, P_j\right\} = f(P^2)\delta_{ij} + F(P^2) P_i P_j, \tag{34}$$

$$\left\{P_i, P_j\right\} = 0. \tag{35}$$

Let us study the weak equivalence principle in the quantized space represented by Eqs. 33–35. Considering a particle in a gravitational field $V(\mathbf{X})$ with Hamiltonian

$$H = \sum_i \frac{P_i^2}{2m} + mV(\mathbf{X}), \tag{36}$$

and taking into account the deformation of the Poisson brackets Eqs. 33–35, we find the following equations of motion

$$\dot{X}_i = \frac{P_i}{m} f(P^2) + m \sum_j \frac{\partial V(\mathbf{X})}{\partial X_j} G(P^2)\left(X_i P_j - X_j P_i\right), \tag{37}$$

$$\dot{P}_i = -m \frac{\partial V(\mathbf{X})}{\partial X_i} \tilde{f}(\beta P^2) - m \sum_j \frac{\partial V(\mathbf{X})}{\partial X_j} F(P^2) P_i P_j. \tag{38}$$

On the basis of dimensional considerations the functions $f(P^2)$, $F(P^2)$, $G(P^2)$ can be rewritten as $f(P^2) = \tilde{f}(\beta P^2)$, $F(P^2) = \beta \tilde{F}(\beta P^2)$ and $G(P^2) = \beta \tilde{G}(\beta P^2)$, where $\tilde{f}(\beta P^2)$, $\tilde{F}(\beta P^2)$ and $\tilde{G}(\beta P^2)$ are dimensionless functions. Taking this into account, and considering the condition Eq. 26, one can rewrite the equations of motion of a particle in a gravitational field as follows

$$\dot{X}_i = P'_i \tilde{f}\left(\gamma^2 (P')^2\right) + \gamma^2 \sum_j \frac{\partial V(\mathbf{X})}{\partial X_j} \tilde{G}\left(\gamma^2 (P')^2\right) \times \left(X_i P'_j - X_j P'_i\right), \tag{39}$$

$$\dot{P}'_i = -\frac{\partial V(\mathbf{X})}{\partial X_i} \tilde{f}\left(\gamma^2 (P')^2\right) - \gamma^2 \sum_j \frac{\partial V(\mathbf{X})}{\partial X_j} \tilde{F} \times \left(\gamma^2 (P')^2\right) P'_i P'_j, \tag{40}$$

where $P'_i = P_i/m$. It is important that Eqs. 39, 40 do not depend on mass. So, the weak equivalence principle is preserved in quantized space Eqs. 33–35 if the relation of the parameter of deformation with mass Eq. 26 is satisfied (Gnatenko and Tkachuk, 2020).

It is also important to mention that the relation Eq. 26 gives a possibility to preserve the additivity property of the kinetic energy in a space with GUP and to solve the problem of the significant effect of the GUP on the kinetic energy of a macroscopic body [for details see (Gnatenko and Tkachuk, 2020)].

3 Influence of noncommutativity of coordinates and noncommutativity of momenta on the motion in a gravitational field

Let us study the motion of a particle in a uniform gravitational field in the context of a noncommutative algebra of canonical type (2D)

$$[X_1, X_2] = i\hbar\theta, \tag{41}$$

$$\left[X_i, P_j\right] = i\hbar\delta_{ij}, \tag{42}$$

$$[P_1, P_2] = i\hbar\eta, \tag{43}$$

where the parameters of noncommutativity θ, η are constants and $i, j = (1, 2)$. In the classical limit we obtain the following deformed Poisson brackets

$$\{X_1, X_2\} = \theta, \tag{44}$$

$$\left\{X_i, P_j\right\} = \delta_{ij}, \tag{45}$$

$$\{P_1, P_2\} = \eta. \tag{46}$$

Let us examine the motion of a particle in a gravitational field in the space Eqs. 44–46 and find the way to preserve the weak equivalence principle (Gnatenko, 2013; Gnatenko Kh. and Tkachuk V., 2017b, Gnatenko Kh. and Tkachuk, V. 2018a). The equations of motion of a particle with mass m in a uniform gravitational field with Hamiltonian

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} - mgX_1, \tag{47}$$

read

$$\dot{X}_1 = \{X_1, H\} = \frac{P_1}{m}, \tag{48}$$

$$\dot{X}_2 = \{X_2, H\} = \frac{P_2}{m} + mg\theta, \tag{49}$$

$$\dot{P}_1 = \{P_1, H\} = mg + \eta\frac{P_2}{m}, \tag{50}$$

$$\dot{P}_2 = \{P_2, H\} = -\eta\frac{P_1}{m}. \tag{51}$$

In Eq. 47 one considers the field directed along the X_1 axis. Note that in the two-dimensional case the noncommutative algebra of canonical type Eqs. 41–43 is rotationally invariant, so the results and conclusions we obtain, considering this particular case, can be generalized to the case of the arbitrary direction of the field.

The solution of Eqs. 48–51 with initial conditions $X_1(0) = X_{01}$, $X_2(0) = X_{02}$, $\dot{X}_1(0) = \upsilon_{01}$, $\dot{X}_2(0) = \upsilon_{02}$ is the following

$$X_1(t) = \frac{m\upsilon_{01}}{\eta}\sin\frac{\eta}{m}t + \left(\frac{m^2 g}{\eta^2} - \frac{m^2 g\theta}{\eta} + \frac{m\upsilon_{02}}{\eta}\right)\left(1 - \cos\frac{\eta}{m}t\right) + X_{01}, \tag{52}$$

$$X_2(t) = \left(\frac{m^2 g}{\eta^2} - \frac{m^2 g\theta}{\eta} + \frac{m\upsilon_{02}}{\eta}\right)\sin\frac{\eta}{m}t - \frac{m\upsilon_{01}}{\eta}\left(1 - \cos\frac{\eta}{m}t\right) - \frac{mg}{\eta}t + mg\theta t + X_{02}. \tag{53}$$

The obtained expressions Eqs. 52, 53 depend on mass, if we assume that the parameters of noncommutativity θ, η are the same for different particles. In this case the weak equivalence principle is violated in the noncommutative phase space of canonical type. The way to solve this problem is to consider, as in the previous section, that the parameters of noncommutativity are dependent upon mass (Gnatenko Kh. and Tkachuk V., 2017b).

The trajectory of a particle in the uniform gravitational field depends on $m\theta$ and η/m. So, if these values do not depend on mass then the weak equivalence principle is recovered. So, let us consider the following conditions

$$\theta m = \gamma = \text{const}, \tag{54}$$

$$\frac{\eta}{m} = \alpha = \text{const}, \tag{55}$$

where γ, α are the same for different particles. Taking into account Eqs. 54, 55, the trajectory Eqs. 52, 53 transforms to

$$X_1(t) = \frac{\upsilon_{01}}{\alpha}\sin\alpha t + \left(\frac{g}{\alpha^2} - \frac{g\gamma}{\alpha} + \frac{\upsilon_{02}}{\alpha}\right)(1 - \cos\alpha t) + X_{01}, \tag{56}$$

$$X_2(t) = \left(\frac{g}{\alpha^2} - \frac{g\gamma}{\alpha} + \frac{\upsilon_{02}}{\alpha}\right)\sin\alpha t - \frac{\upsilon_{01}}{\alpha}(1 - \cos\alpha t) - \frac{g}{\alpha}t + \gamma g t + X_{02}. \tag{57}$$

The trajectory of a particle in the gravitational field Eqs. 56, 57 is determined by its initial coordinates and velocities and does not depend on its mass. So, the weak equivalence principle is recovered in the noncommutative phase space of canonical type due to the relations Eqs. 54, 55 (Gnatenko Kh. and Tkachuk V., 2017b).

It is worth also mentioning that for $\eta \to 0$ the expressions Eqs. 52, 53 reduce to the well known result for the trajectory of a particle in a uniform gravitational field in ordinary space, $X_1(t) = gt^2/2 + \upsilon_{01}t + X_{01}$, $X_2(t) = \upsilon_{02}t + X_{02}$. At the same time, the noncommutativity of the coordinates affects the relation between the momenta and velocities, such that

$$P_1 = m\dot{X}_1, \quad P_2 = m\left(\dot{X}_2 + mg\theta\right). \tag{58}$$

In the case when the parameter of coordinate noncommutativity is inversely proportional to the mass on the basis of Eq. 58 we obtain that the momentum P_2 is proportional to mass, as it is in ordinary space $P_2 = m(\dot{X}_2 + \gamma g)$.

In the more general case of a particle in a nonuniform gravitational field $V(X_1, X_2)$ with Hamiltonian

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + mV(X_1, X_2), \tag{59}$$

The equations of motion read

$$\dot{X}_1 = \{X_1, H\} = \frac{P_1}{m} + m\theta\frac{\partial V(X_1, X_2)}{\partial X_2}, \tag{60}$$

$$\dot{X}_2 = \{X_2, H\} = \frac{P_2}{m} - m\theta\frac{\partial V(X_1, X_2)}{\partial X_1}, \tag{61}$$

$$\dot{P}_1 = \{P_1, H\} = -m\frac{\partial V(X_1, X_2)}{\partial X_1} + \eta\frac{P_2}{m}, \tag{62}$$

$$\dot{P}_2 = \{P_2, H\} = -m\frac{\partial V(X_1, X_2)}{\partial X_2} - \eta\frac{P_1}{m}. \tag{63}$$

To obtain Eqs. 60–63 we take into account Eqs. 44–46. From Eqs. 60–63 we can conclude that the weak equivalence principle is violated, if the parameters of noncommutativity are the same for different particles. In the case when the conditions on the parameters of noncommutativity Eqs. 54, 55 hold, introducing the notation $P_i' = P_i/m$, we can write

$$\dot{X}_1 = P_1' + \gamma\frac{\partial V(X_1, X_2)}{\partial X_2}, \quad \dot{X}_2 = P_2' - \gamma\frac{\partial V(X_1, X_2)}{\partial X_1}, \tag{64}$$

$$\dot{P}_1' = -\frac{\partial V(X_1, X_2)}{\partial X_1} + \alpha P_2', \quad \dot{P}_2' = -\frac{\partial V(X_1, X_2)}{\partial X_2} - \alpha P_1'. \tag{65}$$

Eqs. 64, 65 depend on the parameters γ and α and do not depend on mass. As a result, $X_i = X_i(t)$, $P_i' = P_i'(t)$ also do not depend on mass. So, conditions Eqs. 54, 55 give a possibility to preserve the weak equivalence principle also in the case of motion in a nonuniform gravitational field (Gnatenko Kh. and Tkachuk V., 2017b).

It is worth noting that in this section we consider the two-dimensional case of the noncommutative algebra of canonical type Eqs. 41–43, because it is rotationally-invariant. In the three-dimensional noncommutative phase space of canonical type one faces the problem of rotational symmetry breaking. A 3D algebra which is rotationally invariant and equivalent to the noncommutative algebra of canonical type was proposed in (Gnatenko K. P. and Tkachuk V. M., 2017a). It is important to mention that to recover the weak equivalence principle in the context of this algebra the idea to relate the parameters of noncommutativity with mass has to be considered [for details see (Gnatenko, 2018)].

4 Weak equivalence principle in noncommutative space of Lie type

Let us also study the motion of a composite system in a gravitational field in a space with a noncommutative algebra of Lie type and examine the weak equivalence principle. We consider the following algebra

$$\{X_i, X_j\} = \theta^0_{ij}t + \theta^k_{ij}X_k, \tag{66}$$

$$\{X_i, P_j\} = \delta_{ij} + \bar{\theta}^k_{ij}X_k + \tilde{\theta}^k_{ij}P_k, \quad \{P_i, P_j\} = 0, \tag{67}$$

where $i, j, k = (1, 2, 3)$, θ^0_{ij}, θ^k_{ij}, $\bar{\theta}^k_{ij}$, $\tilde{\theta}^k_{ij}$ are the parameters of noncommutativity, that are antisymmetric in their lower indexes $\theta^0_{ij} = -\theta^0_{ji}$, $\bar{\theta}^k_{ij} = -\bar{\theta}^k_{ji}$, $\tilde{\theta}^k_{ij} = -\tilde{\theta}^k_{ij}$ (Miao et al., 2011).

From the Jacobi identity it follows that the parameters θ^0_{ij}, θ^k_{ij}, $\bar{\theta}^k_{ij}$, $\tilde{\theta}^k_{ij}$ can not be arbitrary. In the particular case when

$$\theta^0_{kl} = -\theta^0_{ky} = \frac{1}{\kappa}, \quad \theta^0_{ly} = \frac{1}{\kappa}, \tag{68}$$

$$\theta^l_{ky} = -\theta^k_{ly} = \bar{\theta}^l_{ky} = -\bar{\theta}^k_{ly} = \frac{1}{\tilde{\kappa}}, \tag{69}$$

the noncommutative algebra of Lie type reads

$$\{X_k, X_y\} = -\frac{t}{\kappa} + \frac{X_l}{\tilde{\kappa}}, \quad \{X_l, X_y\} = \frac{t}{\kappa} - \frac{X_k}{\tilde{\kappa}}, \tag{70}$$

$$\{X_k, X_l\} = \frac{t}{\kappa}, \quad \{P_k, X_y\} = \frac{P_l}{\tilde{\kappa}}, \tag{71}$$

$$\{P_l, X_y\} = -\frac{P_k}{\tilde{\kappa}}, \quad \{X_i, P_j\} = \delta_{ij}, \tag{72}$$

$$\{X_y, P_y\} = 1 \quad \{P_m, P_n\} = 0, \tag{73}$$

[see (Miao et al., 2011)]. Choosing

$$\theta^0_{ly} = -\theta^0_{ky} = \frac{1}{\kappa}, \quad \theta^l_{ky} = -\theta^k_{ly} = \frac{1}{\tilde{\kappa}}, \tag{74}$$

$$\tilde{\theta}^l_{ky} = -\tilde{\theta}^k_{ly} = \frac{1}{\tilde{\kappa}}, \tag{75}$$

$$\bar{\theta}^l_{ky} = -\bar{\theta}^k_{ly} = \frac{1}{\tilde{\kappa}}, \tag{76}$$

We obtain the following noncommutative algebra

$$\{X_k, X_y\} = -\frac{t}{\kappa} + \frac{X_l}{\tilde{\kappa}}, \quad \{X_l, X_y\} = \frac{t}{\kappa} - \frac{X_k}{\tilde{\kappa}}, \tag{77}$$

$$\{X_k, X_l\} = 0, \quad \{P_k, X_y\} = \frac{X_l}{\tilde{\kappa}} + \frac{P_l}{\tilde{\kappa}}, \tag{78}$$

$$\{P_l, X_y\} = \frac{X_k}{\tilde{\kappa}} - \frac{P_k}{\tilde{\kappa}}, \quad \{X_i, P_j\} = \delta_{ij}, \tag{79}$$

$$\{X_y, P_y\} = 1, \quad \{P_m, P_n\} = 0, \tag{80}$$

[see (Miao et al., 2011)].

The equations of motion of a particle with mass m in a gravitational field $V = V(X_1, X_2, X_3)$ with Hamiltonian $H = \frac{\mathbf{P}^2}{2m} + mV(X_1, X_2, X_3)$ in a space with a noncommutative algebra of Lie type read

$$\dot{X}_i = \frac{P_i}{m} + \bar{\theta}^k_{ij}\frac{P_jX_k}{m} + \tilde{\theta}^k_{ij}\frac{P_jP_k}{m} + m\left(\theta^0_{ij}t + \theta^k_{ij}X_k\right)\frac{\partial V}{\partial X_j}, \tag{81}$$

$$\dot{P}_i = -m\frac{\partial V}{\partial X_i} - m\left(\bar{\theta}^k_{ij}X_k + \tilde{\theta}^k_{ij}P_k\right)\frac{\partial V}{\partial X_j}. \tag{82}$$

The equivalence principle is recovered if the following conditions are satisfied (Gnatenko, 2019)

$$\theta^{0(b)}_{ij}m_b = \gamma^0_{ij} = \text{const}, \quad \theta^{k(b)}_{ij}m_b = \gamma^k_{ij} = \text{const}, \tag{83}$$

$$\tilde{\theta}^{k(b)}_{ij}m_b = \tilde{\gamma}^k_{ij} = \text{const}, \quad \bar{\theta}^{k(b)}_{ij} = \bar{\theta}^k_{ij}. \tag{84}$$

The constants γ^0_{ij}, γ^k_{ij}, $\tilde{\gamma}^k_{ij}$ are the same for different particles, $\gamma^0_{ij} = -\gamma^0_{ji}$, $\gamma^k_{ij} = -\gamma^k_{ji}$, $\tilde{\gamma}^k_{ij} = -\tilde{\gamma}^k_{ji}$. Taking into account (83), (84) and using the notation $P'_i = P_i/m$ the equations of motion of a particle in an arbitrary gravitational field can be rewritten as

$$\dot{X}_i = P'_i + \bar{\theta}^k_{ij} P'_j X_k + \tilde{\gamma}^k_{ij} P'_j P'_k + \left(\gamma^0_{ij} t + \gamma^k_{ij} X_k\right) \frac{\partial V}{\partial X_j}, \quad (85)$$

$$\dot{P}'_i = -\frac{\partial V}{\partial X_i} - \left(\bar{\theta}^k_{ij} X_k + \tilde{\gamma}^k_{ij} P'_k\right) \frac{\partial V}{\partial X_j}. \quad (86)$$

The obtained Eqs. 85, 86 do not contain mass. So, conditions Eqs. 83, 84 give a possibility to recover the weak equivalence principle in a space characterized by a noncommutative algebra of Lie type Eqs. 66, 67.

5 Discussion

The idea to describe features of the spatial structure at the Planck scale (the existence of a minimal length) with the help of deformed algebras has been considered. Deformed algebras of different types have been studied. Among them are deformed algebras with arbitrary functions of deformation that depends on momenta (these algebras are generalizations of the nonrelativistic Snyder and Kempf algebras), algebras with noncommutativity of the coordinates and noncommutativity of the momenta of canonical type, and noncommutative algebras of Lie type. The implementation of the weak equivalence principle has been examined in the quantized spaces described by these deformed algebras.

We have shown that, considering the parameters of the deformed algebras to be the same for different particles (different bodies), one faces the problem of violation of the weak equivalence principle. In this case the motion of a particle in a gravitational field in quantized space depends on its mass and composition. Even in the case of equality between the gravitational and the inertial masses of a body the Eötvös parameter is not equal to zero. Besides space quantization leads to great violation of this principle which should have been seen experimentally (see Eq. 23). To solve this problem the dependence of the parameters of the deformed algebras on mass has been considered. We have shown that if the parameters of the deformed algebras for coordinates and momenta are related to the particle mass the weak equivalence principle is preserved in noncommutative phase spaces of canonical type, in spaces with Lie algebraic noncommutativity, and in spaces with an arbitrary function of deformation dependent on momenta. In addition, the same relations for the parameters of deformation (parameters of noncommutativity) on mass give a possibility to recover the properties of the kinetic energy (its additivity and independence of compositions) and to solve the problem of the great effect of the minimal length on the motion of macroscopic bodies which is well known in the literature as the soccer-ball problem (Gnatenko and Tkachuk, 2020, Gnatenko Kh. and Tkachuk, V. 2017b; Gnatenko, 2019).

Data availability statement

The datasets analyzed during the current study are available from the corresponding author on reasonable request.

Author contributions

KG and VT contributed in equal part to the manuscript.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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